Question 1 [Solution]

Let x, y and z denote the number of participants in the age group ≤ 12 years, 13 to 49 years and ≥ 50 years respectively.

$$x + y + z = 100 \quad \text{--- (1)}$$
$$6x + 9y + 4z = 700 \quad \text{--- (2)}$$
$$z \geq 35$$

From GC,

$$x = \frac{200}{3} - \frac{5}{3}z \geq 0 \quad \Rightarrow \quad z \leq 40 \quad \therefore \quad 35 \leq z \leq 40$$

$$y = \frac{100}{3} + \frac{2}{3}z$$

Method 1

Since $x = \frac{200}{3} - \frac{5}{3}z = \frac{5}{3}(40 - z)$ **must be a positive integer**,

$z = 40$ or $z = 37$

Thus the possible number of participants are $\begin{cases} x = 5 \\ y = 58 \\ z = 37 \end{cases}$ or $\begin{cases} x = 0 \\ y = 60 \\ z = 40 \end{cases}$

Method 2 Using GC

z	x	y
37	5	58
40	0	60

Question 2 [Solution]

(i) $y^2 = 1 - \frac{4}{x^2} \Rightarrow x^2 = \frac{4}{1 - y^2}$

$$\text{Volume} = \pi \int_0^{\frac{\sqrt{3}}{2}} x^2 \, \mathrm{d}y$$
$$= 4\pi \int_0^{\frac{\sqrt{3}}{2}} \frac{1}{1 - y^2} \, \mathrm{d}y$$
$$= 4 \times \frac{1}{2(1)} \pi \left[\ln \left| \frac{1+y}{1-y} \right| \right]_0^{\frac{\sqrt{3}}{2}}$$
$$= 2\pi \ln \left(\frac{1 + \frac{\sqrt{3}}{2}}{1 - \frac{\sqrt{3}}{2}} \right) \text{ units}^3 \quad \text{or} \quad 2\pi \ln \left(\frac{2 + \sqrt{3}}{2 - \sqrt{3}} \right) \text{ units}^3$$
$$\text{or} \quad 2\pi \ln \left(7 + 4\sqrt{3} \right) \text{ units}^3$$

(ii) Volume $= 2\times\left[\pi\left(\frac{\sqrt{3}}{2}\right)^2(4) - \pi\int_2^4 y^2 \, dx\right]$

$$= 2\left[3\pi - \pi\int_2^4\left(1-\frac{4}{x^2}\right) dx\right]$$

$$= 2\pi\left(3 - \left[x+\frac{4}{x}\right]_2^4\right)$$

$$= 2\pi\left(3 - [5-4]\right)$$

$$= 4\pi \text{ units}^3$$

Question 2 [Solution]
(i) $\overrightarrow{OC} = \begin{pmatrix}0\\20\\0\end{pmatrix}, \quad \overrightarrow{OA} = \begin{pmatrix}43\\0\\-1\end{pmatrix}$ A normal is $\overrightarrow{OC}\times\overrightarrow{OA} = \begin{pmatrix}0\\20\\0\end{pmatrix}\times\begin{pmatrix}43\\0\\-1\end{pmatrix} = -20\begin{pmatrix}1\\0\\43\end{pmatrix}$ Since origin lies on the plane *OABC*, Equation of plane *OABC* is $\mathbf{r}\cdot\begin{pmatrix}1\\0\\43\end{pmatrix} = 0$
(ii) Let *N* be the foot of perpendicular from *P* to the plane *OABC*. Line *PN*: $\mathbf{r} = \begin{pmatrix}39.6\\8\\0.8\end{pmatrix} + \lambda\begin{pmatrix}1\\0\\43\end{pmatrix}, \quad \lambda\in\mathbb{R}$ At *N*, $\begin{pmatrix}39.6+\lambda\\8\\0.8+43\lambda\end{pmatrix}.\begin{pmatrix}1\\0\\43\end{pmatrix} = 0$ $39.6+\lambda+34.4+1849\lambda = 0 \Rightarrow \lambda = -\frac{1}{25}$ $\overrightarrow{ON} = \begin{pmatrix}39.6\\8\\0.8\end{pmatrix} - \frac{1}{25}\begin{pmatrix}1\\0\\43\end{pmatrix} = \begin{pmatrix}\frac{989}{25}\\8\\-\frac{23}{25}\end{pmatrix}$

