CEDAR GIRLS' SECONDARY SCHOOL
Preliminary Examination 2023
Secondary Four

CANDIDATE NAME	SOLUTIONS		
CLASS		CLASS INDEX NUMBER	
CENTRE/ INDEX NO	/		

ADDITIONAL MATHEMATICS
Paper 2

4049/02
11 September 2023

2 hours 15 minutes

Candidates answer on the Question Paper.

No Additional Materials are required.

READ THESE INSTRUCTIONS FIRST

Write your centre number, index number and name in the spaces at the top of this page.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved scientific calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

The total number of marks for this paper is 90.

For Examiner's Use
90

Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Binomial expansion

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \ldots + \binom{n}{r} a^{n-r} b^r + \ldots + b^n,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\ \ldots\ (n-r+1)}{r!}$

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\Delta = \frac{1}{2} bc \sin A$$

Answer **all** the questions.

1 The mass, x grams, of a volatile matter from a space mission remaining t days after being exposed to Earth's atmosphere is given by $x = 1.3 + 7e^{-0.5t}$.

(a) Find the initial mass of the matter. [1]

t=0,
$x = 1.3 + 7 = 8.3$ grams

(b) Explain why the mass of the substance can never be lower than 1.3 grams. [2]

For all real values of t ($t \geq 0$),
$e^{-0.5t} > 0$
$7e^{-0.5t} > 0$
$7e^{-0.5t} + 1.3 > 1.3$
Hence the lowest value will be 1.3 grams

(c) Find the least number of days it takes for the matter to be reduced to half of its initial mass. [3]

Half of initial mass $= 8.3 \div 2 = 4.15$ grams
$4.15 = 1.3 + 7e^{-0.5t}$
$\frac{57}{140} = e^{-0.5t}$
$\ln\frac{57}{140} = -0.5t$
$t = \ln\frac{57}{140} \div -0.5$
$t = 1.80$ days (3 s.f)

2 **(a)** Prove that $\dfrac{1}{\sec x-1}-\dfrac{1}{\sec x+1}=2\cot^2 x$. [3]

$LHS=\dfrac{1}{\sec x-1}-\dfrac{1}{\sec x+1}$

$=\dfrac{\sec x+1-(\sec x-1)}{\sec^2 x-1}$

$=\dfrac{2}{\tan^2 x}$

$=2\cot^2 x$

(b) Hence solve $\dfrac{1}{\sec x-1}-\dfrac{1}{\sec x+1}=5\operatorname{cosec} x$, for $0°\le x\le 360°$. [5]

$2\cot^2 x=5\cos ecx$

$2(co\sec^2 x-1)=5\cos ecx$

$2co\sec^2 x-5\cos ecx-2=0$

$\cos ecx=\dfrac{5\pm\sqrt{25-4(2)(-2)}}{4}$

$\sin x=\dfrac{4}{5+\sqrt{41}}$ or $\sin x=\dfrac{4}{5-\sqrt{41}}$

Reference angle = $20.545°$

$x=20.5°,159.5°$

3 The polynomial $\mathrm{f}(x) = ax^3 + bx^2 + 5x - 3$, where a and b are constants, is exactly divisible by $2x - 1$ and leaves a remainder of 39 when divided by $x - 2$.

(a) Find the value of a and of b. [4]

$f(0.5) = a(0.5)^3 + b(0.5)^2 + 5(0.5) - 3$
$0 = 0.125a + 0.25b - 0.5$
$0 = a + 2b - 4$
$a = 4 - 2b$ - - - (1)
$f(2) = a(2)^3 + b(2)^2 + 5(2) - 3$
$39 = 8a + 4b + 7$
$0 = 8a + 4b - 32$ - - - (2)
Sub (1) into (2),
$0 = 8(4 - 2b) + 4b - 32$
$0 = 32 - 16b + 4b - 32$
$b = 0$
$a = 4$

(b) Using these values of a and of b, determine the number of real roots of the equation $\mathrm{f}(x) = 0$. [3]
Show all necessary working.

