HWA CHONG INSTITUTION

2023 JC1 Promotional Examination

Higher 2

9758/01

MATHEMATICS

27 September 2023

3 hours

Candidate Name	

CT Group	23

Write here how many additional pieces of writing paper you have used (if any).	

Candidates answer on the Question Paper.

Additional materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Do not write anything on the List of Formulae (MF26).

Write in dark blue or black pen. You may use HB pencil for any diagrams or graphs. Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions. Write your answers in the spaces provided in the Question Paper.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

You are expected to use an approved graphing calculator.

Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.

Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part of question.

Blank Pages are designated for any extra working that is required. Clearly indicate the question number on the Blank Page where applicable.

For Examiner's Use			
Qn	**Marks**	**Total**	**Remarks**
1		5	
2		3	
3		6	
4		6	
5		6	
6		6	
7		7	
8		7	
9		8	
10		9	
11		11	
12		13	
13		13	
		100	

This document consists of **27** printed pages and **3** blank pages.

Remarks

a) **INSTR**: Follow **instructions** as stated in Question (e.g. correct s.f , exact values, coordinates, similar form etc.)
b) **NOT**: Correct Mathematical **Notations**
c) **ACC**: **Accuracy** of Answers (e.g. affected by early rounding off, not writing +C for indefinite integrals etc.)

1. Without using a calculator, solve the inequality

$$\frac{4x^2+6x-12}{2x-1} \leq 3.$$ [3]

Deduce the solution of the inequality $\dfrac{4e^{2x}+6e^{x}-12}{2e^{x}-1} \leq 3$ in exact form. [2]

2. The function $y = \mathrm{f}(x)$ is a **one-to-one** function.

The graphs of $y = \mathrm{f}(|x|)$ and $y = |\mathrm{f}(x)|$ are shown below in Figure 1 and 2 respectively.

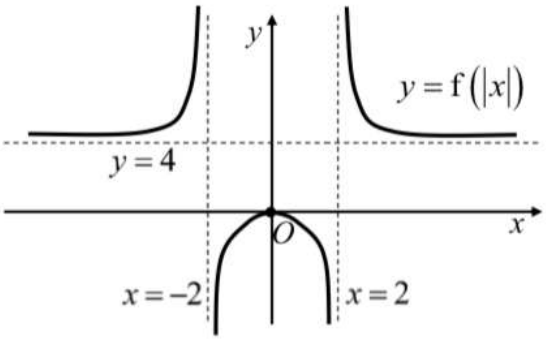


Figure 1

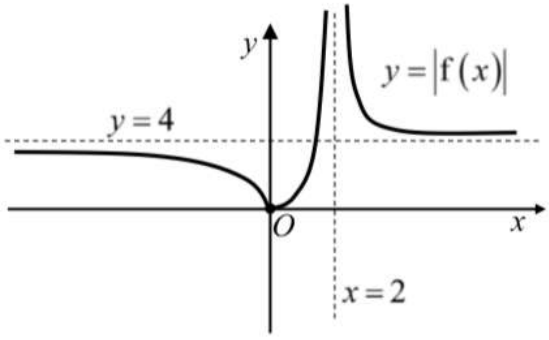


Figure 2

(a) State the range of values of x for which $\mathrm{f}(x) \le 0$. [1]

(b) State the equation(s) of the asymptote(s) of the graph of $y = \dfrac{1}{\mathrm{f}(x)}$. [2]

3. The function h is defined by

$$\mathrm{h}: x \mapsto \ln\left(3x^2 - 9\right) \quad \text{for } x \in \mathbb{R}, \quad x < -\sqrt{3} \text{ or } x > \sqrt{3}.$$

(a) Explain why h does not have an inverse. [1]

If the domain of h is further restricted to $-3 \leq x < \lambda$, where $\lambda \in \mathbb{R}$ such that h^{-1} will exist.