(iii) $\overrightarrow{OM} = \begin{pmatrix} 43 \\ 10 \\ -1 \end{pmatrix}$

$$\overrightarrow{PM} = \begin{pmatrix} 43 \\ 10 \\ -1 \end{pmatrix} - \begin{pmatrix} 39.6 \\ 8 \\ 0.8 \end{pmatrix} = \begin{pmatrix} 3.4 \\ 2 \\ -1.8 \end{pmatrix}$$

<u>Method 1</u>

Let θ be the acute angle between line *PM* and the plane *OABC*.

$$\sin\theta = \frac{\left|\begin{pmatrix} 3.4 \\ 2 \\ -1.8 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 43 \end{pmatrix}\right|}{\sqrt{18.8}\sqrt{1850}} = \frac{74}{\sqrt{34780}}$$

$$\theta = 23.4^\circ \quad \text{or} \quad 0.408 \text{ rad}$$

<u>Method 2</u>

Let ϕ be the acute angle between line *PM* and the normal of plane *OABC*.

$$\cos\phi = \frac{\left|\begin{pmatrix} 3.4 \\ 2 \\ -1.8 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 43 \end{pmatrix}\right|}{\sqrt{18.8}\sqrt{1850}} = \frac{74}{\sqrt{34780}}$$

$$\phi = 66.6^\circ \quad \text{or} \quad 1.162 \text{ rad}$$

Therefore, required angle $= 90^\circ - 66.6^\circ = 23.4^\circ$ or 0.408 rad

Question 4 [Solution]

(i) $u = \frac{1}{2}\left(1 + \left(2\cos^2\frac{\theta}{2} - 1\right) + \mathrm{i}\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)\right)$

$$= \cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + \mathrm{i}\sin\frac{\theta}{2}\right)$$

$\therefore$ $|u| = \cos\left(\frac{\theta}{2}\right)$, and $\arg u = \frac{\theta}{2}$

<u>Alternative</u>

$$u = \frac{1}{2}(1 + \cos\theta + \mathrm{i}\sin\theta)$$

$$= \frac{1}{2}\left(1 + \mathrm{e}^{i\theta}\right)$$

$$= \frac{1}{2}\mathrm{e}^{i\frac{\theta}{2}}\left(\mathrm{e}^{-i\frac{\theta}{2}} + \mathrm{e}^{i\frac{\theta}{2}}\right)$$

$$= \frac{1}{2} e^{i\frac{\theta}{2}} \left(2\cos\frac{\theta}{2} \right)$$

$$= \cos\frac{\theta}{2} \left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2} \right)$$

$\therefore$ $|u| = \cos\left(\frac{\theta}{2}\right)$, and $\arg u = \frac{\theta}{2}$

(ii) $|v| = \sec\left(\frac{\theta}{2}\right)$, and $\arg v = -\frac{\theta}{2}$

$$v = \sec\left(\frac{\theta}{2}\right)\left(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\right) = 1 - i\tan\frac{\theta}{2}$$

$\therefore$ real part of v is 1 (shown)

(iii) $|w| = \sec^2\left(\frac{\theta}{2}\right)$, and $\arg w = -\theta$

$$w = \sec^2\left(\frac{\theta}{2}\right)(\cos\theta - i\sin\theta)$$

$$= \sec^2\left(\frac{\theta}{2}\right)\left(2\cos^2\frac{\theta}{2} - 1 - i\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)\right)$$

$$= 2 - \sec^2\left(\frac{\theta}{2}\right) - i\left(2\tan\frac{\theta}{2}\right)$$

