1 **Do not use a calculator in answering this question.**

It is given that $2-\mathrm{i}$ is a root of the equation $2z^3+az^2-2z+b=0$.

(a) Find the values of the real numbers a and b and the remaining roots of the equation. [4]

(b) Using these values of a and b, deduce the roots of the equation

$$bz^3-2z^2+az+2=0.$$ [2]

2 Figure 1 shows a sketch of the curve $y=\mathrm{f}(x)$ and A is the region under the curve between $x=1$ and $x=3$. Yvonne and Irene use different ways to draw 4 rectangles of equal width, h, to estimate the area of A.

Figure 2 shows 4 rectangles drawn by Yvonne, with the curve intersecting each rectangle at the mid-point of its width. The x-coordinates of the mid-points are m_1, m_2, m_3 and m_4.

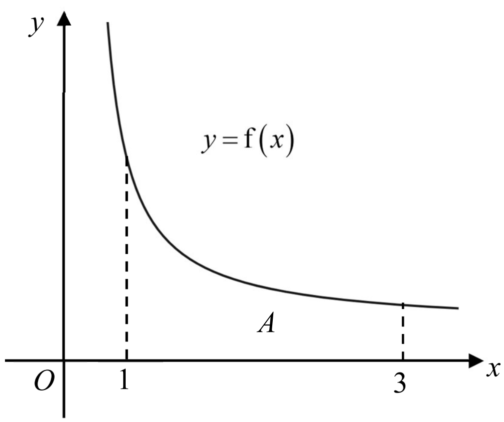

Figure 1

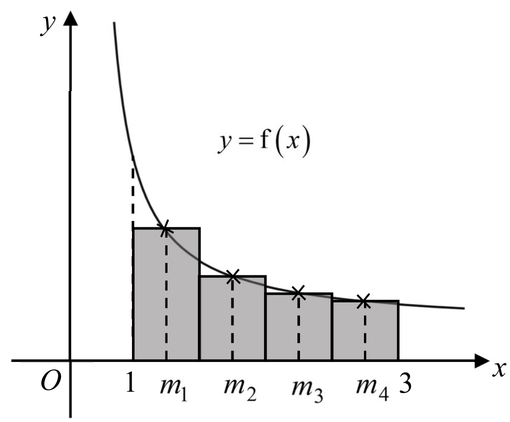

Figure 2

(a) State the values of h, m_1, m_2, m_3 and m_4. [2]

(b) The sum of area of rectangles using Yvonne's method is denoted by B. Show that $B=h\sum_{r=0}^{3}\left(\mathrm{f}\left(a+rh\right)\right)$, where the value of a is to be determined. [2]

(c) Irene finds that the sum of area of 4 rectangles that she has drawn is $C=h\sum_{r=0}^{3}\left(\mathrm{f}\left(1+rh\right)\right)$. Draw these rectangles in **Figure 1**. [1]

You are now given that $\mathrm{f}(x)=\dfrac{1}{x}+1$.

(d) By finding the numerical values of B, C and the actual area of region A, explain how these values **and** the rectangles drawn in Figures 1 and 2 show that Yvonne's estimation of the area of A is better than Irene's. [2]

3 **(a)** On the same axes, sketch the graphs of $y=\left|x(x-5)\right|$ and $y=\sqrt{2}\left|x-5\right|$. [2]

(b) Hence, or otherwise, solve exactly the inequality $\left|x(x-5)\right|>\sqrt{2}\left|x-5\right|$. [4]

4 The points P, Q and R have position vectors $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ respectively where $\mathbf{p}$ and $\mathbf{q}$ are non-zero and non-parallel. The points P and Q are fixed and R varies.

