Paper 1 Solution with Markers' Comments

<table>
<tr><td>1</td><td>
Since the coefficients of the cubic equation are all real and $2-\mathrm{i}$ is a root of the equation, another root would be $2+\mathrm{i}$.

<u>Method 1</u>
$\left(z-2+\mathrm{i}\right)\left(z-2-\mathrm{i}\right)=\left(z-2\right)^2-\left(\mathrm{i}\right)^2$
$=z^2-4z+4+1$
$=z^2-4z+5$
$2z^3+az^2-2z+b=0$
$\left(z^2-4z+5\right)\left(2z+\frac{b}{5}\right)=0$ (By inspection)
Comparing coefficients of z: $-2=-\frac{4b}{5}+10 \Rightarrow b=15$
Comparing coefficients of z^2: $a=\frac{b}{5}-8=-5$
The other roots are $z=2+\mathrm{i}$ and $z=-\frac{3}{2}$.
</td></tr>
<tr><td></td><td>
<u>Method 2 (Not recommended)</u>
Since $2-\mathrm{i}$ is a root to the equation,
$2\left(2-\mathrm{i}\right)^3+a\left(2-\mathrm{i}\right)^2-2\left(2-\mathrm{i}\right)+b=0$
$2\left(8-12\mathrm{i}-6+\mathrm{i}\right)+a\left(4-4\mathrm{i}-1\right)-2\left(2-\mathrm{i}\right)+b=0$
$4-22\mathrm{i}+3a-4a\mathrm{i}-4+2\mathrm{i}+b=0$
$(3a+b)-(20+4a)\mathrm{i}=0$

Comparing the imaginary part: $20+4a=0 \Rightarrow a=-5$
Comparing the real part: $3a+b=0 \Rightarrow b=15$

Since the coefficients of the cubic equation are all real and $2-\mathrm{i}$ is a root of the equation, another root would be $2+\mathrm{i}$.

$\left(z-2+\mathrm{i}\right)\left(z-2-\mathrm{i}\right)=\left(z-2\right)^2-\left(\mathrm{i}\right)^2$
$=z^2-4z+4+1$
$=z^2-4z+5$
$2z^3-5z^2-2z+15=0$
$\left(z^2-4z+5\right)\left(2z+3\right)=0$
The other roots are $z=2+\mathrm{i}$ and $z=-\frac{3}{2}$.
</td></tr>
<tr><td></td><td>
$bz^3-2z^2+az+2=0.$
</td></tr>
</table>

Divide z^3 throughout, $b-2\left(\frac{1}{z}\right)+a\left(\frac{1}{z^2}\right)+2\left(\frac{1}{z^3}\right)=0$

$$2\left(\frac{1}{z^3}\right)+a\left(\frac{1}{z^2}\right)-2\left(\frac{1}{z}\right)+b=0$$

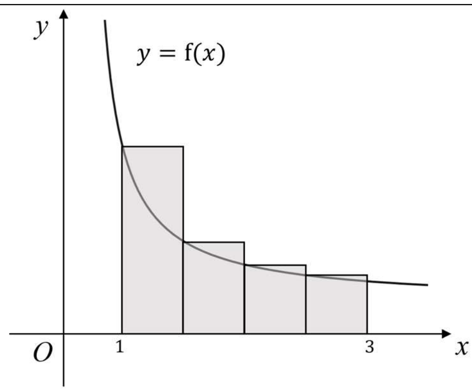
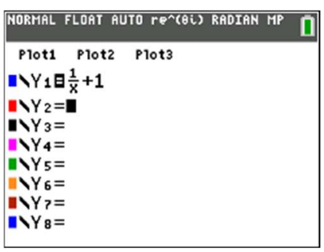
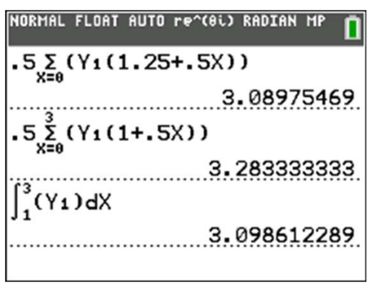
Replace z with $\frac{1}{z}$,