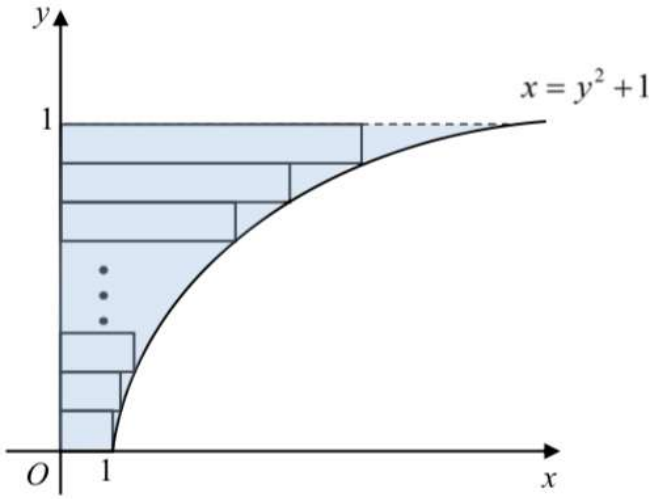
(b) State the largest exact value of λ. [1]

Use the value of λ found in part **(b)** for the rest of the parts of this question.

(c) Find $\mathrm{h}^{-1}(x)$ and state the domain of h^{-1}. [3]

(d) Find the set of values of x for which $\mathrm{h}^{-1}\mathrm{h}(x) = \mathrm{h}\,\mathrm{h}^{-1}(x)$. [1]

4. The shaded region R is bounded by the x-axis, the y-axis, the line $y=1$ and the curve $x=y^2+1$. There are n rectangles of equal width drawn as shown in the diagram below. The total area of all the n rectangles, A_n, approximates the area of the shaded region R.



It is given that $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$.

(a) Show that $A_n = 1 + \frac{(n-1)(2n-1)}{6n^2}$. [3]

(b) Find the exact value of $\lim_{n\to\infty} A_n$, showing your working. [3]

5. The position vectors of points P and R with respect to the origin O are $\mathbf{p}$ and $\mathbf{r}$ respectively, and $\mathbf{q}$ is the position vector of the point Q on PR such that $2\overrightarrow{PQ} = \overrightarrow{QR}$.

(a) Write down $\mathbf{q}$ in terms of $\mathbf{p}$ and $\mathbf{r}$. [1]

(b) It is given that $OR = 3OP$ and the angle between $\mathbf{p}$ and $\mathbf{r}$ is θ where $\cos\theta = \frac{4}{5}$. F is the foot of the perpendicular from Q to the line OP.

(i) By using a suitable scalar product, find the exact value of OF in terms of $|\mathbf{p}|$. [4]

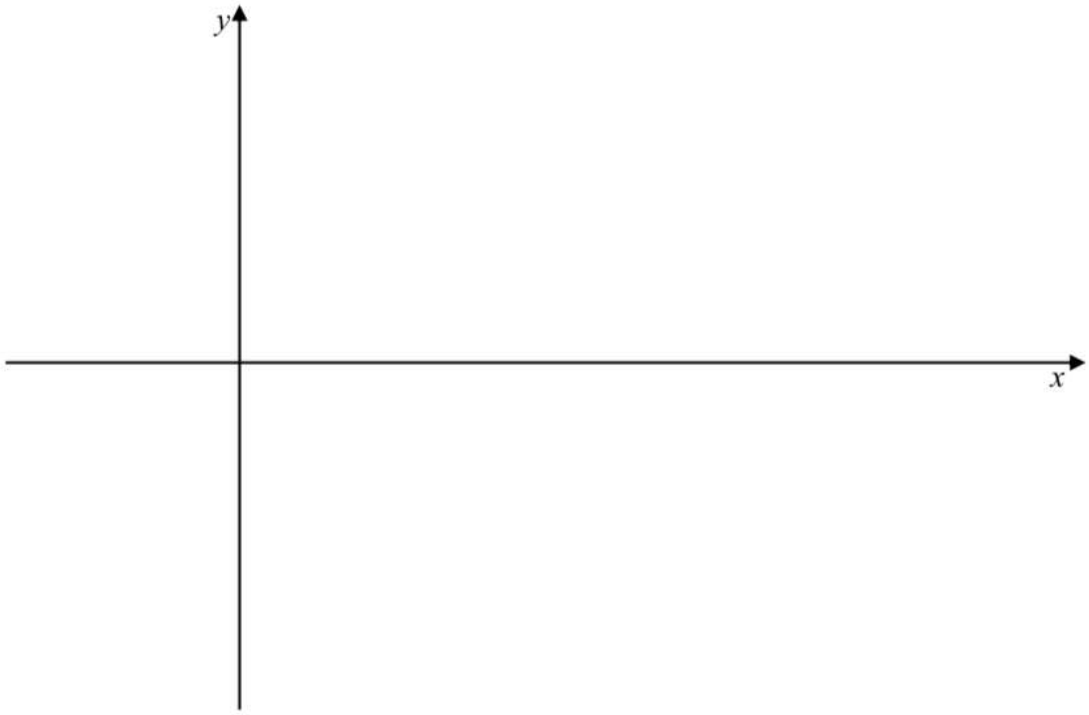
(ii) Hence determine if F lies within the line segment OP, justifying your answer. [1]