$\mathrm{f}(x) = 4x^3 + 5x - 3$
$\mathrm{f}(x) = (2x - 1)(2x^2 + x + 3)$
$(2x - 1)(2x^2 + x + 3) = 0$
$b^2 - 4ac = 1 - 4(2)(3) = -23 < 0$
Therefore there is only 1 real root, $x = \frac{1}{2}$

4 (a) If $y=(4x-3)\sqrt{2x+1}$, show that $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{12x+1}{\sqrt{2x+1}}$. [3]

$$y=(4x-3)\sqrt{2x+1}$$
$$\frac{dy}{dx}=(4)\sqrt{2x+1}+\frac{1}{2}(4x-3)(2x+1)^{-\frac{1}{2}}(2)$$
$$\frac{dy}{dx}=(4)\sqrt{2x+1}+\frac{(4x-3)}{\sqrt{2x+1}}$$
$$\frac{dy}{dx}=\frac{4(2x+1)+(4x-3)}{\sqrt{2x+1}}$$
$$\frac{dy}{dx}=\frac{12x+1}{\sqrt{2x+1}}$$

(b) Hence find the value of $\int\frac{12x+3}{\sqrt{2x+1}}\,\mathrm{d}x$ expressing your answer in the form $\sqrt{2x+1}(ax+b)$ where a and b are integers. [4]

$$\int\frac{12x+3}{\sqrt{2x+1}}\,\mathrm{d}x=\int\frac{12x+1}{\sqrt{2x+1}}+\frac{2}{\sqrt{2x+1}}\mathrm{d}x+C$$
$$=(4x-3)\sqrt{2x+1}+\int\frac{2}{\sqrt{2x+1}}\mathrm{d}x+C$$
$$=(4x-3)\sqrt{2x+1}+\int 2(2x+1)^{-\frac{1}{2}}\mathrm{d}x+C$$
$$=(4x-3)\sqrt{2x+1}+\frac{2(2x+1)^{\frac{1}{2}}}{\frac{1}{2}\times 2}+C$$
$$=(4x-3)\sqrt{2x+1}+2\sqrt{2x+1}+C$$
$$=\sqrt{2x+1}(4x-1)+C$$

5 **(a)** The area of a quadrilateral is given as $25(\tan 15°)\ \text{cm}^2$.
Without using a calculator, express the area in the form $(a + b\sqrt{3})\ \text{cm}^2$. [4]

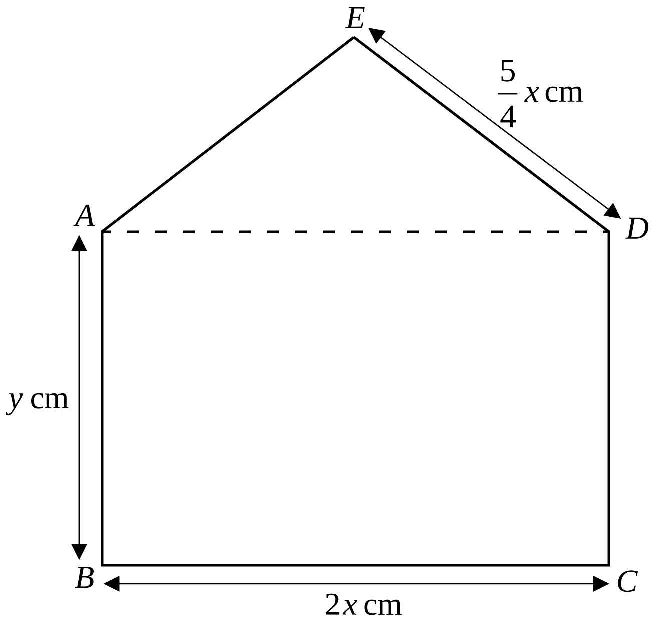
$$25(\tan 15^\circ)$$
$$= 25(\tan(45^\circ - 30^\circ))$$
$$= 25\left(\frac{\tan 45^\circ - \tan 30^\circ}{1 + \tan 45^\circ \tan 30^\circ}\right)$$
$$= 25\left(\frac{1 - \frac{\sqrt{3}}{3}}{1 + \frac{\sqrt{3}}{3}}\right)$$
$$= 25\left(\frac{\frac{3-\sqrt{3}}{3}}{\frac{3+\sqrt{3}}{3}}\right)$$
$$= 25\left(\frac{3-\sqrt{3}}{3+\sqrt{3}}\right)$$
$$= 25\left(\frac{3-\sqrt{3}}{3+\sqrt{3}}\right) \cdot \frac{3-\sqrt{3}}{3-\sqrt{3}}$$
$$= 25\left(\frac{12 - 6\sqrt{3}}{6}\right)$$
$$= 50 - 25\sqrt{3}$$