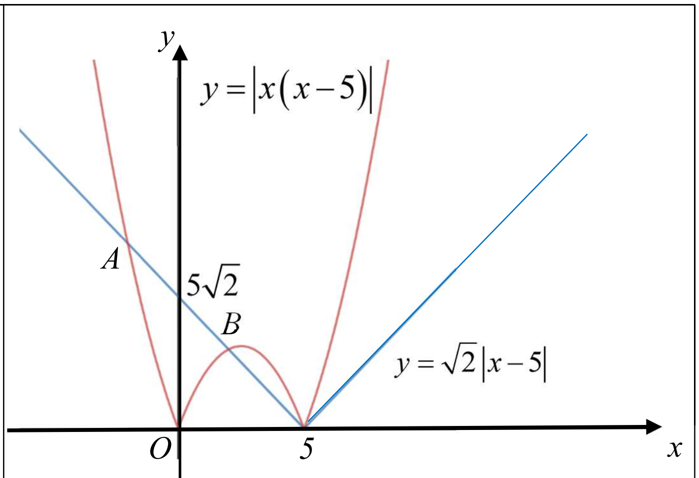
$\frac{1}{z}=2-\mathrm{i}$, $\frac{1}{z}=2+\mathrm{i}$ or $\frac{1}{z}=-\frac{3}{2}$

$z=\frac{1}{2-\mathrm{i}}\times\frac{2+\mathrm{i}}{2+\mathrm{i}}$ $\quad z=\frac{1}{2+\mathrm{i}}\times\frac{2-\mathrm{i}}{2-\mathrm{i}}$ $\quad z=-\frac{2}{3}$

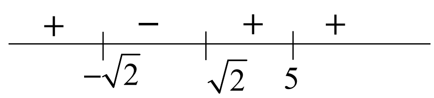
$=\frac{2+\mathrm{i}}{5}$ $\quad =\frac{2-\mathrm{i}}{5}$

$\therefore z=\frac{2}{5}+\frac{1}{5}\mathrm{i},\ \frac{2}{5}-\frac{1}{5}\mathrm{i}$ or $-\frac{2}{3}$

2a	$h=\dfrac{3-1}{4}=0.5$ Since m_i are the mid-points of the intervals, $m_1=1.25,\ m_2=1.75,\ m_3=2.25,\ m_4=2.75.$
2b	$B=\mathrm{f}(m_1)\times 0.5+\mathrm{f}(m_2)\times 0.5+\mathrm{f}(m_3)\times 0.5+\mathrm{f}(m_4)\times 0.5$ $=0.5\times\left[\mathrm{f}(1.25)+\mathrm{f}(1.75)+\mathrm{f}(2.25)+\mathrm{f}(2.75)\right]$ $=0.5\times\left[\mathrm{f}(1.25)+\mathrm{f}(1.25+1\times 0.5)+\mathrm{f}(1.25+2\times 0.5)+\mathrm{f}(1.25+3\times 0.5)\right]$ $=h\sum_{r=0}^{3}\mathrm{f}(1.25+rh)$ $a=m_1=1.25$.
2c	
2d	$\mathrm{f}(x)=\dfrac{1}{x}+1$   $B\approx 3.09$ (3 s.f.) $C\approx 3.28$ (3 s.f.) Actual area of $A=\int_1^3 \dfrac{1}{x}+1\,\mathrm{d}x$ ≈ 3.10 (3 s.f.) • The value of B is closer to the actual area of A • Comparing Figures 1 & 2, the gaps between the curve and the top of Irene's rectangles are larger than those of Yvonne's. ***(or other equivalent explanations)*** Thus Yvonne's estimation of the actual area of A is better.

