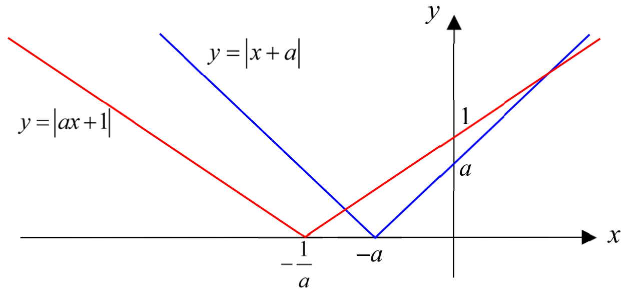
Question 1 [Solution]
(i) $\int \sin^{-1} 2x \ \mathrm{d}x = \int 1.\sin^{-1} 2x \ \mathrm{d}x$ $= x\sin^{-1} 2x - \int x.\frac{2}{\sqrt{1-4x^2}} \ \mathrm{d}x$ $= x\sin^{-1} 2x + \frac{1}{4}\int (-8x)(1-4x^2)^{-\frac{1}{2}} \ \mathrm{d}x$ $= x\sin^{-1} 2x + \frac{1}{4}.\frac{(1-4x^2)^{\frac{1}{2}}}{\frac{1}{2}} + c$ $= x\sin^{-1} 2x + \frac{1}{2}\sqrt{1-4x^2} + c$
(ii) $\int \frac{1}{x^2 - nx + n^2} \ \mathrm{d}x = \int \frac{1}{\left(x-\frac{1}{2}n\right)^2 - \frac{1}{4}n^2 + n^2} \ \mathrm{d}x$ $= \int \frac{1}{\left(x-\frac{1}{2}n\right)^2 + \frac{3}{4}n^2} \ \mathrm{d}x$ $= \frac{2}{\sqrt{3}n}\tan^{-1}\left(\frac{x-\frac{1}{2}n}{\frac{\sqrt{3}}{2}n}\right) + c$ or $\frac{2}{\sqrt{3}n}\tan^{-1}\left(\frac{2x-n}{\sqrt{3}n}\right) + c$

Question 2 [Solution]
(i) $3(1 + \mathrm{i})^2 - (5 + \mathrm{i})(1 + \mathrm{i}) = k$ $3(1 + 2\mathrm{i} - 1) - (5 + 5\mathrm{i} + \mathrm{i} - 1) = k$ $k = -4$
(ii) Using sum of roots : $1+\mathrm{i}+\beta = \frac{5+\mathrm{i}}{3}$ $\therefore \beta = \frac{5+\mathrm{i}}{3} - 1 - \mathrm{i} = \frac{2}{3} - \frac{2}{3}\mathrm{i}$ Thus the other root is $z = \frac{2}{3} - \frac{2}{3}\mathrm{i}$

Question 3 [Solution]

(i)



At the intersection points, $|x+a|=|ax+1|$

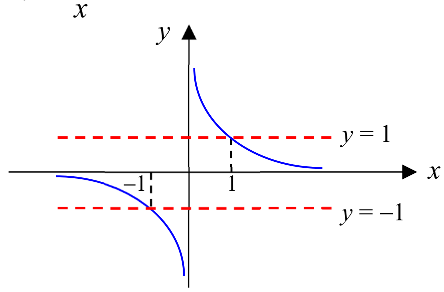
$x+a=ax+1$ or $x+a=-(ax+1)$

$x=1$ or $x=-1$

From the graph, the solution is $-1<x<1$

(ii) Replacing x by $\frac{1}{x}$, $-1<\frac{1}{x}<1$

From the graph of $y=\frac{1}{x}$,



The solution is $x<-1$ or $x>1$

Question 4 [Solution]

(i) $\ln y=1+\tan^{-1}(2x)$

Differentiating w.r.t. x,

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2}{1+(2x)^2}=\frac{2}{1+4x^2}$$

$$\Rightarrow \quad (1+4x^2)\frac{\mathrm{d}y}{\mathrm{d}x}=2y$$

Differentiating w.r.t. x,

$$(1+4x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+8x\frac{\mathrm{d}y}{\mathrm{d}x}=2\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$(1+4x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+(8x-2)\frac{\mathrm{d}y}{\mathrm{d}x}=0 \qquad \text{(shown)}$$

