RAFFLES INSTITUTION
H2 Mathematics (9758)
2022 Year 5

2022 Year 5 H2 Mathematics Promotion Examination: Solutions with Comments

1 Elly runs a home bakery business and sells three different flavors of cakes. A 0.5 kg Strawberrylicious, Cheeseburst and Chocofanatic cake is priced at x, y and z dollars respectively. A 1 kg cake costs 40% more than a 0.5 kg cake of the same flavour.

Customer A bought one 0.5 kg Strawberrylicious cake, one 0.5 kg Cheeseburst cake and one 0.5 kg Chocofanatic cake. She paid a total of \$175.

Customer B bought one 1 kg Strawberrylicious cake, three 1 kg Cheeseburst cakes and two 0.5 kg Chocofanatic cakes. She paid a total of \$463.

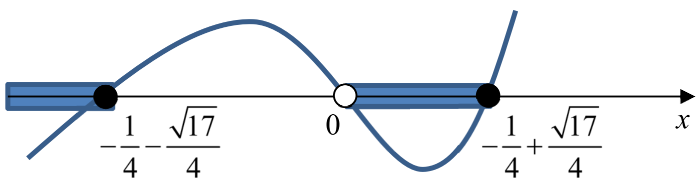
During a 11.11 sale, Customer C bought one 1 kg Strawberrylicious cake, one 1 kg Cheeseburst cake and two 1 kg Chocofanatic cakes at 10% off the total bill. He paid a total of \$296.10 after the discount.

(i) Find the values of x, y and z. [3]

(ii) Find the total amount paid by Customer D, who bought three 0.5 kg Strawberrylicious cakes, two 0.5 kg Cheeseburst cakes and two 1 kg Chocofanatic cakes at 25% off the total bill during a 12.12 sale. [1]

	Solution	Comments
1(a) **[3]**	A 1 kg cake costs 40% more than a 0.5 kg cake of the same flavour, which means the price for a 1 kg cake for the flavours Strawberrylicious, Cheeseburst and Chocofanatic is $1.4x, 1.4y$ and $1.4z$ dollars respectively. Customer A: $x+y+z=175$ -- (1) Customer B: $1.4x+3(1.4y)+2z=463$ -- (2) Customer C: $0.9[1.4x+1.4y+2(1.4z)]=296.10$ -- (3) Solving equations (1), (2) and (3) with the GC, we have $x=50, y=65$ and $z=60$	Generally well done. Those who did not get full credit mainly misread the question or made careless algebraic manipulations.
(b) **[1]**	Total amount paid by Customer D $=0.75[3x+2y+2(1.4z)]$ $=0.75[3(50)+2(65)+2(1.4)(60)]$ $=\$336$	

2 Without using a calculator, solve the inequality $3x+1\le\frac{x^2+2}{x}$. [5]

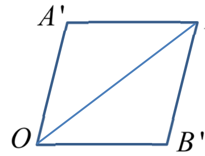
	Solution	**Comments**
2 **[5]**	$3x+1\le\frac{x^2+2}{x}$ $3x+1-\left(\frac{x^2+2}{x}\right)\le 0$ $\frac{2x^2+x-2}{x}\le 0 \quad \cdots(*)$ $x\left(2x^2+x-2\right)\le 0, \quad x\ne 0 \quad \cdots(**)$ Let $2x^2+x-2=0$ $x=\frac{-1\pm\sqrt{1^2-4(2)(-2)}}{2(2)}$ $=-\frac{1}{4}\pm\frac{\sqrt{17}}{4}$  $\therefore$ Solution to inequality is $x\le-\frac{1}{4}-\frac{\sqrt{17}}{4}$ or $0<x\le-\frac{1}{4}+\frac{\sqrt{17}}{4}$	To convert from (*) to (**), **multiply both sides of (*) by x^2 which is non-negative**, not by x. Note that inequality signs will be "reversed" when they are multiplied or divided by a negative number. Note that $ab\le 0$ $\Leftrightarrow a\le 0$ and $b\ge 0$ (Case 1) or $a\ge 0$ and $b\le 0$ (Case 2) NOT $a\le 0$ or $b\le 0$ Remember to exclude "0" from final solution since $x\ne 0$ in (*). $x\le -1.28$ or $0<x\le 0.781$ is not acceptable since you are not supposed to use a calculator.

