Section A: Pure Mathematics [40 marks]
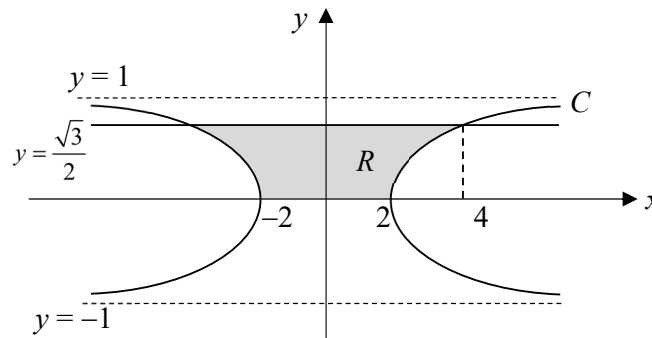
1   A movie theatre supports a charity event by providing discounted movie tickets at the following prices based on age group.

Age group	Discounted ticket price
≤ 12 years	\$6
13 to 49 years	\$9
≥ 50 years (senior citizen)	\$4

The event organiser plans to spend \$700 to sponsor 100 participants of which at least 35 of them are senior citizens. Find the possible number of participants for each age group.
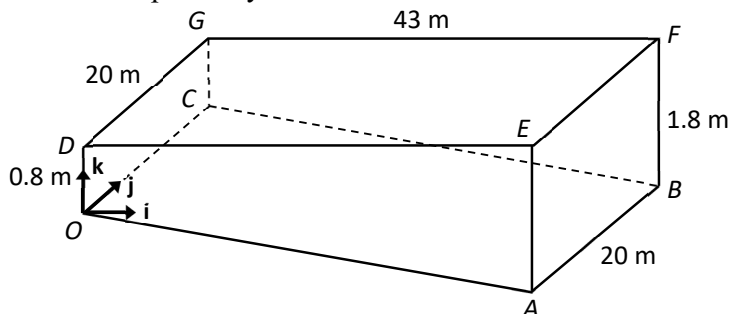
[4]

2

The diagram shows the curve C with equation $y^2 = 1 - \dfrac{4}{x^2}$. The shaded region R is enclosed by C, the x-axis and the line $y = \dfrac{\sqrt{3}}{2}$.

(i)   Find the exact volume of the solid generated when R is rotated through π radians about the y-axis. [3]

(ii)   Find the exact volume of the solid generated when R is rotated through 2π radians about the x-axis. [4]

3 The diagram **(not drawn to scale)** shows a swimming pool, with a horizontal rectangular surface $DEFG$, where $DG = 20$ m and $FG = 43$ m. The walls of the pool are perpendicular to the surface $DEFG$. The sloping floor $OABC$ is a rectangle with $OC = 20$ m. The shallowest and the deepest ends of the pool are at a depth of 0.8 m and 1.8 m respectively. The point O is taken as the origin, with unit vectors **i**, **j** and **k** in the directions of $\overrightarrow{DE}$, $\overrightarrow{OC}$ and $\overrightarrow{OD}$ respectively.

(i) Show that a vector equation of the plane $OABC$ is $\mathbf{r} \cdot \begin{pmatrix} 1 \\ 0 \\ 43 \end{pmatrix} = 0$. [1]

(ii) Given that the point P with coordinates $(39.6, 8, 0.8)$ lies on the plane $DEFG$, find the exact position vector of the foot of perpendicular from P to the plane $OABC$. [4]

(iii) Point M is the midpoint of AB. Find the acute angle between the line PM and the plane $OABC$. [3]

4 The complex numbers u, v and w are given by

$$u = \frac{1}{2}(1 + \cos\theta + i\sin\theta), \quad \text{where} -\pi < \theta < \pi,$$
$$v = u^{-1},$$
$$w = u^{-2}.$$

(i) Show that $|u| = \cos\left(\dfrac{\theta}{2}\right)$, and $\arg u = \dfrac{\theta}{2}$. [2]

(ii) Using the result in part **(i)**, write down $|v|$ and $\arg v$ in term of θ. Show that the real part of v is 1. [2]

(iii) Write down $|w|$ and $\arg w$ in term of θ. Show that the real part of w is $2 - |w|$. [3]

(iv) Given that $\theta \neq 0$, find the possible values of $\arg\left(\dfrac{v - w}{v}\right)$. [3]

5 The curve C has parametric equations

$$x = \cos 3\theta, \quad y = \sin\theta, \quad \text{for } 0 \leq \theta \leq \frac{\pi}{6}.$$

2

(i) Sketch C, showing clearly the coordinates of the end points. [1]

It is given that C cuts the y- and x-axis at points A and B respectively.