(ii) Differentiating w.r.t. x,

$$\left(1+4x^2\right)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+8x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(8x-2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+8\frac{\mathrm{d}y}{\mathrm{d}x}=0$$

i.e. $\left(1+4x^2\right)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+\left(16x-2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+8\frac{\mathrm{d}y}{\mathrm{d}x}=0$

When $x=0$, $\quad \ln y = 1+\tan^{-1}(0)=1 \quad \Rightarrow \quad y=\mathrm{e}$

$$(1+0)\frac{\mathrm{d}y}{\mathrm{d}x}=2\mathrm{e} \quad \Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x}=2\mathrm{e}$$

$$(1+0)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-2\left(2\mathrm{e}\right)=0 \quad \Rightarrow \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2}=4\mathrm{e}$$

$$(1+0)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+\left(0-2\right)\left(4\mathrm{e}\right)+8\left(2\mathrm{e}\right)=0 \quad \Rightarrow \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-8\mathrm{e}$$

Maclaurin series is $y=\mathrm{e}+2\mathrm{e}x+4\mathrm{e}\left(\frac{x^2}{2!}\right)-8\mathrm{e}\left(\frac{x^3}{3!}\right)+\ldots$

i.e., $y=\mathrm{e}+2\mathrm{e}x+2\mathrm{e}x^2-\frac{4}{3}\mathrm{e}x^3+\ldots$

(iii) $\ln y = 1+\tan^{-1}(2x)$

$$y=\mathrm{e}^{1+\tan^{-1}(2x)}=\mathrm{e}\ \mathrm{e}^{\tan^{-1}(2x)} \Rightarrow \mathrm{e}^{\tan^{-1}(2x)}=\frac{y}{\mathrm{e}}$$

Thus Maclaurin series is $\mathrm{e}^{\tan^{-1}(2x)}=1+2x+2x^2-\frac{4}{3}x^3+\ldots$

Question 5 [Solution]

(i) Line AB: $\mathbf{r}=\mathbf{a}+\lambda(\mathbf{b}-\mathbf{a})$, $\lambda\in\mathbb{R}$

Line OC: $\mathbf{r}=\mu(9\mathbf{a}-6\mathbf{b})$, $\mu\in\mathbb{R}$

At the intersection point,

$$\mathbf{a}+\lambda(\mathbf{b}-\mathbf{a})=\mu(9\mathbf{a}-6\mathbf{b})$$

$$(1-\lambda-9\mu)\mathbf{a}=(-6\mu-\lambda)\mathbf{b}$$

Since $\mathbf{a}$ and $\mathbf{b}$ are non-zero and non-parallel,

$$1-\lambda-9\mu=0 \quad \text{--- (1)}$$

$$-6\mu-\lambda=0 \quad \text{--- (2)}$$

Solving (1) and (2), $\mu=\frac{1}{3}$, $\lambda=-2$

Position vector of the intersection point is $\mathbf{r}=\frac{1}{3}(9\mathbf{a}-6\mathbf{b})=3\mathbf{a}-2\mathbf{b}$

(ii) $\mathbf{d}=t\begin{pmatrix}1\\1\\0\end{pmatrix}+\left(1-t\right)\begin{pmatrix}4\\-2\\6\end{pmatrix}=\begin{pmatrix}-3t+4\\3t-2\\-6t+6\end{pmatrix}$

$$\cos 60^0 = \frac{\mathbf{a}\cdot\mathbf{d}}{|\mathbf{a}||\mathbf{d}|} = \frac{\begin{pmatrix}1\\1\\0\end{pmatrix}\cdot\begin{pmatrix}-3t+4\\3t-2\\-6t+6\end{pmatrix}}{\sqrt{1^2+1^2}\sqrt{(3t-4)^2+(3t-2)^2+(6t-6)^2}}$$

$$\frac{1}{2} = \frac{2}{\sqrt{2}\sqrt{54t^2-108t+56}}$$

$$\sqrt{2}\sqrt{54t^2-108t+56} = 4$$

$$54t^2-108t+56 = 8$$

$$54t^2-108t+48 = 0$$

$$9t^2-18t+8 = 0$$

$$(3t-2)(3t-4) = 0$$

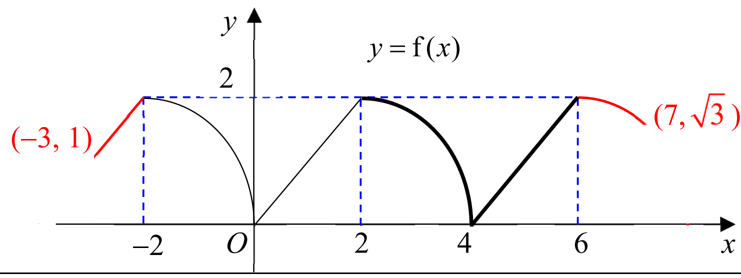
$$t = \frac{2}{3} \quad \text{or} \quad t = \frac{4}{3}$$