3 The origin O and the points A, B and C lie in the same plane, where $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$.

(a) Show that if $\mathbf{c} = |\mathbf{a}|\mathbf{b} + |\mathbf{b}|\mathbf{a}$, then OC bisects the angle AOB. [3]

(b) Given that $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$, show that $\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c} = \mathbf{c} \times \mathbf{a}$. [2]

	Solution	Comments
3 (a) **[3]**	$$\begin{aligned}\cos(\angle AOC) &= \frac{\mathbf{a}\cdot\mathbf{c}}{\lvert\mathbf{a}\rvert\lvert\mathbf{c}\rvert} \\ &= \frac{\mathbf{a}\cdot\left(\lvert\mathbf{a}\rvert\mathbf{b} + \lvert\mathbf{b}\rvert\mathbf{a}\right)}{\lvert\mathbf{a}\rvert\lvert\mathbf{c}\rvert} \\ &= \frac{\lvert\mathbf{a}\rvert\mathbf{a}\cdot\mathbf{b} + \lvert\mathbf{b}\rvert\mathbf{a}\cdot\mathbf{a}}{\lvert\mathbf{a}\rvert\lvert\mathbf{c}\rvert} \\ &= \frac{\lvert\mathbf{a}\rvert\mathbf{a}\cdot\mathbf{b} + \lvert\mathbf{b}\rvert\lvert\mathbf{a}\rvert^2}{\lvert\mathbf{a}\rvert\lvert\mathbf{c}\rvert} \\ &= \frac{\lvert\mathbf{a}\rvert\left(\mathbf{a}\cdot\mathbf{b} + \lvert\mathbf{b}\rvert\lvert\mathbf{a}\rvert\right)}{\lvert\mathbf{a}\rvert\lvert\mathbf{c}\rvert} \\ &= \frac{\mathbf{a}\cdot\mathbf{b} + \lvert\mathbf{b}\rvert\lvert\mathbf{a}\rvert}{\lvert\mathbf{c}\rvert}\end{aligned}$$ Similarly, $$\begin{aligned}\cos(\angle BOC) &= \frac{\mathbf{b}\cdot\mathbf{c}}{\lvert\mathbf{b}\rvert\lvert\mathbf{c}\rvert} \\ &= \frac{\mathbf{b}\cdot\left(\lvert\mathbf{a}\rvert\mathbf{b} + \lvert\mathbf{b}\rvert\mathbf{a}\right)}{\lvert\mathbf{b}\rvert\lvert\mathbf{c}\rvert} \\ &= \frac{\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert^2 + \lvert\mathbf{b}\rvert\mathbf{b}\cdot\mathbf{a}}{\lvert\mathbf{b}\rvert\lvert\mathbf{c}\rvert} \\ &= \frac{\lvert\mathbf{b}\rvert\lvert\mathbf{a}\rvert + \mathbf{a}\cdot\mathbf{b}}{\lvert\mathbf{c}\rvert} \quad \text{since } \mathbf{b}\cdot\mathbf{a} = \mathbf{a}\cdot\mathbf{b}\end{aligned}$$ Since $\angle BOC$ and $\angle AOC$ are both less than 180° and $\cos(\angle BOC) = \cos(\angle AOC)$, $\angle BOC = \angle AOC$ $\therefore$ OC bisects the angle AOB (shown).	As with all show questions, clear working/explanation is required. Students should also take note of the presentation in proving questions. Properties that were tested here : (i) $\mathbf{a}\cdot\mathbf{a} = \lvert\mathbf{a}\rvert^2$ (ii) $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$ Note that it is not necessary to substitute $\mathbf{c} = \lvert\mathbf{a}\rvert\mathbf{b} + \lvert\mathbf{b}\rvert\mathbf{a}$ in the denominator. Some students did that and confused themselves unnecessarily.