6. A piecewise function is given by

$$\mathrm{f}(x)=\begin{cases}\cos\left(\dfrac{x}{2}\right) & \text{for } 0\le x<2\pi,\\ \dfrac{2}{\pi}x-5 & \text{for } 2\pi\le x<3\pi,\end{cases}$$

and that $\mathrm{f}(x)=\mathrm{f}(x+3\pi)$ for all real values of x.

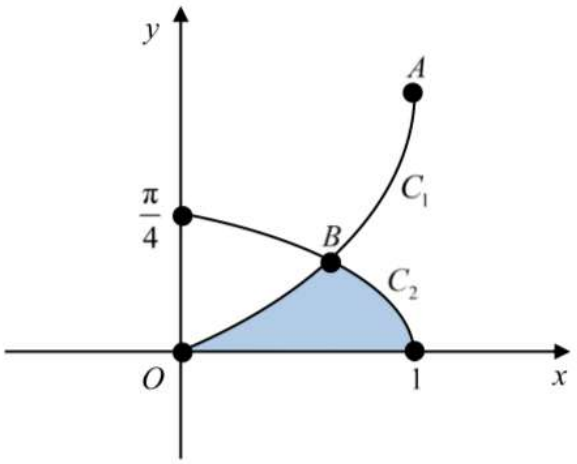
(a) Sketch the graph of $y=\mathrm{f}(x)$ for $-\dfrac{\pi}{2}\le x<4\pi$, labelling the coordinates of the end-points clearly. [3]



(b) Find the exact area of the region bounded by the graph of $y = \mathrm{f}(x)$ and the x-axis for $-\frac{\pi}{2} \leq x < 4\pi$, leaving your answer in terms of π. [3]

7. **(a)** Find $\int \sin^2 \theta \, \mathrm{d}\theta$. [2]

(b) The diagram below shows curves C_1 and C_2 with equations $y = \sin^{-1} x$ and $\frac{\pi}{2} - 2y = \sin^{-1} x$ respectively, for $0 \le x \le 1$. Points A and B lie on C_1 where $A\left(1, \frac{\pi}{2}\right)$ and $B\left(\frac{1}{2}, \frac{\pi}{6}\right)$.



Using integration, find the exact volume generated when the shaded region bounded by the x-axis and the curves C_1 and C_2 in the first quadrant is rotated completely about the y-axis. [5]

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8. **(a)** It is given that

$$1-a+a^2-a^3+...$$

is a convergent geometric series, where a is a real constant.

Find the range of values of a such that its sum to infinity is not more than 5. [4]

(b) It is given that

U_n is the n^{th} term of the geometric sequence $\{1, -a, a^2, -a^3, ...\}$,

T_n is the n^{th} term of the geometric sequence $\{1, a, a^2, a^3, ...\}$ and

$W_n = U_n + T_n$.

A student commented that W_n is a geometric progression with first term 2 and common ratio a^2. State, with justification, if the student is correct. [1]

(c) Express W_{2n+1} in the form pa^{qn}, where p and q are real constants. [2]

9. **(a)** Find $\int \frac{1}{\sqrt{2-x}}\,\mathrm{d}x$, where $x<2$. [2]

(b) Use the substitution $u^2=2-x$ to find $\int x\sqrt{2-x}\,\mathrm{d}x$, where $x<2$. [3]