(ii) A particle moves along C from A to B with its x-coordinate increasing at a constant rate of 0.1 units per second. Find the rate of decrease of its y-coordinate when $x = \dfrac{1}{2}$. [3]

(iii) Find the equation of the tangent to C at A, giving your answer in the form $y = a + bx$, where a and b are exact values to be determined. [2]

(iv) Find the exact area of the finite region between C, the tangent in **(iii)** and the x-axis. [5]

Section B: Probability and Statistics [60 marks]

6 A group of 3 married couples, 2 men and 2 women are to sit in a row of 10 adjacent seats to watch a movie. Find the number of different seating arrangements if
(i) the men and women are to sit at alternate seats, [2]

(ii) two particular men and one particular woman were to not turn up for the movie and people of the same gender are to sit together. [2]

After the movie, all the 10 people sit at a round table for dinner.
(iii) Find the number of different seating arrangements if each married couple are to sit together and the seats are numbered. [3]

7 A biased tetrahedral die has its faces marked with numbers 2, 3, 4 and 5. Let X be the number on the face in contact with the table when the die is thrown. The probability of X being 2, 3, 4 and 5 are $\dfrac{1}{10}$, α, β and $\dfrac{2}{5}$ respectively. A player throws the die and his score, Y, is given by $Y = |X - 3|$.
(i) Given that $\text{Var}(Y) = 0.56$, find the values of α and β. [6]
(ii) Find the probability that a player's mean score in 50 throws is more than 1.3. [2]

8 A study is carried out to investigate the time to failure of the batteries of Talsa electric cars after travelling different distances. The table shows the time to failure, t days, for 9 Talsa car batteries after travelling distances of s thousand kilometres.

s	105	110	114	120	128	132	143	155	164
t	118	102	90	79	78	55	68	62	58

(i) Sketch a scatter diagram of t against s. [1]

(ii) Circle the point on the scatter diagram that does not seem to follow the trend and label it as P. Suggest a possible reason for it. [2]

Omit point P identified in part (ii) for the rest of this question.

Below are three models proposed to describe the relationship between s and t.

Model I : $t = a + bs^2$ where $a > 0$ and $b < 0$

Model II: $t = a + b\sqrt{s}$ where $a > 0$ and $b < 0$

Model III: $t = a + be^{-\sqrt{s}}$ where $a > 0$ and $b > 0$

(iii) Comment on why model I is not a suitable model for the given set of data. Explain, with justification, why model III is a better choice than model II. [3]

(iv) Find the equation of the least square regression line for model III, giving the values of a and b correct to two decimal places. Hence, estimate the time to failure of a battery of a Talsa electric car after travelling 135 000 km. Give two reasons why you would expect this estimate to be reliable. [3]

9 It is found that $100p\%$ (where $0 < p < 1$) of all standard size packets of potato chips produced by a snack company contain a winning coupon. The potato chips are sold as a family pack which contains 10 randomly chosen standard size packets of potato chips.

(i) Find the range of values of p such that the most likely number of winning coupons in a family pack is 2. [3]

For the rest of the question, let $p = 0.2$.

(ii) A birthday party organizer buys N family packs. Find the least value of N if he wants to be more than 99% sure that there are at least 30 family packs with at least 1 winning coupon. [4]

The family packs are delivered to supermarkets in cartons. Each carton contains 12 randomly chosen family packs.

(iii) (a) Find the probability that a randomly chosen carton contains exactly 24 winning coupons. [2]

(b) Find the probability that every family pack in a randomly chosen carton contains exactly 2 winning coupons each. [1]

(c) Explain why the answer for **(b)** is smaller than that for **(a)**. [1]

10   A workshop employs craftspeople to make wooden souvenirs for sale at a tourist attraction. Typically, the time taken to make a wooden souvenir by a skilled craftsperson follows a normal distribution with a mean of 1.2 hours and a standard deviation of 10 minutes. A workday at the workshop is 8 hours long (with a lunch break given separately). It may be assumed that a skilled craftsperson will start making the next souvenir once the previous one is completed.

   (i)   Find the probability that a skilled craftsperson is able to make 7 wooden souvenirs in a workday.   [2]

   (ii) Two skilled craftspeople start work at the same time. Find the probability that they make their first wooden souvenir of the workday within 15 minutes of each other.   [3]

   (iii) Find the probability that a skilled craftsperson is unable to make the 7th souvenir in a workday. Hence find the probability that a skilled craftsperson makes exactly 6 wooden souvenirs in a workday.   [3]

   (iv) The time taken by an apprentice of a particular skilled craftsperson to make a wooden souvenir is 1.5 times of the time taken by the skilled craftsperson. Find the probability that the total time taken to make 3 wooden souvenirs by the apprentice is longer than 4 times the time taken by the skilled craftsperson to make a wooden souvenir.
   State the assumption needed for your working.   [4]

11   The tensile strength of a thread is the maximum stress it can withstand before it breaks and is measured in Newtons (N).

   (i)   Company A manufactures cotton threads whose tensile strength is normally distributed with mean 9.0 N. A random sample of 50 pieces of threads was taken and the tensile strength, x N, of each thread was tested. The following data was obtained:
   $$\sum x = 441.6, \quad \sum (x - \bar{x})^2 = 20.9.$$

   (a)   Explain what is meant in this context by the term 'a random sample'.   [1]
   (b)   Test at the 2% level of significance, whether the population mean tensile strength of the threads is 9.0 N. State clearly the test statistic used.   [6]

   (ii) Company B claims to manufacture nylon thread with mean tensile strength of k N. 60 pieces of nylon threads were randomly chosen to be tested. The sample mean and standard deviation were found to be 35.2 N and 3.3 N respectively.

   (a)   Find the range of values of k for which there is sufficient evidence for the company to have overstated the mean tensile strength at the 2% level of significance.   [4]
   (b)   Explain, in this context, the meaning of 'at the 2% level of significance'.   [1]
   (c)   Explain if it is necessary to make any assumptions about the distribution of the tensile strengths of the nylon threads.   [1]