(b) Given that $\tan 15°$ is a root to the equation $x^2 + px + q = 0$, where p and q are integers, find the value of p and q. [3]

$$\tan 15^\circ = 2 - \sqrt{3}$$
$$(2-\sqrt{3})^2 + p(2-\sqrt{3}) + q = 0$$
$$7 - 4\sqrt{3} + 2p - p\sqrt{3} + q = 0$$
$$7 + 2p - 4\sqrt{3} = -q + p\sqrt{3}$$
$$p = -4$$
$$7 + 2p = -q$$
$$7 + 2(-4) = -q$$
$$q = 1$$

6 The figure below consists of a rectangle $ABCD$ and an isosceles triangle AED, where $AB = y$ cm, $BC = 2x$ cm and $ED = \frac{5}{4}x$ cm.

Given that the perimeter of $ABCDE$ is 70 cm,



(a) show that the area of figure is $A = 70x - \frac{15}{4}x^2$. [5]

$$P = 2x + 2y + 2\left(\frac{5}{4}x\right)$$

$$70 = \frac{9}{2}x + 2y$$

$$2y = 70 - \frac{9x}{2}$$

$$y = \frac{140 - 9x}{4}$$

$$\text{Height} = \sqrt{(\frac{5}{4}x)^2 - x^2}$$

$$\text{Height} = \frac{3}{4}x$$

$$A = 2xy + \left(\frac{1}{2} \times 2x \times \left(\frac{3}{4}x\right)\right)$$

$$A = 2x\left(\frac{140 - 9x}{4}\right) + x\left(\frac{3}{4}x\right)$$

$$A = 70x - \frac{9}{2}x^2 + \frac{3}{4}x^2$$

$$A = 70x - \frac{15}{4}x^2$$

(b) Given that x can vary, find the value of x for which the area of the figure is at a maximum. [5]

$$y = \frac{140 - 9x}{4}$$

$$\frac{\mathrm{d}A}{\mathrm{d}x} = 70 - \frac{15}{2}x$$

$$0 = 70 - \frac{15}{2}x$$

$x = 9\frac{1}{3}$ or 9.33 (3 s.f)

$$\frac{\mathrm{d}^2A}{\mathrm{d}x^2} = -\frac{15}{2}$$

By second derivative test, A is a maximum when $x = 9\frac{1}{3}$

7 **(a)** Solve the equation $3^{2x+1}-3^{x+2}+6=0$. [4]

$3^{2x+1}-3^{x+2}+6=0$
$3^{2x}\cdot 3-3^{x}\cdot 3^{2}+6=0$
$3^{x}=y$
$3y^{2}-9y+6=0$
$y^{2}-3y+2=0$
$(y-2)(y-1)=0$
$y=2$ or $y=1$
$3^{x}=2$ or $3^{x}=1$
$x=0.631$ (3 s.f) or $x=0$

(b) Solve $\log_{2}(x+2)-1=\log_{\sqrt{2}}(x-1)$. [4]