	Alternative Solution: $\mathbf{c}=\lvert\mathbf{a}\rvert\mathbf{b}+\lvert\mathbf{b}\rvert\mathbf{a}$ $=\lvert\mathbf{a}\rvert\left(\lvert\mathbf{b}\rvert\hat{\mathbf{b}}\right)+\lvert\mathbf{b}\rvert\left(\lvert\mathbf{a}\rvert\hat{\mathbf{a}}\right)$ $=\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\left(\hat{\mathbf{b}}+\hat{\mathbf{a}}\right)$  Since $\lvert\hat{\mathbf{a}}\rvert=\lvert\hat{\mathbf{b}}\rvert=1$, $OA'PB'$ is a rhombus where $\overrightarrow{OA'}=\hat{\mathbf{a}}$, $\overrightarrow{OB'}=\hat{\mathbf{b}}$ and $\overrightarrow{OP}=\hat{\mathbf{b}}+\hat{\mathbf{a}}$. OP is the diagonal of the rhombus and it bisects angle $A'OB'$. Since $\mathbf{c}$ is parallel to $\overrightarrow{OP}$, OC bisects the angle AOB. (shown)	
(b) **[2]**	Given $\mathbf{a}+\mathbf{b}+\mathbf{c}=\mathbf{0}$, $\mathbf{a}=-\mathbf{b}-\mathbf{c}$ $\mathbf{a}\times\mathbf{b}=(-\mathbf{b}-\mathbf{c})\times\mathbf{b}$ $=-\mathbf{b}\times\mathbf{b}-\mathbf{c}\times\mathbf{b}$ $=\mathbf{0}+\mathbf{b}\times\mathbf{c}$ $=\mathbf{b}\times\mathbf{c}$ $=(-\mathbf{a}-\mathbf{c})\times\mathbf{c}$ since $\mathbf{a}+\mathbf{b}+\mathbf{c}=\mathbf{0}\Rightarrow\mathbf{b}=-\mathbf{a}-\mathbf{c}$ $=-\mathbf{a}\times\mathbf{c}-\mathbf{c}\times\mathbf{c}$ $=\mathbf{c}\times\mathbf{a}-\mathbf{0}$ $=\mathbf{c}\times\mathbf{a}$ Hence, $\mathbf{a}\times\mathbf{b}=\mathbf{b}\times\mathbf{c}=\mathbf{c}\times\mathbf{a}$ (shown).	Clear working/explanation must be evident in a show question. Properties that were tested here : (i) $\mathbf{a}\times\mathbf{a}=\mathbf{0}$ (ii) $\mathbf{a}\times\mathbf{b}=-(\mathbf{b}\times\mathbf{a})$

4 **(i)** Find $\displaystyle\sum_{r=1}^{n}\frac{1}{(r+1)(r+3)}$, where $n \geq 3$. (There is no need to express your answer as a single algebraic fraction.) [5]

(ii) Explain why $\displaystyle\sum_{r=1}^{\infty}\frac{1}{(r+1)(r+3)}$ is a convergent series, and state the value of the sum to infinity. [2]