$\therefore$ real part of $w = 2 - \sec^2\left(\frac{\theta}{2}\right) = 2 - |w|$ (shown)

(iv) $\arg\left(\frac{v-w}{v}\right) = \arg\left(\frac{u^{-1} - u^{-2}}{u^{-1}}\right)$

$$= \arg\left(1 - u^{-1}\right)$$

$$= \arg(1 - v)$$

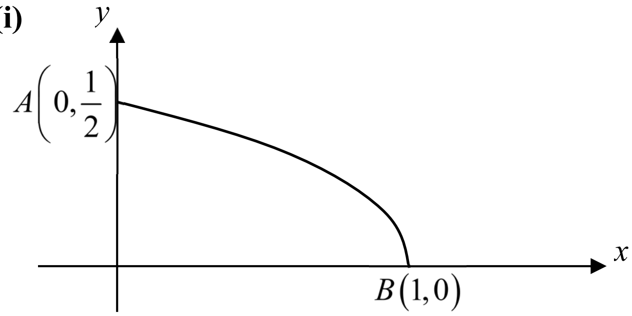
$$= \arg\left(1 - \left(1 - i\tan\frac{\theta}{2}\right)\right)$$

$$= \arg\left(i\tan\frac{\theta}{2}\right)$$

$$= \frac{\pi}{2} \text{ or } -\frac{\pi}{2}$$

Question 5 [Solution]

(i)



(ii) $\dfrac{dx}{d\theta}=-3\sin 3\theta$, $\dfrac{dy}{d\theta}=\cos\theta$

$$\frac{dy}{dx}=\frac{\cos\theta}{-3\sin 3\theta}$$

$$\frac{dy}{dt}=\frac{dy}{dx}.\frac{dx}{dt}=\frac{\cos\theta}{-3\sin 3\theta}(0.1)$$

When $x=\frac{1}{2}$, $\cos 3\theta=\frac{1}{2}\Rightarrow 3\theta=\frac{\pi}{3}\Rightarrow\theta=\frac{\pi}{9}$

$$\frac{dy}{dt}=\frac{\cos\frac{\pi}{9}}{-3\sin\frac{\pi}{3}}(0.1)=-0.0362$$

The rate of decrease of its y-coordinate is 0.0362 units per second.

(iii) At A, when $x = 0$, $\cos 3\theta = 0$ $\Rightarrow 3\theta=\frac{\pi}{2}$, i.e. $\theta=\frac{\pi}{6}$

$$\Rightarrow \frac{dy}{dx}=\frac{\cos\frac{\pi}{6}}{-3\sin\frac{\pi}{2}}=-\frac{\sqrt{3}}{6}$$

Equation of the tangent at A:

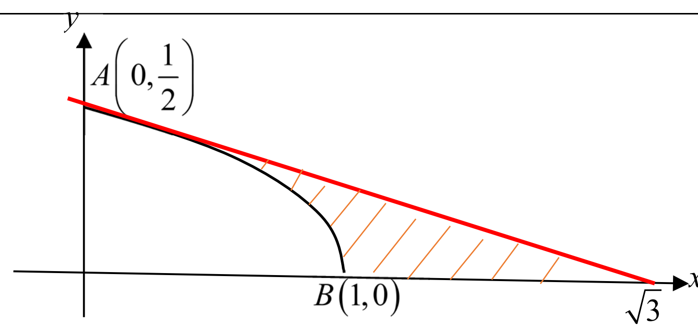
$$y-\frac{1}{2}=-\frac{\sqrt{3}}{6}(x-0) \quad \text{i.e} \quad y=-\frac{\sqrt{3}}{6}x+\frac{1}{2}$$

(iv)

$$-\frac{\sqrt{3}}{6}x+\frac{1}{2}=0\Rightarrow -\frac{\sqrt{3}}{6}x=-\frac{1}{2}\Rightarrow x=\sqrt{3}$$