	Question 6 [Solution]
(i)	For $r \ge 1$, $\frac{1}{S_{r+1}} - \frac{1}{S_r} = \frac{1+(2r-1)S_r}{S_r} - \frac{1}{S_r}$ $= \frac{1}{S_r} + (2r-1) - \frac{1}{S_r}$ $= 2r-1$
(ii)	$\sum_{r=1}^{n-1}\left(\frac{1}{S_{r+1}} - \frac{1}{S_r}\right) = \sum_{r=1}^{n-1}(2r-1)$ $\text{RHS} = \sum_{r=1}^{n-1}(2r-1) = \frac{n-1}{2}(1+2(n-1)-1)$ $= (n-1)^2$ $\text{LHS} = \sum_{r=1}^{n-1}\left(\frac{1}{S_{r+1}} - \frac{1}{S_r}\right) = \cancel{\frac{1}{S_2}} - \frac{1}{S_1}$ $+ \cancel{\frac{1}{S_3}} - \cancel{\frac{1}{S_2}}$ $+ \frac{1}{S_4} - \cancel{\frac{1}{S_3}}$ $+ \ldots\ldots$ $+ \cancel{\frac{1}{S_{n-1}}} - \cancel{\frac{1}{S_{n-2}}}$ $+ \frac{1}{S_n} - \cancel{\frac{1}{S_{n-1}}}$ $= \frac{1}{S_n} - \frac{1}{S_1} = \frac{1}{S_n} - 1$

	Equating, $\dfrac{1}{S_n}-1=(n-1)^2$ $\dfrac{1}{S_n}=(n-1)^2+1 \Rightarrow S_n=\dfrac{1}{(n-1)^2+1}$ for $n\geq 2$
(iii)	As $n\to\infty$, $(n-1)^2+1\to\infty$, $S_n=\dfrac{1}{(n-1)^2+1}\to 0$ Thus the sequence converges to 0.
(iv)	For $n\geq 2$, $u_n=S_n-S_{n-1}=\dfrac{1}{(n-1)^2+1}-\dfrac{1}{(n-2)^2+1}$ **<u>Method 1</u>** Since $(n-1)^2+1>(n-2)^2+1$ for $n\geq 2$, $\dfrac{1}{(n-1)^2+1}<\dfrac{1}{(n-2)^2+1}$ $u_n=\dfrac{1}{(n-1)^2+1}-\dfrac{1}{(n-2)^2+1}<0$ i.e. for $n\geq 2$, $u_n<0$ **<u>Method 2</u>** $u_n=\dfrac{1}{(n-1)^2+1}-\dfrac{1}{(n-2)^2+1}$ $=\dfrac{(n^2-4n+4+1)-(n^2-2n+1+1)}{((n-1)^2+1)((n-2)^2+1)}$ $=\dfrac{3-2n}{((n-1)^2+1)((n-2)^2+1)}$ For $n\geq 2$, $3-2n<0$ and $((n-1)^2+1)((n-2)^2+1)>0$ So $u_n<0$

Question 7 [Solution]

(i)



(ii)

Smallest value of $k = 2$

Let $y=\sqrt{4x-x^2}$ for $2 \le x < 4$

$y^2 = 4-(x-2)^2$

$(x-2)^2 = 4-y^2$

$x = 2 \pm \sqrt{4-y^2}$

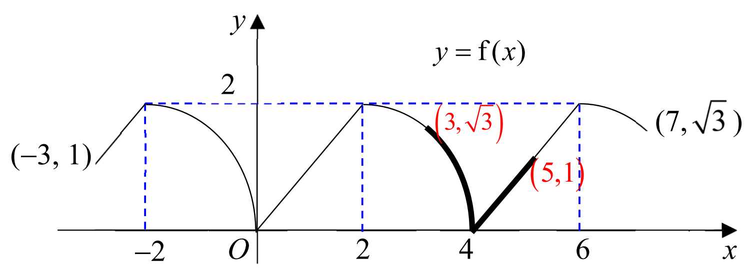
Since $x \ge 2$, $x = 2+\sqrt{4-y^2}$

$\mathrm{f}^{-1}(x) = 2+\sqrt{4-x^2}$, $0 < x \le 2$

(iii)

$R_g = (3, 5)$, $D_f = [2, 6]$

Since $R_g \subseteq D_f$, fg exists.