3(a)	
3(b)	**Method 1** $\lvert x(x-5)\rvert=\sqrt{2}\lvert x-5\rvert$ $x(x-5)=\pm\sqrt{2}(x-5)$ $(x-5)(x\pm\sqrt{2})=0$ $x=5,\ \sqrt{2}\ or\ -\sqrt{2}$ **Method 2** $\lvert x(x-5)\rvert=\sqrt{2}\lvert x-5\rvert$ $x^2(x-5)^2=2(x-5)^2$ $(x-5)^2(x^2-2)=0$ $x-5=0\quad or\quad x^2=2$ $x=5,\ \sqrt{2}\ or\ -\sqrt{2}$ Thus at point $A, x=-\sqrt{2}$ and at point $B,\ x=\sqrt{2}$ From the graph, for $\lvert x(x-5)\rvert>\sqrt{2}\lvert x-5\rvert$, $x<-\sqrt{2}$ or $\sqrt{2}<x<5$ or $x>5$ **Otherwise method:** **Solving inequality algebraically,**

$$|x(x-5)| > \sqrt{2}\,|x-5|$$
$$x^2(x-5)^2 > 2(x-5)^2$$
$$(x-5)^2(x^2-2) > 0$$
$$(x-5)^2(x-\sqrt{2})(x+\sqrt{2}) > 0$$



$x < -\sqrt{2}$ or $\sqrt{2} < x < 5$ or $x > 5$

4 (a)

$\underset{\sim}{r} \times \underset{\sim}{q} = -\underset{\sim}{q} \times \underset{\sim}{p}$

$\underset{\sim}{r} \times \underset{\sim}{q} + \underset{\sim}{q} \times \underset{\sim}{p} = \underset{\sim}{0}$

$\underset{\sim}{r} \times \underset{\sim}{q} - \underset{\sim}{p} \times \underset{\sim}{q} = \underset{\sim}{0}$

$(\underset{\sim}{r} - \underset{\sim}{p}) \times \underset{\sim}{q} = \underset{\sim}{0}$

Since $\underset{\sim}{q} \neq \underset{\sim}{0}$,

Either $(\underset{\sim}{r} - \underset{\sim}{p}) = \underset{\sim}{0}$ $\quad\cdots\cdots(1)$

or $(\underset{\sim}{r} - \underset{\sim}{p}) // \underset{\sim}{q} \Rightarrow (\underset{\sim}{r} - \underset{\sim}{p}) = k\underset{\sim}{q}$, $k \in \mathbb{R} \setminus \{0\}$ $\quad\cdots\cdots(2)$

Combining (1) & (2),

$\Rightarrow \underset{\sim}{r} = \underset{\sim}{p} + k\underset{\sim}{q}$, $k \in \mathbb{R}$

The set of all possible positions of the point R form the line (OR R is any point on the line) passing through P and parallel to the vector $\underset{\sim}{q}$.

(b)

$\underset{\sim}{q} \bullet (\underset{\sim}{p} - \underset{\sim}{r}) = 0$

$\underset{\sim}{q} \bullet \underset{\sim}{p} - \underset{\sim}{q} \bullet \underset{\sim}{r} = 0$

$\underset{\sim}{r} \bullet \underset{\sim}{q} = \underset{\sim}{p} \bullet \underset{\sim}{q}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \bullet \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$$

$q_1x + q_2y + q_3z = q_1 - 2q_2 + 3q_3$

The set of all possible positions of the point R form the plane (OR R is any point on the plane) that contains the point $P(1, -2, 3)$ with a normal vector **q** (or perpendicular to **q)**

(c)	$\lvert \mathbf{q}\cdot(\mathbf{p}-\mathbf{c}) \rvert$ is the shortest distance from point C to the plane in (b). **OR** It is the length of projection of $\overrightarrow{CP}$ onto $\mathbf{q}$.

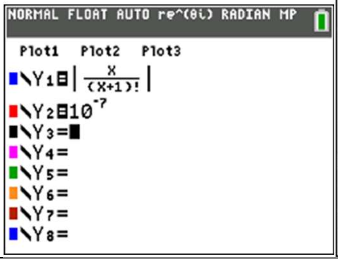
5(a)	$\frac{1}{(r-1)!}-\frac{2}{r!}+\frac{1}{(r+1)!}=\frac{r(r+1)}{(r+1)!}-\frac{2(r+1)}{(r+1)!}+\frac{1}{(r+1)!}$ $=\frac{r^2+r-2r-2+1}{(r+1)!}$ $=\frac{r^2-r-1}{(r+1)!}$ $\mathrm{f}(r)=r^2-r-1$ [*Optional*]