(a) Given that $\mathbf{r}\times\mathbf{q}=-\mathbf{q}\times\mathbf{p}$, describe geometrically the set of all possible positions of the point R. [4]

(b) Given instead that $\mathbf{r}=\begin{pmatrix}x\\y\\z\end{pmatrix}$, $\mathbf{p}=\begin{pmatrix}1\\-2\\3\end{pmatrix}$, $\mathbf{q}=\begin{pmatrix}q_1\\q_2\\q_3\end{pmatrix}$, and that $\mathbf{q}\bullet(\mathbf{p}-\mathbf{r})=0$, find the relationship between x, y, and z in terms of q_1, q_2 and q_3. Describe the set of all possible positions of the point R in this case. [3]

(c) It is now given that $\left|\mathbf{q}\right|=1$ and C is a point with position vector $\mathbf{c}$ such that $\mathbf{q}\bullet(\mathbf{p}-\mathbf{c})\neq 0$. Give a geometrical meaning of $\left|\mathbf{q}\bullet(\mathbf{p}-\mathbf{c})\right|$. [1]

5 **(a)** Show that $\dfrac{1}{(r-1)!}-\dfrac{2}{r!}+\dfrac{1}{(r+1)!}=\dfrac{\mathrm{f}(r)}{(r+1)!}$, where $\mathrm{f}(r)$ is a function in r to be found. [1]

The sum $\displaystyle\sum_{r=2}^{N}\frac{\mathrm{f}(r)}{(r+1)!}$ is denoted by S_N.

(b) Using your answer in part **(a)**, find S_N in terms of N. [3]

(c) Give a reason why S_N converges and find the exact value of S_∞. [2]

(d) Find the smallest value of N such that S_N is within 10^{-7} of S_∞. [2]

6 **(a)** Show that $1+\mathrm{e}^{-\mathrm{i}\alpha}=2\cos\dfrac{\alpha}{2}\mathrm{e}^{-\mathrm{i}\frac{\alpha}{2}}$, where $-\pi<\alpha\leq\pi$. [2]

(b) Hence or otherwise, show that

$$\left(1+\mathrm{e}^{-\mathrm{i}\alpha}\right)^3-\left(1+\mathrm{e}^{\mathrm{i}\alpha}\right)^3=-16\mathrm{i}\cos^3\left(\frac{\alpha}{2}\right)\sin\left(\frac{3\alpha}{2}\right).$$ [3]

(c) Given further that $0<\alpha<\dfrac{2}{3}\pi$ and $z=\left(1+\mathrm{e}^{-\mathrm{i}\alpha}\right)^3-\left(1+\mathrm{e}^{\mathrm{i}\alpha}\right)^3$, deduce the modulus and argument of z. Express your answers in terms of α whenever applicable. [2]

7 **(a)** It is given that $\mathrm{f}(x)=\ln(1+\sin 2x)+2$, where $0\le x\le\frac{\pi}{2}$. By using differentiation, find $\mathrm{f}'(0)$ and $\mathrm{f}''(0)$. Hence write down the Maclaurin series for $\mathrm{f}(x)$, up to and including the term in x^2. [5]

(b) Given that x is a sufficiently small angle, find the series expansion of $\frac{1}{\cos 2x+\sin x}$, up to and including the term in x^2. [4]

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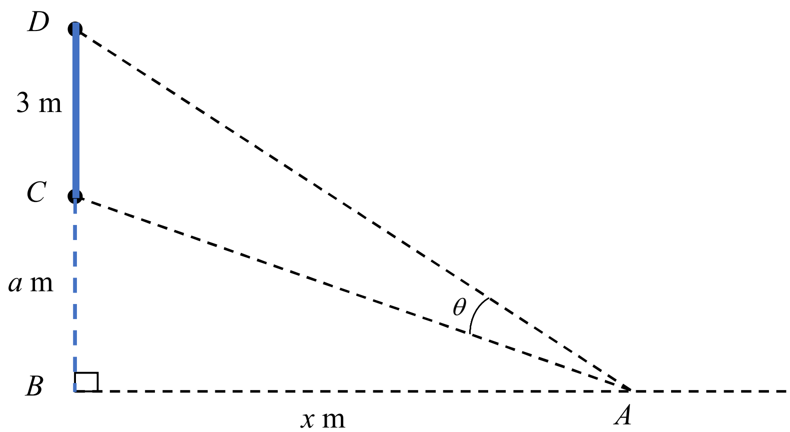

The diagram shows a cross-sectional view of an advertisement sign CD hung against a vertical wall, where the point C is a metres above the eye level AB of an observer who is x metres from B. The distance CD is 3 metres and angle CAD is θ.

