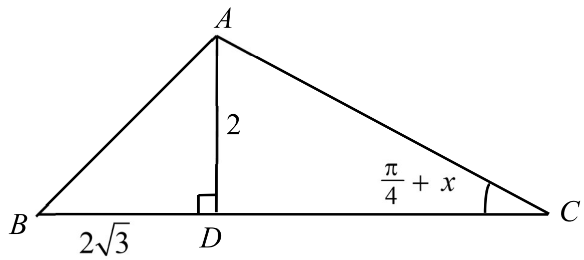
1 A curve has equation $2x + y + 2 = (x+y)^2 + \dfrac{x^2}{1+x^2}$.

Find the equation of the normal to the curve at the point $(0, 2)$. [5]

2 The diagram shows triangle ABC, where angle $ACD = \left(\dfrac{\pi}{4} + x\right)$ radians. Point D is on BC such that $AD = 2$ and $BD = 2\sqrt{3}$.



Show that if x is sufficiently small for x^3 and higher powers of x to be negligible, then

$$BC \approx k(1 + \sqrt{3} - 2x + 2x^2),$$

where k is a constant to be determined. [5]

3 **(a)** Find $\int (\ln x)^2 \, dx$. [3]

(b) Find $\displaystyle\int \frac{(\sin x + \cos x)^2}{\cos(2x) - 2x} \, dx$. [3]

4 A curve has equation $y = f(x)$, where

$$f(x) = \begin{cases} 2 - |x+1| & \text{for } -3 < x < 1, \\ 2 - 2(x-2)^2 & \text{for } 1 \leqslant x < 2. \end{cases}$$

(i) Sketch the curve for $-3 < x < 2$. [3]

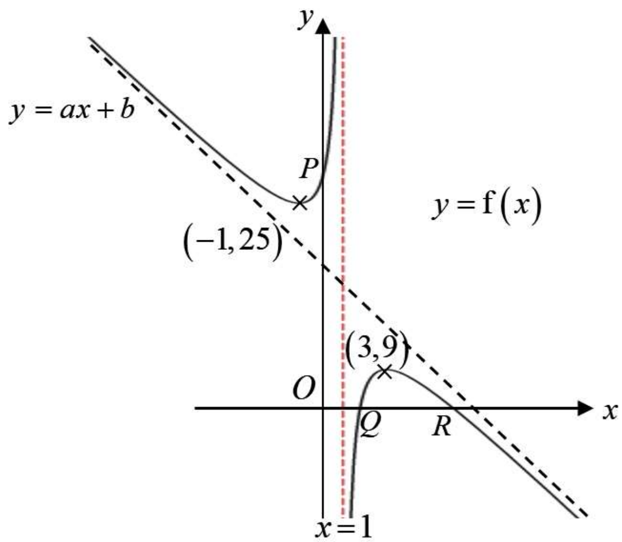
(ii) Hence, solve the inequality $f(x) \leqslant 0.1(x-1)^2$ for $-3 < x < 2$, leaving your answers in an exact form. [4]

5 **Do not use a calculator in answering this question.**
The complex number z satisfies the equation

$$z^2-(4+\mathrm{i})z+2(\mathrm{i}-t)=0,$$

where t is a real number. It is given that one root is of the form $k-k\mathrm{i}$, where k is real and positive. Find t and k, and the other root of the equation. [7]

6



It is given that $\mathrm{f}(x)=ax+b+\dfrac{c}{x-1}$, where a, b and c are constants. The diagram shows the curve with equation $y=\mathrm{f}(x)$. The curve crosses the axes at points P, Q and R, and has stationary points at $(-1,\ 25)$ and $(3,\ 9)$.
Find the values of the constants a, b and c. [4]

It is now given that points P, Q and R have coordinates $(0, 27)$, $\left(\dfrac{3}{2}, 0\right)$ and $(9, 0)$ respectively.

Sketch the curve

(i) $y=\mathrm{f}(|x|)$, [2]

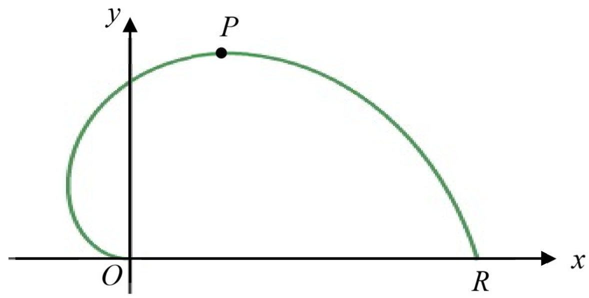
(ii) $y=\dfrac{1}{\mathrm{f}(x)}$, [3]

stating the equations of any asymptotes, the coordinates of any points where the curve crosses the axes and of any turning point(s).

7 The diagram below shows the curve C with parametric equations given by

$$x = -3\theta\cos 3\theta, \quad y = 4\theta\sin 3\theta, \qquad \text{for } 0 \leqslant \theta \leqslant \frac{\pi}{3}.$$

Point P lies on C with parameter θ and C crosses the x-axis at the origin O and the point R.