$D_g = (e^3, e^5) \xrightarrow{g} (3,\ 5) \xrightarrow{f} [0, \sqrt{3}) = R_{fg}$

Question 8 [Solution]

(a)

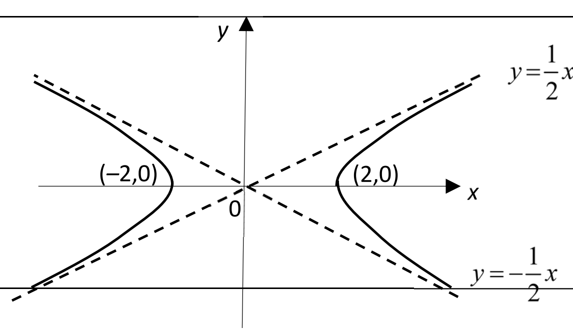
$C_2 : x^2 = a^2(1-y^2) \Rightarrow \left(\frac{x}{a}\right)^2 + y^2 = 1$

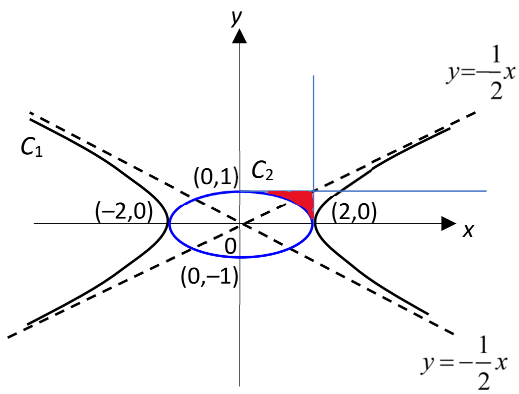
$(x+1)^2 + y^2 = 1 \xrightarrow{T} x^2 + y^2 = 1 \xrightarrow{S} \left(\frac{x}{a}\right)^2 + y^2 = 1$

1. Translation of 1 unit in the positive direction of the x-axis.
2. Scaling parallel to the x-axis by factor a.

(b)(i)

$C_1 : \frac{x^2}{4} - y^2 = 1$



(ii)	$C_2:\ \left(\frac{x}{a}\right)^2+y^2=1$ is an ellipse with centre $(0,0)$. For C_1 and C_2 to intersect exactly twice, $a=2$.
(iii)	
(iv)	Exact area $=2(1)-\int_0^2\sqrt{1-\frac{x^2}{4}}\ \mathrm{d}x$ $\boxed{x=2\cos\theta \quad \frac{\mathrm{d}x}{\mathrm{d}\theta}=-2\sin\theta}$ $=2(1)-\int_{\frac{\pi}{2}}^{0}\sqrt{1-\frac{4\cos^2\theta}{4}}\ (-2\sin\theta)\mathrm{d}\theta$ $=2(1)-\int_{\frac{\pi}{2}}^{0}\sqrt{\sin^2\theta}\ (-2\sin\theta)\mathrm{d}\theta$ $=2(1)+2\int_{\frac{\pi}{2}}^{0}\sin^2\theta\ \mathrm{d}\theta$ $=2(1)+\int_{\frac{\pi}{2}}^{0}(1-\cos 2\theta)\ \mathrm{d}\theta$ $=2(1)+\left[\theta-\frac{\sin 2\theta}{2}\right]_{\frac{\pi}{2}}^{0}$ $=2(1)+\left[0-\left(\frac{\pi}{2}-0\right)\right]$ $=\left(2-\frac{\pi}{2}\right)\text{units}^2$

Question 9 [Solution]

(a)

<u>Team A</u>

Day	Depth drilled (m)
1	190
2	$190\left(\frac{r}{100}\right)$
3	$190\left(\frac{r}{100}\right)^2$
…	
n	$190\left(\frac{r}{100}\right)^{n-1}$

GP: first term = 190 and common ratio $=\frac{r}{100}$

Since $0<r<100$, $0<\frac{r}{100}<1$, sum to infinity S_∞ exists.