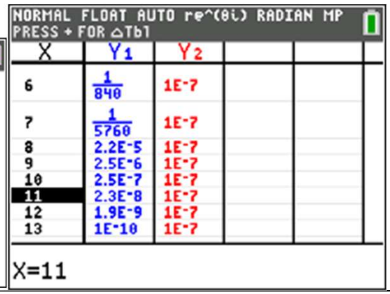
(b)	$\displaystyle\sum_{r=2}^{N}\frac{r^2-r-1}{(r+1)!}=\sum_{r=2}^{N}\left(\frac{1}{(r-1)!}-\frac{2}{r!}+\frac{1}{(r+1)!}\right)$ $=\left(\frac{1}{1!}-\frac{2}{2!}+\cancel{\frac{1}{3!}}\right)$ $+\left(\frac{1}{2!}-\cancel{\frac{2}{3!}}+\cancel{\frac{1}{4!}}\right)$ $+\left(\cancel{\frac{1}{3!}}-\cancel{\frac{2}{4!}}+\cancel{\frac{1}{5!}}\right)$ $+\left(\cancel{\frac{1}{4!}}-\cancel{\frac{2}{5!}}+\cancel{\frac{1}{6!}}\right)$ $\ldots$ $+\left(\cancel{\frac{1}{(N-3)!}}-\cancel{\frac{2}{(N-2)!}}+\cancel{\frac{1}{(N-1)!}}\right)$ $+\left(\cancel{\frac{1}{(N-2)!}}-\cancel{\frac{2}{(N-1)!}}+\frac{1}{N!}\right)$ $+\left(\cancel{\frac{1}{(N-1)!}}-\frac{2}{N!}+\frac{1}{(N+1)!}\right)$ $=\frac{1}{2}+\frac{1}{N!}-\frac{2}{N!}+\frac{1}{(N+1)!}$ $=\frac{1}{2}-\frac{1}{N!}+\frac{1}{(N+1)!}$ $=\frac{1}{2}-\frac{N}{(N+1)!}$
(c)	As $N\to\infty$, $-\frac{1}{N!}+\frac{1}{(N+1)!}\to 0$, hence $S_N\to\frac{1}{2}$, which is a finite value. Therefore S_N converges. **OR** As $N\to\infty$, $\frac{N}{(N+1)!}\to 0$, hence $S_N\to\frac{1}{2}$, which is a finite value. Therefore S_N converges. $S_\infty=\frac{1}{2}$
(d)	$S_\infty=\frac{1}{2}$

$$|S_\infty - S_N| = \left|\frac{1}{N!} - \frac{1}{(N+1)!}\right| < 10^{-7}$$

OR $|S_\infty - S_N| = \left|\frac{N}{(N+1)!}\right| < 10^{-7}$

Using GC, smallest $N = 11$





6 (a)

Method 1

$$\begin{aligned}
1+e^{-i\alpha} &= e^{-i\frac{\alpha}{2}}\left(e^{i\frac{\alpha}{2}} + e^{-i\frac{\alpha}{2}}\right) \\
&= e^{-i\frac{\alpha}{2}}\left[\cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2} + \cos\left(-\frac{\alpha}{2}\right) + i\sin\left(-\frac{\alpha}{2}\right)\right] \\
&= e^{-i\frac{\alpha}{2}}\left[\cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2} + \cos\frac{\alpha}{2} - i\sin\frac{\alpha}{2}\right] \\
&= 2\cos\frac{\alpha}{2}\, e^{-i\frac{\alpha}{2}}
\end{aligned}$$

Method 2

$$\begin{aligned}
1+e^{-i\alpha} &= 1+\cos(-\alpha) + i\sin(-\alpha) \\
&= 1+\cos\alpha - i\sin\alpha \\
&= 1+\left(2\cos^2\frac{\alpha}{2} - 1\right) - 2i\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} \\
&= 2\cos\frac{\alpha}{2}\left(\cos\frac{\alpha}{2} - i\sin\frac{\alpha}{2}\right) \\
&= 2\cos\frac{\alpha}{2}\left(\cos\left(-\frac{\alpha}{2}\right) + i\sin\left(-\frac{\alpha}{2}\right)\right) \\
&= 2\cos\frac{\alpha}{2}\, e^{-i\frac{\alpha}{2}}
\end{aligned}$$