$\log_{2}(x+2)-1=\log_{\sqrt{2}}(x-1)$

$\log_{2}(x+2)-1=\dfrac{\log_{2}(x-1)}{\log_{2}\sqrt{2}}$

$\log_{2}(x+2)-1=2\log_{2}(x-1)$

$\log_{2}(x+2)-2\log_{2}(x-1)=1$

$\log_{2}(x+2)-\log_{2}(x-1)^{2}=1$

$\log_{2}\dfrac{(x+2)}{(x-1)^{2}}=1$

$\dfrac{(x+2)}{(x-1)^{2}}=2$

$(x+2)=2x^{2}-4x+2$
$2x^{2}-5x=0$
$x(2x-5)=0$
$x=0$ (rej.) or $x=2\frac{1}{2}$

8 A circle with centre C and radius r has an equation of $x^2+y^2-4x-6y-12=0$.

(a) Find the coordinates of C and the value of r. [3]

$-2f=-4$
$f=2$
$-2g=-6$
$g=3$
$C(2,3)$
$r=\sqrt{(2)^2+(3)^2-(-12)}$
$r=\sqrt{4+9+12}$
$r=5$

The line $4y=3x+31$ is tangent to the circle at the point T.

(b) Find the coordinates of the point T. [4]

$m_{\tan gent}=\frac{3}{4}$

$m_{AT}=-\frac{4}{3}$

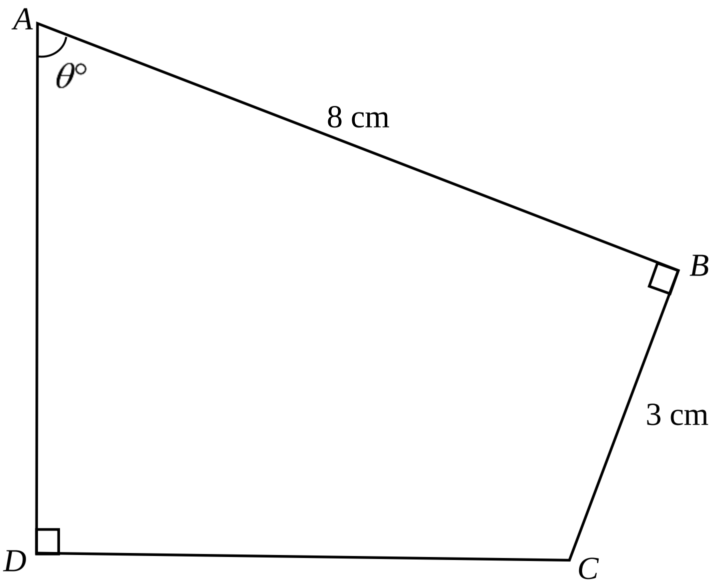
Let T (p,q)

$\frac{3-q}{2-p}=-\frac{4}{3}$
$9-3q=4p-8$
$4p+3q=17$ ---- (1)
$4q=3p+31$ ----(2)
Solving for p and q
T(-1,7)

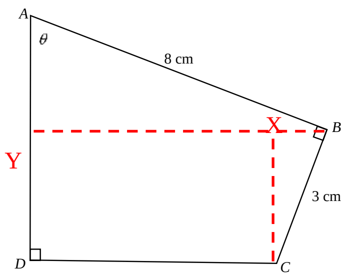
(c) Determine, with working, if S (0,8) lies within the circle. [2]

$S=(0,8)$
$CS=\sqrt{(2-0)^2+(3-8)^2}$
$CS=\sqrt{29}>5$
Since CS is greater than the radius, the point lies outside the circle.

9 The diagram shows a quadrilateral $ABCD$ in which $\angle ABC = \angle ADC = 90°$.
$AB = 8$ cm, $BC = 3$ cm and $\angle BAD = \theta°$.