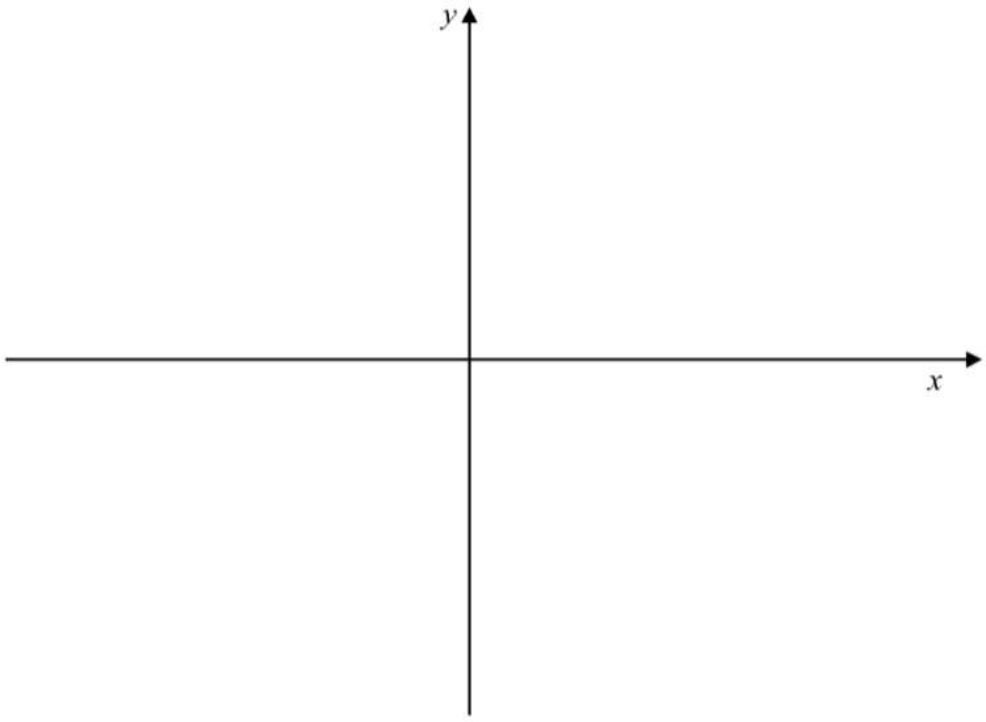
(c) Hence use integration by parts to find $\int \frac{x^2}{\sqrt{2-x}}\,\mathrm{d}x$, where $x < 2$. [3]

10. It is given that curve C has the equation $y = ax + \frac{2a}{x+1}$, where $a > 0, x \neq -1$.

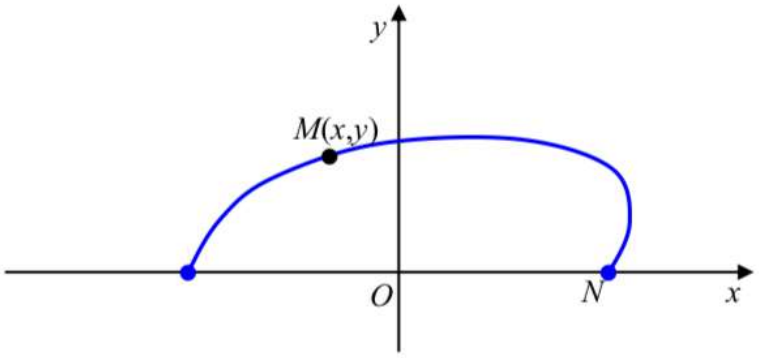
(a) Using an algebraic method, find the exact range of values of y for which there are points on C, leaving your answer in terms of a. [4]

(b) Find, by differentiation, the exact x-coordinates of the turning point(s) on C. [2]

(c) Sketch C, clearly indicating the coordinates of the axial intercept(s), stationary point(s) and equations of any asymptote(s). [3]



11. The diagram below shows the curve C with equation $x^2 - xy + y^3 = 16$. N is a fixed point where C intersects the positive x-axis and M is a variable point (x, y) on C where $y > 0$.



(a) Show that $\left(3y^2 - x\right)\dfrac{\mathrm{d}y}{\mathrm{d}x} - y + 2x = 0$. [2]

(b) Show that $\left(3y^2 - x\right)\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + 6y\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - 2\dfrac{\mathrm{d}y}{\mathrm{d}x} + 2 = 0$. [2]

(c) Show that A, the area of triangle OMN, is given by $2y$. [2]

(d) Hence find the value of x for which A has a stationary value. Determine the nature of this stationary value. [5]

12. Points (x, y, z) are defined relative to an origin $(0,0,0)$ on a horizontal ground, where units are measured in metres. Engineers are installing tension wires on a construction site. The tension wires with negligible thickness are constructed in straight lines. A slanted platform $\prod$ is built on top of the horizontal ground of the construction site. The equation of $\prod$ is $5x - y - 11z + 38 = 0$.