(b)	$\left(1+e^{-i\alpha}\right)^3-\left(1+e^{i\alpha}\right)^3$ $=\left(2\cos\frac{\alpha}{2}e^{-i\frac{\alpha}{2}}\right)^3-\left(2\cos\left(-\frac{\alpha}{2}\right)e^{-i\left(-\frac{\alpha}{2}\right)}\right)^3$ $=8\cos^3\frac{\alpha}{2}e^{-i\frac{3\alpha}{2}}-8\cos^3\frac{\alpha}{2}e^{i\frac{3\alpha}{2}}$ $=8\cos^3\frac{\alpha}{2}\left(e^{-i\frac{3\alpha}{2}}-e^{i\frac{3\alpha}{2}}\right)$ $=8\cos^3\frac{\alpha}{2}\left\{\left[\cos\left(-\frac{3\alpha}{2}\right)+i\sin\left(-\frac{3\alpha}{2}\right)\right]-\left[\cos\left(\frac{3\alpha}{2}\right)+i\sin\left(\frac{3\alpha}{2}\right)\right]\right\}$ $=8\cos^3\frac{\alpha}{2}\left[\cos\left(\frac{3\alpha}{2}\right)-i\sin\left(\frac{3\alpha}{2}\right)-\cos\left(\frac{3\alpha}{2}\right)-i\sin\left(\frac{3\alpha}{2}\right)\right]$ $=8\cos^3\frac{\alpha}{2}\left(-2i\sin\frac{3\alpha}{2}\right)$ $=-16i\cos^3(\frac{\alpha}{2})\sin\left(\frac{3\alpha}{2}\right)$
(c)	Given that $0<\alpha<\frac{2}{3}\pi$, $0<\frac{\alpha}{2}<\frac{\pi}{3} \quad \Rightarrow \cos\frac{\alpha}{2}>0$ $0<\frac{3}{2}\alpha<\pi \quad \Rightarrow \sin\frac{3}{2}\alpha>0$ $z=-16i\cos^3\left(\frac{\alpha}{2}\right)\sin\left(\frac{3\alpha}{2}\right)$ $=16\cos^3\left(\frac{\alpha}{2}\right)\sin\left(\frac{3\alpha}{2}\right)e^{-i\frac{\pi}{2}}$ $\lvert z\rvert=16\cos^3\left(\frac{\alpha}{2}\right)\sin\left(\frac{3\alpha}{2}\right);\quad \arg(z)=-\frac{\pi}{2}$

7 (a)	$\mathrm{f}(x)=\ln(1+\sin 2x)+2$ **<u>Method 1</u>** $\mathrm{f}'(x)=\dfrac{2\cos 2x}{1+\sin 2x}$ $(1+\sin 2x)\mathrm{f}'(x)=2\cos 2x$ $(1+\sin 2x)\mathrm{f}''(x)+(2\cos 2x)\mathrm{f}'(x)=-4\sin 2x$ ---- (*) $\mathrm{f}'(0)=\dfrac{2\cos 0}{1+\sin 0}=2$, $(1+\sin 0)\mathrm{f}''(0)+(2\cos 0)(2)=-4\sin 0$ $\mathrm{f}''(0)=-4$ $\mathrm{f}(0)=\ln(1+\sin 0)+2=2$ $\therefore\ \mathrm{f}(x)=2+2x-2x^2+\ldots$ **<u>Method 2</u>** $\mathrm{f}'(x)=\dfrac{2\cos 2x}{1+\sin 2x}$ $\mathrm{f}''(x)=\dfrac{(1+\sin 2x)(-4\sin 2x)-(2\cos 2x)(2\cos 2x)}{(1+\sin 2x)^2}$ $=\dfrac{-4\sin 2x-4\sin^2 2x-4\cos^2 2x}{(1+\sin 2x)^2}$ $=\dfrac{-4(1+\sin 2x)}{(1+\sin 2x)^2}$ $=\dfrac{-4}{(1+\sin 2x)}$ $\mathrm{f}'(0)=\dfrac{2\cos 0}{1+\sin 0}=2$, $\mathrm{f}''(0)=-4$ $\mathrm{f}(0)=\ln(1+\sin 0)+2=2$ $\therefore\ \mathrm{f}(x)=2+2x-2x^2+\ldots$