(a) Show that the sum of the lengths of AD and CD is given by $11\sin\theta + 5\cos\theta$ cm. [4]



$\sin\theta = \dfrac{BY}{8}$

$BY = 8\sin\theta$

$\cos\theta = \dfrac{BX}{3}$

$BX = 3\cos\theta$

$CD = 8\sin\theta - 3\cos\theta$

$\cos\theta = \dfrac{AY}{8}$

$AY = 8\cos\theta$

$\sin\theta = \dfrac{CX}{3}$

$CX = 3\sin\theta$

$AD = 8\cos\theta + 3\sin\theta$

$AD + CD = 8\cos\theta + 3\sin\theta + 8\sin\theta - 3\cos\theta$

$AD + CD = 11\sin\theta + 5\cos\theta$

(b) Express $11\sin\theta + 5\cos\theta$ in the form $R\sin(\theta+\alpha)$, where R is a positive constant and α is acute. [3]

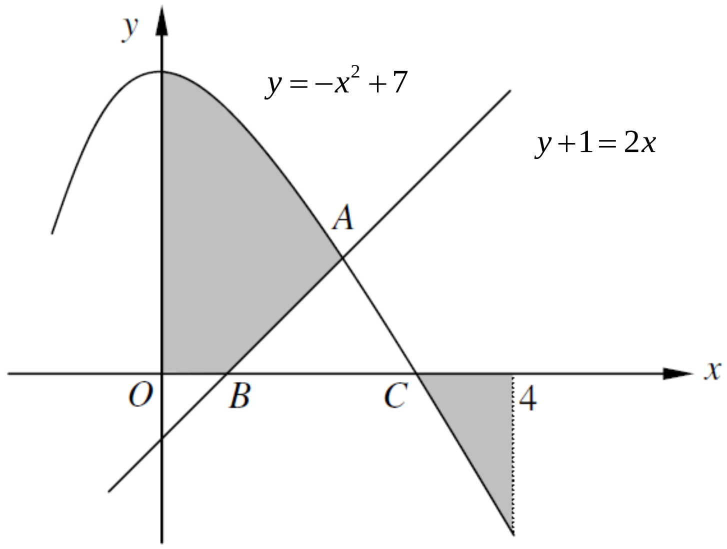
$R = \sqrt{11^2 + 5^2}$
$R = \sqrt{146}$
$\alpha = \tan^{-1}\left(\dfrac{5}{11}\right)$
$\alpha = 24.444^\circ$

$11\sin\theta + 5\cos\theta = \sqrt{146}\sin(\theta + 24.4^\circ)$

(c) Find the maximum value of the sum of the lengths of *AD* and *CD* and the corresponding value of θ. [2]

Maximum value is $\sqrt{146}$
$\sin(\theta + 24.4^\circ) = 1$
$ref\angle = 90^\circ$
$ref\angle = 90^\circ - 24.444^\circ = 65.6^\circ$ (1 d.p)

10 The figure below shows part of the curve $y=-x^2+7$ and the line $y+1=2x$. The curve and the line intersect at the point *A*. The points *B* and *C* lie on the x-axis.



(a) Find the coordinates of *A*, *B* and *C*. [4]

Let $y=0$, $x=\frac{1}{2}$

$B(\frac{1}{2},0)$

Let $y=0$, $x=\sqrt{7}$

$C(\sqrt{7},0)$

Sub $y=2x-1$ into $y=-x^2+7$

$2x-1=-x^2+7$

$x^2+2x-8=0$

$(x+4)(x-2)=0$

$x=-4$ (rej.) or $x=2$

$y=3$

$A(2,3)$

(b) Calculate the area of the shaded region. [5]

$$\text{Area} = \int_0^{\frac{1}{2}} -x^2 + 7dx + \int_{\frac{1}{2}}^{2} \left((-x^2+7)-(2x-1)\right)dx + \left|\int_{\sqrt{7}}^{4} -x^2+7dx\right|$$

$$= \left[\frac{-x^3}{3} + 7x\right]_0^{\frac{1}{2}} + \left[\frac{-x^3}{3} + 8x - x^2\right]_{\frac{1}{2}}^{2} + \left|\left[\frac{-x^3}{3} + 7x\right]_{\sqrt{7}}^{4}\right|$$