If team A never reaches the oil deposit,

$$S_\infty=\frac{190}{1-\frac{r}{100}}<8000$$

$$\frac{190}{8000}<1-\frac{r}{100}$$

$$\frac{r}{100}<\frac{781}{800}$$

$$0<r<\frac{781}{8} \quad (\text{or } 0<r<97.625)$$

(b)

<u>Team A</u>

GP: first term = 190 and common ratio $=0.99$

Consider $S_{n,A}=\frac{190\left(1-0.99^n\right)}{1-0.99}=8000$

$$1-0.99^n=\frac{80}{190}$$

$$n=\frac{\ln\left(\frac{11}{19}\right)}{\ln 0.99}=54.38$$

Team A will reach the oil deposit on Day 55.

Alternatively, $S_{n,A}=\frac{190\left(1-0.99^n\right)}{1-0.99}\le 8000$

From GC,

Day	Depth drilled (m)
54	7957.8
55	8068.3
…	…

Team A will reach the oil deposit on Day 55.

(c)

Team B

AP: first term = 180 and common difference $=-1$

Depth drilled on day n for Team B, $b_n = 180+(n-1)(-1) = 181-n$

Depth drilled on day n for Team A, $a_n = 190\left(0.99^{n-1}\right)$

$a_n < b_n$

$190\left(0.99^{n-1}\right) - 181 + n < 0$

From GC,

n	$a_n - b_n$
13	$0.413 > 0$
14	$-0.271 < 0$

Thus the first day is Day 14.

(d)

Team B

AP: first term = 180 and common difference $=-1$

Consider $S_{n,B} = \frac{n}{2}\left(2(180)+(n-1)(-1)\right) \geq 8000$

$$n(361-n) \geq 16000$$

$$n^2 - 361n + 16000 \leq 0$$

Using GC,

n	$n^2 - 361n + 16000$
51	$190 > 0$
52	$-68 < 0$

Team B will reach the oil deposit on Day 52.

Thus **team B** will reach the oil deposit first.

Question 10 [Solution]

(a)

$$\frac{v}{(16+v)(9-v)} = \frac{a}{(16+v)} + \frac{b}{(9-v)}$$

$$v = a(9-v) + b(16+v)$$

Subst $v = 9$, $b = \frac{9}{25}$

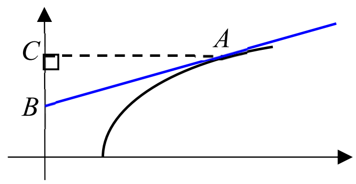
Subst $v = -16$, $a = -\frac{16}{25}$

$$\therefore \frac{v}{(16+v)(9-v)} = -\frac{16}{25}\frac{1}{(16+v)} + \frac{9}{25}\frac{1}{(9-v)}$$

(b)(i)

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{(16+v)(9-v)}{320v}$$

	$\int \frac{v}{(16+v)(9-v)}\,\mathrm{d}v = \frac{1}{320}\int 1\,\mathrm{d}t$ $-\frac{16}{25}\int \frac{1}{(16+v)}\,\mathrm{d}v + \frac{9}{25}\int \frac{1}{(9-v)}\,\mathrm{d}v = \frac{1}{320}t + c$ using (a) $-16\ln\lvert 16+v\rvert \;-9\ln\lvert 9-v\rvert = \frac{5}{64}t + c$ When $t=0,\ v=0,\ c=-16\ln 16-9\ln 9$ or $-\ln\left(9^9\right)\left(16^{16}\right)$ $-16\ln\lvert 16+v\rvert \;-9\ln\lvert 9-v\rvert = \frac{5}{64}t-16\ln 16-9\ln 9$ $t=\frac{64}{5}\left(9\ln\left\lvert\frac{9}{9-v}\right\rvert+16\ln\left\lvert\frac{16}{16+v}\right\rvert\right)$ or $t=\frac{64}{5}\ln\left(\frac{9^9 16^{16}}{\lvert 9-v\rvert^9\lvert 16+v\rvert^{16}}\right)$
(ii)	Her theoretical maximum speed occurs when $\frac{\mathrm{d}v}{\mathrm{d}t}=0$ $\frac{(16+v)(9-v)}{320v}=0$ $v=9$ or $v=-16$ (rejected $\because v\geq 0$) Her theoretical maximum speed is 9 m s^{-1}. When $v=4.5$, $t=\frac{64}{5}\left(9\ln\left\lvert\frac{9}{9-4.5}\right\rvert+16\ln\left\lvert\frac{16}{16+4.5}\right\rvert\right)=29.1$ The time taken is 29.1 seconds.
(iii)	$v=\frac{\mathrm{d}x}{\mathrm{d}t}=\sqrt{\frac{9t}{10}}$ $x=\int\sqrt{\frac{9t}{10}}\,\mathrm{d}t$ $=\frac{3}{\sqrt{10}}\left(\frac{2}{3}t^{\frac{3}{2}}\right)+d$ $=\frac{2}{\sqrt{10}}t^{\frac{3}{2}}+d$ When $t=0,\ x=0,\ d=0$ $\therefore\ x=\frac{2}{\sqrt{10}}t^{\frac{3}{2}}$ When $x=10,\ 10=\frac{2}{\sqrt{10}}t^{\frac{3}{2}} \Rightarrow t=6.30$ The time taken is approximately 6.30 seconds.