(a) Find the area of the region bounded by C and the line $y = \frac{2\pi}{3} - \frac{2x}{3}$, giving your answer correct to 2 decimal places. [4]

(b) Use differentiation to find the maximum value of the area of triangle OPR as θ varies, proving that it is a maximum. [6]

8 **(a)** The sum of the first n terms of a sequence is given by $S_n = \frac{n^2 + 5n}{8}$. Show that the sequence follows an arithmetic progression with common difference d, where d is to be determined.

In a geometric progression, the first term is 100 and its common ratio is $3d$. Find the smallest value of k such that the sum of the first k terms of the arithmetic progression is greater than the sum of the first 30 terms of the geometric progression. [6]

(b) The first and second terms of a geometric sequence are $u_1 = a$ and $u_2 = a^2 - a$. If all the terms of the sequence are positive, find the set of values of a for which $\sum_{r=1}^{\infty} u_r$ converges. [2]

For this sequence, it is known that the sum of all the terms after the nth term is equal to the nth term. Find the value of a and hence the value of $\sum_{r=1}^{\infty} u_r$. [3]

9 The curve C has equation $\frac{1}{3}x^2 + y^2 - 2y = 0$.

(i) Sketch C. [2]

(ii) Use the substitution $x = p\sin\theta$ to show that

$$\int_0^p \sqrt{p^2 - x^2}\ \mathrm{d}x = \frac{p^2\pi}{4},$$

where p is a positive constant. [4]

(iii) The region R is bounded by C, the line $x = \sqrt{3}$ and the x-axis. Find the exact area of R. [3]

(iv) R is rotated completely about the y-axis. Find the exact volume of the solid obtained. [3]

(v) Describe a pair of transformations which transforms the graph of C onto the graph of $x^2 + y^2 = 1$. [2]

10 Workers are installing zip lines at an adventure park. The points (x, y, z) are defined relative to the entrance at $(0,0,0)$ on ground level, where units are in metres. The ticketing booth at $(100,100,1)$ and lockers at $(200,120,0)$ are also on ground level. Zip lines are laid in straight lines and the widths of zip lines can be neglected. The ground level of the park is modelled as a plane.

(i) Find a cartesian equation of the plane that models the ground level of the park. [2]

A zip line connects the points $P(300,120,30)$ and $Q(300,320,25)$, and is modelled as a segment of the line l. The façade of a building nearby can be modelled as part of the plane with equation $\mathbf{r}.\begin{pmatrix}1\\-5\\100\end{pmatrix} = 0$. As a safety requirement, every point on the zip line must be at a distance of at least 10 metres away from the façade of the building.

(ii) Write down a vector equation of l. Hence, or otherwise, determine if the zip line passes the safety requirement. [4]

The workers need to install another zip line from Q to $R(127,220,a)$, where $0 < a < 30$, and the angle PQR is given to be $60°$.

(iii) Find the value of a, leaving your answer to 3 decimal places. [3]

The façade of the building meets the ground level of the park at line m. A worker sets up a transmitter at point S on line m such that S is nearest to Q.

(iv) Find a vector equation of m and the distance from Q to S. [4]

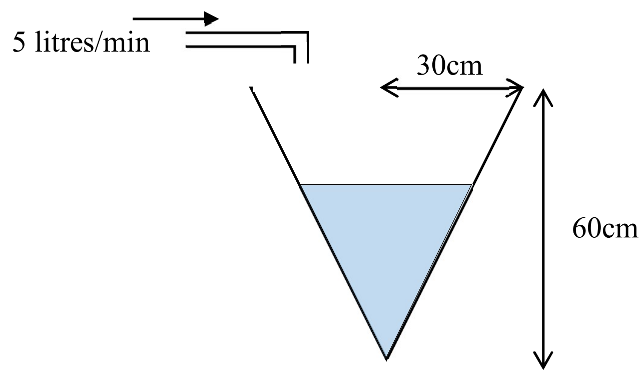
11 A tank contains 500 litres of water in which 100 g of a poisonous chemical called Prokrastenate is dissolved. A solution containing 0.1 g of Prokrastenate per litre is pumped into the tank at a rate of 5 litres per minute, and the well-mixed solution is pumped out at the same rate. By letting x grams be the amount of Prokrastenate in the tank after t minutes, show that

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{k - x}{100},$$

where k is a constant to be determined. [2]

Find x in terms of t and find the value of t when $x = 75$. [5]

The well-mixed solution that is pumped out flows into an empty container, in the form of an open inverted cone with a height of 60 cm and base radius 30 cm, at the same rate (see diagram).



Given that 1 litre = 1000 cm^3,

(i) Show that the volume of the well-mixed solution in the container, V cm^3 can be expressed as $V = \dfrac{\pi h^3}{12}$, where h cm is the depth of the solution at that instant. [2]

(ii) Hence, or otherwise, find the rate of change of the depth of solution after 5 minutes. [4]

[The volume of a cone of base radius r and height h is given by $V = \dfrac{1}{3}\pi r^2 h$.]