It is given that points $A(6, 2, 6)$ and $B(1, 3, 17)$ lie on tension wire R.

(a) Find a vector equation of tension wire R. What can you say about tension wire R and the slanted platform $\prod$? [3]

The cartesian equation of tension wire Q is given by

$$21(x-1)=15(y-3)=7(z-17).$$

(b) Find the acute angle between tension wires Q and R, giving your answer in degrees. [3]

12 [Continued]

(c) Find the coordinates of the point C at which tension wire Q meets the slanted platform $\prod$. [3]

Shining a light source on the tension wires can help improve visibility, making it easier to inspect wires, especially in dimly lit area.

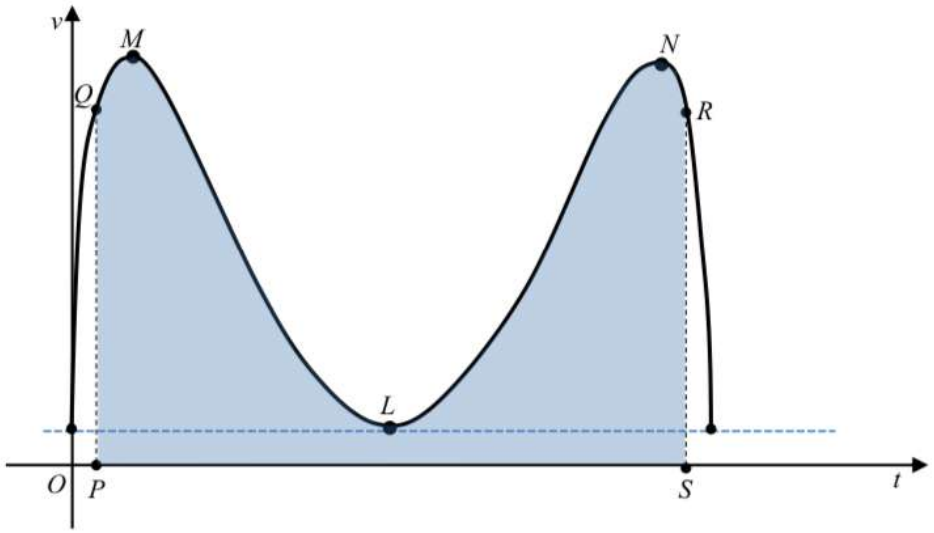
(d) A fixed light source at point L located above the horizontal ground with coordinates $(-6, -1, 5)$ is to be shone on tension wire Q. Find the shortest distance from L to tension wire Q. [4]

13. The diagram below shows the change in speed of a car on a roller coaster ride where the vertical axis represents the speed, v in metres per second (m/s) of the car on the roller coaster and the horizontal axis represents the time, t in seconds (s).

The change in speed of the car of the roller coaster ride from point Q to R forms part of the curve $QMLNR$ with parametric equations

$$t = 2(\theta - \sin\theta), \qquad v = a + b(1 - \cos 2\theta), \quad \text{for } 0 \le \theta \le 2\pi,$$

where a and b are positive constants.



PQ and SR are parallel to the vertical axis, and $PQ = SR$.

At the point Q, $\theta = \alpha$ and at the point R, $\theta = 2\pi - \alpha$.

(a) **(i)** Show that the area of the shaded region can be expressed as

$$\int_{\alpha}^{2\pi-\alpha} \left[(2a+2b) - (2a+b)\cos\theta - 2b\cos 2\theta + b\cos 3\theta\right] \mathrm{d}\theta. \qquad [4]$$

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13 [Continued]

(ii) Simplify $\sin(2k\pi - \beta)$, where $k \in \mathbb{Z}$ and $0 < \beta < 2\pi$.

Leave your answer in terms of β. [1]

(iii) Find a simplified expression for the exact area of the shaded region found in part **(a)(i)** in terms of a, b, α and π. [3]

(b) The car of the roller coaster ride will reach the maximum and minimum speed at M and L respectively.

(i) Find the speed of the car of the roller coaster at M and L in terms of a and/or b. [2]

(ii) Using your answer from part **(a)(iii)**, or otherwise, find the distance covered by the roller coaster from M to L, leaving your answer in terms of a, b and π. [3]

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