(a) By expressing θ as the difference of two angles, or otherwise, show that

$$\tan\theta=\frac{3x}{x^2+3a+a^2}\ .$$ [3]

(b) Find, in terms of a, the value of x which maximises $\tan\theta$, simplifying your answer. Find also the corresponding value of $\tan\theta$. You do not need to show that $\tan\theta$ is maximum. [4]

(c) Find $\tan ADB$ when $\tan\theta$ is maximum, expressing your answer in terms of a. Find the approximate value of angle ADB when a is much greater than 3. [3]

9 A curve C has parametric equations

$$x = a\cos 2t, \qquad y = 2a\cos t,$$

for $0 \le t \le \frac{\pi}{2}$, where a is a positive constant.

(a) Show that the equation of normal to the curve at the point $P\left(a\cos 2p,\, 2a\cos p\right)$ is

$$y = -2\cos p\left(x - 2a\cos^2 p\right).$$ [3]

(b) The normal at P meets the x-axis at the point R. Show that the area enclosed by the x-axis, the normal at P and C is given by

$$4a^2\int_{t_1}^{t_2} \cos t \sin 2t \; \mathrm{d}t \; + \; a^2 \cos p\,,$$

where the values of t_1 and t_2 should be stated. [6]

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(c) Hence find in terms of a, the exact area in part **(b)** given now that $p = \frac{\pi}{3}$. [3]

10 Alan and Betty bought an apartment at \$450, 000. They are eligible to take a housing loan, up to 85% of the cost of the apartment, for a maximum of 30 years.

After careful consideration, the couple decides to borrow 85% of the cost of the apartment. They will make a cash repayment of \$$x$ at the beginning of each month, starting 1st July 2022. Interest will be charged with effect from 31st July 2022 at a monthly interest rate of 0.2% for the remaining amount owed at the end of each month.

(a) Find the amount of money owed on 31st of July 2022 after the interest for the month has been added. Express your answer in terms of x. [1]

(b) Show that the total amount of money owed after the nth repayment at the beginning of the month is

$$1.002^{n-1}(382500) - 500x(1.002^{n} - 1)\,.$$ [4]

(c) Find the earliest date on which the couple will be able to pay off the loan completely if $x = 2000$, and state the amount of repayment on this date. [4]

(d) If the couple wishes to pay off the loan completely on 1st Jan 2050 (after the repayment on this day), what should the monthly repayment be? [3]

11 Naturalists are managing a wildlife reserve to increase the number of plants of a rare species. The number of plants at time t years is denoted by N, where N is treated as a continuous variable. It is given that the rate of increase of N with respect to t is proportional to $(N-120)$.

(a) Write down a differential equation relating N and t. [1]

Initially, the number of plants was 600. It is noted that at a time when there were 750 plants, the number of plants was increasing at a rate of 63 per year.

(b) Express N in terms of t. [6]

(c) The naturalist has a target of increasing the number of plants from 600 to 2500 within 15 years. Justify whether this target will be met. [2]

Alongside the monitoring of the number of plants of this rare species, naturalists also study the rate of increase of its height, h cm, with respect to time, t years after planting. The height of a plant is modelled by the differential equation

$$\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{1}{2}\sqrt{\left(24-\frac{1}{3}h\right)}.$$

The plant is planted as a seedling of negligible height, so that $h=0$ when $t=0$.

(d) State the maximum height of the plant, according to this model. [1]

(e) Find an expression for t in terms of h, and hence find the time the plant takes to reach 24 cm. [5]