	Method 3 $$y=\ln(1+\sin 2x)+2$$ $$e^{y-2}=1+\sin 2x$$ $$e^{y-2}\frac{dy}{dx}=2\cos 2x$$ $$e^{y-2}\frac{d^2y}{dx^2}+e^{y-2}\left(\frac{dy}{dx}\right)^2=-4\sin 2x$$ $e^0\frac{dy}{dx}=2 \Rightarrow f'(0)=2$ $\frac{d^2y}{dx^2}+2^2=0 \Rightarrow f''(0)=-4$, $f(0)=\ln(1+\sin 0)+2=2$ $\therefore f(x)=2+2x-2x^2+\dots$
(b)	$\frac{1}{\cos 2x+\sin x}\approx\frac{1}{1-\frac{(2x)^2}{2}+x}$ $=\left[1+\left(x-2x^2\right)\right]^{-1}$ $=1-\left(x-2x^2\right)+\frac{(-1)(-2)}{2!}\left(x-2x^2\right)^2+\dots$ $=1-x+2x^2+x^2+\dots$ $=1-x+3x^2+\dots$

8(a)	$\tan\theta$ $=\tan(\angle BAD-\angle BAC)$ $=\frac{\tan\angle BAD-\tan\angle BAC}{1+(\tan\angle BAD)(\tan\angle BAC)}$ $=\frac{\frac{a+3}{x}-\frac{a}{x}}{1+\frac{a+3}{x}\cdot\frac{a}{x}}$ $=\frac{\frac{3}{x}}{1+\frac{a(a+3)}{x^2}}\times\frac{x^2}{x^2}$ $=\frac{3x}{x^2+3a+a^2}$

(b)

Let $y = \tan\,\theta = \dfrac{3x}{x^2 + 3a + a^2}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\left(x^2 + 3a + a^2\right)3 - 3x\left(2x\right)}{\left(x^2 + 3a + a^2\right)^2}$$

$$= \frac{-3x^2 + 3a^2 + 9a}{\left(x^2 + 3a + a^2\right)^2}$$

At stationary value of tan θ,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$\frac{-3x^2 + 3a^2 + 9a}{\left(x^2 + 3a + a^2\right)^2} = 0$$

$$-3x^2 + 3a^2 + 9a = 0$$

$$3x^2 = 9a + 3a^2$$

$x = \sqrt{a\left(3 + a\right)}$ (reject $-\sqrt{a\left(3 + a\right)}$ as $x > 0$)

$$\tan\theta = \frac{3\sqrt{a(3+a)}}{a(3+a) + 3a + a^2}$$

$$= \frac{3\sqrt{a(3+a)}}{a(3+a) + a(3+a)}$$

$$= \frac{3\sqrt{a(3+a)}}{2a(a+3)} \quad or \quad \frac{3}{2\sqrt{a(a+3)}}$$

(c)

$$\tan\angle ADB = \frac{x}{a+3}$$

$$= \frac{\sqrt{a\left(a+3\right)}}{a+3}$$

$$= \sqrt{\frac{a}{a+3}}$$

$$\tan\angle ADB = \sqrt{\frac{\frac{a}{a}}{\frac{a+3}{a}}} = \sqrt{\frac{1}{1 + \frac{3}{a}}}$$

Since a is much greater than 3, then $\dfrac{3}{a} \approx 0$

$\tan\angle ADB \approx 1$. Thus $\angle ADB \approx 45°$.