Area required

$$=\frac{1}{2}\left(\frac{1}{2}\right)\left(\sqrt{3}\right)-\int_0^1 y\,dx$$



$$=\frac{1}{2}\left(\frac{1}{2}\right)\left(\sqrt{3}\right)-\int_{\frac{\pi}{6}}^{0}(\sin\theta)(-3\sin 3\theta)\ \mathrm{d}\theta$$

$$=\frac{\sqrt{3}}{4}-3\int_{0}^{\frac{\pi}{6}}\sin 3\theta\sin\theta\ \mathrm{d}\theta$$

$$=\frac{\sqrt{3}}{4}+\frac{3}{2}\int_{0}^{\frac{\pi}{6}}(\cos 4\theta-\cos 2\theta)\ \mathrm{d}\theta$$

$$=\frac{\sqrt{3}}{4}+\frac{3}{2}\left[\frac{1}{4}\sin 4\theta-\frac{1}{2}\sin 2\theta\right]_{0}^{\frac{\pi}{6}}$$

$$=\frac{\sqrt{3}}{4}+\frac{3}{2}\left[\frac{1}{4}\left(\frac{\sqrt{3}}{2}\right)-\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)\right]$$

$$=\frac{\sqrt{3}}{4}-\frac{3\sqrt{3}}{16}$$

$$=\frac{\sqrt{3}}{16}\text{ units}^2$$

Question 6 [Solution]

(i) Number of arrangements = $5! \times 5! \times 2 = 28800$

(ii) Number of arrangements = $\frac{5!}{3!}\times 3!\times 4! = 2880$

(iii) Number of arrangements = $(7-1)!\times 2!\times 2!\times 2!\times 10 = 57600$

Question 7 Solution

(i)

y	0	1	2
$P(Y=y)$	α	$\beta+\frac{1}{10}$	$\frac{2}{5}$

$$E(Y)=\left(\beta+\frac{1}{10}\right)+2\times\frac{2}{5}=\beta+\frac{9}{10}$$

$$E(Y^2)=\left(\beta+\frac{1}{10}\right)+2^2\times\frac{2}{5}=\beta+\frac{17}{10}$$

$$\text{Var}(Y)=E(Y^2)-(E(Y))^2$$

$$=\left(\beta+\frac{17}{10}\right)-\left(\beta+\frac{9}{10}\right)^2$$

Given Var (Y) = 0.56, $\left(\beta+\frac{17}{10}\right)-\left(\beta+\frac{9}{10}\right)^2=0.56$

From GC, $\beta=\frac{3}{10}$ or $\beta=-\frac{11}{10}$ (rejected as $\because$ β>0)

$\frac{1}{10}+\alpha+\beta+\frac{2}{5}=1 \quad \Rightarrow \alpha=\frac{1}{5}$

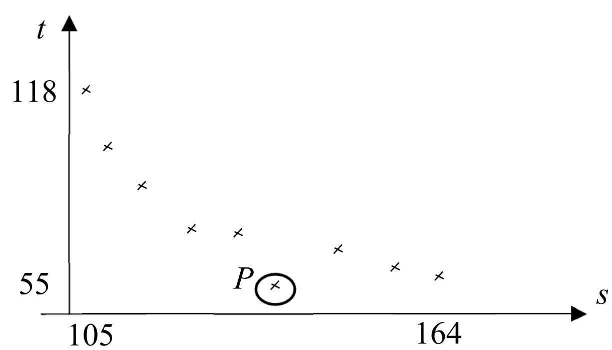
(ii) Since sample size of 50 is large, by **Central Limit Theorem**,

$\overline{Y}\sim \mathrm{N}\left(\frac{6}{5},\ \frac{0.56}{50}\right)$ approximately

$\mathrm{P}\left(\overline{Y}>1.3\right)=0.172$

Question 8 Solution

(i)



(ii) When s = 132, the time to failure t = 55 days seems to be much lower than the expected value based on the trend. This could be due to a faulty car battery.