	Solution	**Comments**
4(i) **[5]**	$\dfrac{1}{(r+1)(r+3)}=\dfrac{A}{r+1}+\dfrac{B}{r+3}$ where $A=\dfrac{1}{2}$ and $B=-\dfrac{1}{2}$ $\therefore \dfrac{1}{(r+1)(r+3)}=\dfrac{1}{2}\left[\dfrac{1}{r+1}-\dfrac{1}{r+3}\right]$ $\displaystyle\sum_{r=1}^{n}\frac{1}{(r+1)(r+3)}=\frac{1}{2}\sum_{r=1}^{n}\left(\frac{1}{r+1}-\frac{1}{r+3}\right)$ $=\dfrac{1}{2}\begin{bmatrix} \frac{1}{2} & - & \cancel{\frac{1}{4}} \\ +\frac{1}{3} & - & \cancel{\frac{1}{5}} \\ +\cancel{\frac{1}{4}} & - & \cancel{\frac{1}{6}} \\ +\cancel{\frac{1}{5}} & - & \cancel{\frac{1}{7}} \\ +\ldots & \ldots & \ldots \\ +\cancel{\frac{1}{n-1}} & - & \cancel{\frac{1}{n+1}} \\ +\cancel{\frac{1}{n}} & - & \frac{1}{n+2} \\ +\cancel{\frac{1}{n+1}} & - & \frac{1}{n+3} \end{bmatrix}$ $=\dfrac{1}{2}\left[\dfrac{1}{2}+\dfrac{1}{3}-\dfrac{1}{n+2}-\dfrac{1}{n+3}\right]$ $=\dfrac{1}{2}\left[\dfrac{5}{6}-\dfrac{1}{n+2}-\dfrac{1}{n+3}\right]$	Generally well done. Students are encouraged to combine the 2 algebraic fractions to check that they have done their partial fractions correctly before proceeding. There must be 2 full cancellations at the start and 1 full cancellation at the end. There is no need to express the answer as a single algebraic fraction as stated in the question but students should simplify $\dfrac{1}{2}+\dfrac{1}{3}$ to $\dfrac{5}{6}$.

(ii) **[2]**	$\displaystyle\sum_{r=1}^{n}\frac{1}{(r+1)(r+3)}=\frac{1}{2}\left[\frac{5}{6}-\frac{1}{n+2}-\frac{1}{n+3}\right]$ As $n\to\infty, \dfrac{1}{n+2}\to 0$ and $\dfrac{1}{n+3}\to 0,$ hence $\displaystyle\sum_{r=1}^{n}\frac{1}{(r+1)(r+3)}\to\frac{5}{12}$ $\displaystyle\therefore\sum_{r=1}^{\infty}\frac{1}{(r+1)(r+3)}$ is a convergent series. $\displaystyle\therefore\sum_{r=1}^{\infty}\frac{1}{(r+1)(r+3)}=\frac{5}{12}$	Generally well done.

5 **Do not use a calculator in answering this question.**

(a) Solve the equation $z^3-4z^2+6z-4=0$. [3]

(b) The complex numbers w and z are given by $-1+\mathrm{i}\sqrt{3}$ and $9\sin\alpha+9\mathrm{i}\cos\alpha$, where $0<\alpha<\frac{\pi}{2}$, respectively.

It is further given that $\arg(wz)=\frac{5}{6}\pi$.

Find z^* in the form $re^{\mathrm{i}\theta}$, where $r>0$ and $0\le\theta<2\pi$. [5]

	Solution	**Comments**
5 (a) [3]	$z^3-4z^2+6z-4=0$ … (1) Checking $z=2$: LHS = 8 – 16 + 12 – 4 = 0 = RHS $z-2$ is a factor. $\Rightarrow z^3-4z^2+6z-4=(z-2)(z^2+az+2)$ for some $a\in\mathbb{R}$ Comparing coefficient of z, $-4=a-2\Rightarrow a=-2$ $z^3-4z^2+6z-4=0$ $(z-2)(z^2-2z+2)=0$ $z=2$ or $z=\frac{2\pm\sqrt{4-4(1)(2)}}{2}$ $=\frac{2\pm\sqrt{-4}}{2}=\frac{2\pm2\mathrm{i}}{2}=1\pm\mathrm{i}$	Long division may be applied to obtain factorization of cubic polynomial in (1) Answers obtained via GC Plysmlt 2 and/or with no justification/working does not earn any credit Some concluded that discriminant of $z^2-2z+2=0$ is negative so (1) only has a real root, $z=2$. Note that non-real roots are also considered as solutions.
(b) [5]	$w=-1+\mathrm{i}\sqrt{3}$ $\Rightarrow \arg(w)=\pi-\theta$ $=\pi-\tan^{-1}\left\|\frac{\sqrt{3}}{-1}\right\|$ $=\frac{2\pi}{3}$	To work out arg(w), students are strongly encouraged to 1) locate the quadrant containing $(-1,\sqrt{3})$ which correspond to $w=-1+\mathrm{i}\sqrt{3}$, 2) visualize arg(w), 3) calculate basic angle θ followed by arg(w). Note : $\arg(a+\mathrm{i}b)\ne\tan^{-1}\left(\frac{b}{a}\right)$ in general