9 (a)	$x = a\cos 2t, \qquad y = 2a\cos t$ $\dfrac{dx}{dt} = -2a\sin 2t, \qquad \dfrac{dy}{dt} = -2a\sin t$ $\dfrac{dy}{dx} = \dfrac{\sin t}{\sin 2t}$ $= \dfrac{\sin t}{2\sin t\cos t}$ $= \dfrac{1}{2\cos t}$ At P, $t = p$. Gradient of normal at $P = -2\cos p$ Equation of normal at P is $y - 2a\cos p = -2\cos p\left(x - a\cos 2p\right)$ $y = -2\cos p\left[x - a\left(2\cos^2 p - 1\right)\right] + 2a\cos p$ $y = -2\cos p\left(x + a - 2a\cos^2 p\right) + 2a\cos p$ $y = -2\cos p\left(x - 2a\cos^2 p\right)$
9 (b)	At R, $y = 0$ $-2\cos p\left(x - 2a\cos^2 p\right) = 0$ $x = 2a\cos^2 p$ When C meets the x-axis, $y = 0 \Rightarrow 2a\cos t = 0$ $t = \dfrac{\pi}{2}$ $x = a\cos\pi$ $= -a$ Required area $= \displaystyle\int_{-a}^{a\cos 2p} y \, dx + \frac{1}{2}\left[2a\cos^2 p - a\cos 2p\right]\left(2a\cos p\right)$ $= \displaystyle\int_{\frac{\pi}{2}}^{p} 2a\cos t\left(-2a\sin 2t\right) dt + \left(a^2\cos p\right)\left[2\cos^2 p - \left(2\cos^2 p\right.\right.$ $= 4a^2\displaystyle\int_{p}^{\frac{\pi}{2}} \cos t\left(\sin 2t\right) dt + a^2\cos p$ (Shown)

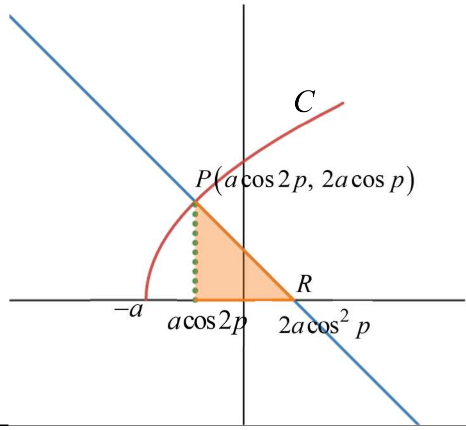


Diagram is not required. For reference only.

9 (c)	**Method 1: (Using double angle formula for sin2t)**

Area

$$= 4a^2 \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos t\left(2\sin t\cos t\right)\,\mathrm{d}t \ + \ a^2\cos\frac{\pi}{3}$$

$$= -8a^2 \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left(-\sin t\right)\left(\cos^2 t\right)\,\mathrm{d}t \ + \ \frac{1}{2}a^2$$

$$= -8a^2\left[\frac{1}{3}\cos^3 t\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} \ + \ \frac{1}{2}a^2$$

$$= -\frac{8}{3}a^2\left[0-\cos^3\frac{\pi}{3}\right] \ + \ \frac{1}{2}a^2$$

$$= \frac{1}{3}a^2 + \frac{1}{2}a^2$$

$$= \frac{5}{6}a^2$$

	<u>Method 2: (Using factor formula for cos*t* sin2*t*</u> $= 4a^2 \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{2}\left(\sin 3t + \sin t\right)\, \mathrm{d}t \;+\; a^2 \cos\frac{\pi}{3}$ $= 2a^2\left[-\frac{1}{3}\cos 3t - \cos t\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} \;+\; \frac{1}{2}a^2$ $= 2a^2\left[0 + \frac{1}{3}\cos\pi + \cos\frac{\pi}{3}\right] + \frac{1}{2}a^2$ $= 2a^2\left(-\frac{1}{3} + \frac{1}{2}\right) \;+\; \frac{1}{2}a^2$ $= \frac{1}{3}a^2 + \frac{1}{2}a^2$ $= \frac{5}{6}a^2$

10 (a)	85% of 450000 = 382 500 The amount owed at the end of July 2022 is (1+ 0.002) ×(382500 – x) = 1.002(382500 – x)
(b)	see table below

No. of repay-ment	Amount of money owed after each repayment (beginning of month)	At the end of the month after adding interest
1	$a-x$, where a = 382500,	$1.002(a-x)$
2	$1.002(a-x)-x$ $=1.002a-1.002x-x$	$1.002(1.002a-1.002x-x)$
3	$1.002(1.002a-1.002x-x)$ $-x$ $=1.002^2 a-1.002^2 x-1.002x$ $-x$	$1.002\left(1.002^2 a-1.002^2 x-1.002x-x\right)$
…		
n	$1.002^{n-1} a-1.002^{n-1} x$ $-1.002^{n-2} x-\ldots-x$	