	Question 11 [Solution]
(i)	$3y^2 = 2x - 1$ $6y\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2$ $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{3y}$  At A, $y = p \Rightarrow\ 3p^2 = 2x - 1\ \Rightarrow\ x = \dfrac{3p^2+1}{2}$ Equation of tangent at A: $y - p = \dfrac{1}{3p}\left(x - \left(\dfrac{3p^2+1}{2}\right)\right)$ $6py = 2x - \left(3p^2+1\right) + 6p^2$ $6py = 2x + 3p^2 - 1$ (shown)
(ii)	At B, $x = 0$, $6py = 3p^2 - 1\ \Rightarrow\ y = \dfrac{3p^2-1}{6p}$ Coordinates of C is $(0, p)$ Area of $\Delta ABC = \dfrac{1}{2} \times AC \times BC$ $\because$ right angled at C $= \dfrac{1}{2}\left(\dfrac{3p^2+1}{2}\right)\left(p - \left(\dfrac{3p^2-1}{6p}\right)\right)$ $= \dfrac{1}{4}\left(3p^2+1\right)\left(\dfrac{6p^2-3p^2+1}{6p}\right)$ $= \dfrac{1}{24p}\left(3p^2+1\right)^2$ $= \dfrac{1}{24p}\left(9p^4+6p^2+1\right)$ $= \dfrac{1}{24}\left(9p^3+6p+\dfrac{1}{p}\right)$ (shown)
(iii)	Let $E = \dfrac{1}{24}\left(9p^3+6p+\dfrac{1}{p}\right)$ Let $\dfrac{\mathrm{d}E}{\mathrm{d}p} = \dfrac{1}{24}\left(27p^2+6-\dfrac{1}{p^2}\right) = 0$ $27p^2 + 6 - \dfrac{1}{p^2} = 0$ $27p^4 + 6p^2 - 1 = 0$ $\left(9p^2-1\right)\left(3p^2+1\right) = 0$ $9p^2 = 1$ since $3p^2 + 1 \neq 0$ $p = \dfrac{1}{3}$ since $p > 0$

$$\frac{\mathrm{d}^2E}{\mathrm{d}p^2}=\frac{1}{24}\left(54p+\frac{2}{p^3}\right)>0 \text{ since } p>0$$

Thus E is minimum when $p=\dfrac{1}{3}$

Alternative: Using first derivative test

p	0.3	$\frac{1}{3}$	0.35
$\frac{\mathrm{d}E}{\mathrm{d}p}$	$-0.112<0$	0	$0.0477>0$
Shape of graph	\	—	/

Minimum value of E

$$=\frac{1}{24}\left(9\left(\frac{1}{3}\right)^3+6\left(\frac{1}{3}\right)+\frac{1}{\left(\frac{1}{3}\right)}\right)$$

$$=\frac{2}{9}\ \text{units}^2$$

(iv)

When $p=\dfrac{1}{3}$, $AC=\dfrac{3p^2+1}{2}=\dfrac{2}{3}$

and $BC=\dfrac{3p^2+1}{6p}=\dfrac{2}{3}$

Thus ΔABC is a right-angled isosceles triangle.