(iii) Model I : $t=a+bs^2$ where a > 0 and b < 0

In model I, as s increases, t decreases at an increasing rate but the data trend shows that as s increases, t decreases at decreasing rate. Hence model I is not suitable.

Model II: $t=a+b\sqrt{s}$ where a > 0 and b < 0

Model III: $t=a+b\mathrm{e}^{-\sqrt{s}}$ where a > 0 and b > 0

Model II, $|r|=0.929$

Model III, $|r|=0.988$

Model III has $|r|$-value which is closer to 1 than that in Model II. Hence model III has a stronger positive linear correlation between t and $\mathrm{e}^{-\sqrt{s}}$ and it is a better linear model than model II.

(iv) The regression line is $t = 54.43 + 1699879.37e^{-\sqrt{s}}$

When s = 135, $t = 54.43 + 1699879.37e^{-\sqrt{135}}$
$t = 69.7 \approx 70$ days
The time to failure of the battery is 70 days.

The estimate is reliable because
(i) s = 135 is within the given data range (105, 164) and
(ii) $|r|$-value = 0.988 is very close to 1 indicating a strong positive linear relation between t and $e^{-\sqrt{s}}$.

Question 9 Solution

(i) Let X be the number of standard size packets (out of 10 in a family pack) that contains a winning coupon. $X \sim \mathrm{B}(10, p)$

Most likely number of winning coupons in a family pack is 2
$\Rightarrow$ mode of X = 2

Thus $\mathrm{P}(X=2) > \mathrm{P}(X=1)$ **and** $\mathrm{P}(X=2) > \mathrm{P}(X=3)$

$$\binom{10}{2}p^2(1-p)^8 > \binom{10}{1}p(1-p)^9 \quad \text{and} \quad \binom{10}{2}p^2(1-p)^8 > \binom{10}{3}p^3(1-p)^7$$

$$45p > 10(1-p) \quad \text{and} \quad 45(1-p) > 120p$$

$$p > \frac{2}{9}(1-p) \quad \text{and} \quad (1-p) > \frac{8}{3}p$$

$$p > \frac{2}{11} \quad \text{and} \quad p < \frac{3}{11}$$

Thus $\frac{2}{11} < p < \frac{3}{11}$

(ii) $X \sim \mathrm{B}(10, 0.2)$

P(a family pack has at least 1 winning coupon) $= \mathrm{P}(X \geq 1) = 1 - \mathrm{P}(X = 0) = 0.89263$ (5 s.f.)

Let Y be the number of family packs (out of N packs) with at least 1 winning coupon.
$Y \sim \mathrm{B}(N, 0.89263)$

Given: $\mathrm{P}(Y \geq 30) > 0.99$

$\Rightarrow$ $1 - \mathrm{P}(Y \leq 29) > 0.99$

Using GC,

N	$1 - \mathrm{P}(Y \leq 29)$
38	$0.9829 < 0.99$
39	$0.9931 > 0.99$

Least N = 39

(iii)(a) Let W be the number of standard size packets (out of $12 \times 10 = 120$ packets in a carton) that contains a winning coupon. $W \sim \mathrm{B}(120, 0.2)$ P(a carton contains exactly 24 winning coupons) $= \mathrm{P}(W = 24) = 0.0907$ (3 s.f.)
(iii)(b)P(every family pack in a carton contains exactly 2 winning coupons each) $= \left[\mathrm{P}(X = 2)\right]^{12} = 5.75 \times 10^{-7}$ (3 s.f.)
(iii)(c) The answer for **(b)** is smaller than that for **(a)** because the case in (b) is only one of the many cases for (a). In addition to the case in **(b)**, **(a)** includes many other cases such as 4 family packs containing 10 winning coupons, 1 family pack containing 8 winning coupons and the remaining family packs do not contain any winning coupons.