	Amount of money owed after the n^{th} repayment at the beginning of the month $=1.002^{n-1}a-1.002^{n-1}x-1.002^{n-2}x-...-x$ --- (*) $=1.002^{n-1}a-x\left(1+1.002+1.002^2+...+1.002^{n-2}+1.002^{n-1}\right)$ $=1.002^{n-1}a-x\left(\dfrac{1.002^n-1}{1.002-1}\right)$ $=1.002^{n-1}(382500)-x\left(\dfrac{1.002^n-1}{0.002}\right)$ $=1.002^{n-1}(382500)-500x(1.002^n-1)$ (Shown)
(c)	When $x = 2000$, $1.002^{n-1}(382500)-500(2000)(1.002^n-1)\le 0$ From GC, $n\ge 240.66$ He will pay off his loan on the 241 repayments, that is, 20 years 1 month. So the earliest date is 1 July 2042 after he makes the 241th repayments. At the end of 240th month after interest is added, amount owed is $1.002\times\left(1.002^{239}a-1.002^{239}x-1.002^{238}x-...-x\right)$ $1.002^{240}\times 382500-2000\left[1.002^{240}+1.002^{239}+\cdots+1.002\right]$ $=1.002^{240}\times 382500-2000\times 1.002\left(\dfrac{1.002^{240}-1}{1.002-1}\right)$ $=1321.71$ (2 d.p.) The amount of repayment on 1 July 2042 is \$1321.71
(d)	From 1 July 2022 to 1 Jan 2050, he would have made 27× 12 + 7 = 331 repayments. $1.002^{331-1}(382500)-500x(1.002^{331}-1)=0$ $x=\dfrac{1.002^{331-1}(382500)}{500(1.002^{331}-1)}\approx 1577.94$

11(a)	$\dfrac{\mathrm{d}N}{\mathrm{d}t}=k\left(N-120\right),\quad k>0$

11(b)	$\int \frac{1}{N-120}\,\mathrm{d}N = \int k\,\mathrm{d}t$ $\ln\lvert N-120\rvert = kt + C$ $N-120 = \pm \mathrm{e}^{kt+C}$ $N = 120 + A\mathrm{e}^{kt}$ where $A = \pm \mathrm{e}^{C}$ When $t=0,\ N=600 \quad \Rightarrow \quad A = 600-120$ $= 480$ When $N=750,\ \frac{\mathrm{d}N}{\mathrm{d}t} = 63 \quad \Rightarrow \quad 63 = k(750-120)$ $k = \frac{1}{10}$ Thus $N = 120 + 480\mathrm{e}^{\frac{1}{10}t}$ *or* $120\left(1+4\mathrm{e}^{\frac{1}{10}t}\right)$
11(c)	**<u>Method 1</u>** When $t=15,\ N = 120+480\mathrm{e}^{1.5} = 2271.21 < 2500$ The target will not be met in 15 years. **<u>Method 2</u>** When $N=2500, \quad 120+480\mathrm{e}^{\frac{1}{10}t} = 2500$ $t = 16.01 > 15$ The target will not be met in 15 years.
11(d)	$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{1}{2}\sqrt{\left(24-\frac{1}{3}h\right)}$ When h is maximum, $\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{1}{2}\sqrt{\left(24-\frac{1}{3}h\right)} = 0$ $h = 24\times 3 = 72$ The maximum height of the plant is 72 cm.

11(e)	$\int\left(24-\frac{1}{3}h\right)^{-1/2}\ \mathrm{d}h=\int\frac{1}{2}\mathrm{d}t$ $\frac{\left(24-\frac{1}{3}h\right)^{1/2}}{\left(\frac{1}{2}\right)\left(-\frac{1}{3}\right)}=\frac{1}{2}t+C$ $\frac{1}{2}t+C=-6\sqrt{\left(24-\frac{1}{3}h\right)}$ When $t=0,\ h=0,\ C=-6\sqrt{24}=-12\sqrt{6}$ $t=24\sqrt{6}-12\sqrt{\left(24-\frac{1}{3}h\right)}$ When $h=24,\ t=24\sqrt{6}-12\sqrt{(24-8)}$ $=10.8$ (3 s.f.) It takes 10.8 years to reach a height of 24 cm.