$$= \left[-\frac{1}{24} + \frac{7}{2}\right] + \left[(-\frac{8}{3} + 16 - 4) - (-\frac{1}{24} + 4 - \frac{1}{4})\right]$$

$$+ \left|\left[(-\frac{64}{3} + 28) - (-\frac{\sqrt{7}^3}{3} + 7(\sqrt{7}))\right]\right|$$

$$= \frac{83}{24} + \frac{45}{8} + \left|\frac{20}{3} + \frac{\sqrt{7}^3}{3} - 7(\sqrt{7})\right|$$

$$= 14.8 \text{ units}^2 \text{ (3 s.f)}$$

11 **(a)** Show that $(\cos x - \sin x)^2 = 1 - \sin 2x$. [2]

$(\cos x - \sin x)^2$
$= \cos^2 x - 2\sin x \cos x + \sin^2 x$
$= 1 - \sin 2x$

(b) Hence find the exact value of $\int_{\frac{\pi}{2}}^{\pi} (\cos x - \sin x)^2 \; \mathrm{d}x$. [4]

$\int_{\frac{\pi}{2}}^{\pi} (\cos x - \sin x)^2 \; \mathrm{d}x$

$\int_{\frac{\pi}{2}}^{\pi} 1 - \sin 2x \; \mathrm{d}x$

$= \left[x + \dfrac{\cos 2x}{2} \right]_{\frac{\pi}{2}}^{\pi}$

$= \pi + \dfrac{\cos 2\pi}{2} - \dfrac{\pi}{2} - \dfrac{\cos \pi}{2}$

$= \pi + \dfrac{(1)}{2} - \dfrac{\pi}{2} - \dfrac{(-1)}{2}$

$= \dfrac{\pi + 2}{2}$ or $= \dfrac{\pi}{2} + 1$

(c) Using the result in **(a)**, find $\frac{\mathrm{d}}{\mathrm{d}x}\ln\left(\frac{\cos x-\sin x}{\cos 2x}\right)^2$. [4]

Method 1

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln\left(\frac{\cos x-\sin x}{\cos 2x}\right)^2$$
$$=\frac{\mathrm{d}}{\mathrm{d}x}\ln\left(\frac{(\cos x-\sin x)^2}{\cos^2 2x}\right)$$
$$=\frac{\mathrm{d}}{\mathrm{d}x}\ln\left(\frac{1-\sin 2x}{\cos^2 2x}\right)$$
$$=\frac{\mathrm{d}}{\mathrm{d}x}\ln\left(\frac{1-\sin 2x}{1-\sin^2 2x}\right)$$
$$=\frac{\mathrm{d}}{\mathrm{d}x}\ln\left(\frac{1}{1+\sin 2x}\right)$$
$$=-\frac{\mathrm{d}}{\mathrm{d}x}\ln\left(1+\sin 2x\right)$$
$$=-\frac{1}{1+\sin 2x}\times(2\cos 2x)$$
$$=\frac{-2\cos 2x}{1+\sin 2x}$$

Method 2

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln\left(\frac{\cos x-\sin x}{\cos 2x}\right)^2$$
$$=\frac{\mathrm{d}}{\mathrm{d}x}\ln\left(\frac{(\cos x-\sin x)^2}{\cos^2 2x}\right)$$
$$=\frac{\mathrm{d}}{\mathrm{d}x}\ln\left(\frac{1-\sin 2x}{\cos^2 2x}\right)$$
$$=\frac{\mathrm{d}}{\mathrm{d}x}\left[\ln(1-\sin 2x)-2\ln(\cos 2x)\right]$$
$$=\frac{1}{1-\sin 2x}\times -2\cos 2x-2\left(\frac{1}{\cos 2x}\times -2\sin 2x\right)$$
$$=\frac{-2\cos 2x}{1-\sin 2x}+\frac{4\sin 2x}{\cos 2x}\ \text{ or }\ =\frac{-2\cos 2x}{1-\sin 2x}+4\tan 2x$$

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