Question 10 Solution
(i) Let X be the time (in minutes) taken by a skilled craftsperson to make a wooden souvenir. $X \sim \mathrm{N}(72, 10^2)$ Let T_n be the time (in minutes) taken by a skilled craftsperson to make n wooden souvenirs. $T_7 = X_1 + X_2 + X_3 + X_4 + X_5 + X_6 + X_7 \sim \mathrm{N}(504, 700)$ $\mathrm{P}(T \le 480) = 0.182$ (3 sf)
(ii) $X_1 - X_2 \sim \mathrm{N}(0, 200)$ $\mathrm{P}(\lvert X_1 - X_2 \rvert < 15) = \mathrm{P}(-15 < X_1 - X_2 < 15)$ $= 0.711$ (3sf)
(iii) P(makes at most 6 souvenirs in a workday) $= \mathrm{P}(T_7 > 480) = 0.81793 = 0.818$ (3sf) P(makes at most 5 souvenirs in a workday) $= \mathrm{P}(T_6 > 480) = 0.02502$ Hence, P(makes exactly 6 souvenirs in a workday) $= 0.81793 - 0.02502$ $= 0.793$ (3sf)

Let $Y = 1.5(X_1 + X_2 + X_3) - 4X$ $Y \sim \mathrm{N}((1.5\times3-4)\times72, (1.5^2\times3+4^2)\times10^2)$ i.e. $Y \sim \mathrm{N}(36, 2275)$ Required probability = $\mathrm{P}(Y > 0) = 0.774$ (3sf) The time taken to make each wooden souvenir is independent of each other.

Question 11 [Solution]
(i)(a) Each piece of thread has an equal chance of being selected and is selected independently of one another. All samples of 50 pieces of threads have the same chance of being selected.
(i)(b) $\bar{x} = \dfrac{441.6}{50} = 8.832$ Unbiased estimate of population variance is $s^2 = \dfrac{n}{n-1}\left(\dfrac{\sum(x-\bar{x})^2}{n}\right) = \dfrac{50}{49}\left(\dfrac{20.9}{50}\right) = \dfrac{20.9}{49} = 0.4265306$ Let μ be the population mean tensile strength of thread $\mathrm{H}_0 : \mu = 9.0$ $\mathrm{H}_1 : \mu \neq 9.0$ Test at 2% level of significance **<u>Under H₀</u>**, the test statistics is $Z = \dfrac{\overline{X} - 9.0}{\frac{s}{\sqrt{n}}} \sim \mathrm{N}(0,1)$ approximately. Reject H_0 if p-value ≤ 0.02 From GC, p-value = $0.0689 > 0.02$ Since the p-value is more than the level of significance, we do not reject H_0. Thus there is insufficient evidence at 2% level of significance that the mean tensile strength of the cotton threads is not 9.0 N.
(ii)(a) Let Y be the tensile strength of a piece of nylon thread manufactured by Company B, and μ_n be the population mean tensile strength of nylon thread. Unbiased estimate for population variance is $s^2 = \dfrac{60}{59}\left(3.3^2\right) = 11.074576$ $H_0 : \mu_n = k$ $H_1 : \mu_n < k$

Test at 2% level of significance

Reject H_0 if $z_{cal} \leq -2.05374$

Under H_0, since the sample size $n = 60$ is large, by Central Limit Theorem,

$\overline{Y} \sim N(k, \frac{11.074576}{60})$ approximately.

For H_0 to be rejected, z_{cal} lies inside the critical region

$$z_{cal} = \frac{35.2 - k}{\sqrt{\frac{11.07458}{60}}} \leq -2.05374$$

$\Rightarrow k \geq 36.1$ (3 s.f.)

(ii)(b) At 2% level of significance means that there is a probability of 0.02 that we wrongly concluded that the company has overstated the mean tensile strength of the threads when the mean tensile strength is actually k N.

(ii)(c) It is <u>not necessary</u> to make any assumption about the population distribution, as sample size of 60 is large enough, the Central Limit Theorem approximates the sample mean tensile strength of the threads, $\overline{Y}$, to a normal distribution.