	$\arg(wz)=\dfrac{5}{6}\pi$ $\Rightarrow \arg(w)+\arg(z)=\dfrac{5}{6}\pi$ $\Rightarrow \arg(z)=\dfrac{5}{6}\pi-\dfrac{2}{3}\pi=\dfrac{1}{6}\pi$ $z=9\sin\alpha+9\mathrm{i}\cos\alpha$ $\Rightarrow \lvert z\rvert=\sqrt{(9\sin\alpha)^2+(9\cos\alpha)^2}=9\sqrt{1}=9$ $\therefore z=9\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}$ $\Rightarrow z^*=9\mathrm{e}^{-\mathrm{i}\frac{\pi}{6}}=9\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{6}+2\pi\right)}=9\mathrm{e}^{\mathrm{i}\frac{11\pi}{6}}$	$z=9(\sin\alpha+\mathrm{i}\cos\alpha)\quad\cdots(*)$ Note: Any complex number u must be of the form $r[\cos\beta+\mathrm{i}\sin\beta]$ where $r>0$ to conclude $\lvert u\rvert=r$ and $\arg(u)=\beta$. $\therefore$ We <u>cannot conclude arg(z) = α</u> from (*) We can re-express z as $9\left[\cos\left(\dfrac{\pi}{2}-\alpha\right)+\mathrm{i}\sin\left(\dfrac{\pi}{2}-\alpha\right)\right]$ $\Rightarrow \lvert z\rvert=9$ and $\arg(z)=\dfrac{\pi}{2}-\alpha$ Since $\arg(z)=\dfrac{1}{6}\pi,\ \alpha=\dfrac{\pi}{3}$ *(it is not necessary to work out value of α for this question)* Note that (1) z^* is to be expressed in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $0\le\theta<2\pi$. (2) arg(z*) = –arg (z) (3) $9\mathrm{e}^{-\mathrm{i}\frac{\pi}{6}}\neq 9\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{6}+\pi\right)}$, the adjustment needs to be done in multiples of 2π.

6 **(i)** Show that $\dfrac{1-\left(1-x^2\right)^{\frac{1}{2}}}{1+\left(1-x^2\right)^{\frac{1}{2}}}=\dfrac{1}{x^2}\left[1-\left(1-x^2\right)^{\frac{1}{2}}\right]^2$ for $0<x<\dfrac{1}{2}$. [1]

(ii) Hence, show that the expansion of $\sqrt{\dfrac{1-\left(1-x^2\right)^{\frac{1}{2}}}{1+\left(1-x^2\right)^{\frac{1}{2}}}}$, up to and including the term in x^3, is $\dfrac{1}{2}x+\dfrac{1}{8}x^3$, for $0<x<\dfrac{1}{2}$. [2]

(iii) By making a suitable substitution for x, deduce that

$$\tan\theta\approx\frac{1}{2}\sin 2\theta+\frac{1}{8}\sin^3 2\theta \quad \text{for } 0<\theta<\frac{\pi}{12}.$$ [3]

(iv) Using standard series from the List of Formulae (MF26), write down the series expansion for $\sin 2\theta$, up to and including the term in θ^3. [1]

(v) Using the results in parts **(iii)** and **(iv)**, find the series expansion for $\tan\theta$, up to and including the term in θ^3. [2]

	Solution	Comments
6(i) **[1]**	$\dfrac{1-\left(1-x^2\right)^{\frac{1}{2}}}{1+\left(1-x^2\right)^{\frac{1}{2}}}=\dfrac{1-\left(1-x^2\right)^{\frac{1}{2}}}{1+\left(1-x^2\right)^{\frac{1}{2}}}\times\dfrac{1-\left(1-x^2\right)^{\frac{1}{2}}}{1-\left(1-x^2\right)^{\frac{1}{2}}}$ $=\dfrac{\left[1-\left(1-x^2\right)^{\frac{1}{2}}\right]^2}{1-\left(1-x^2\right)}$ $=\dfrac{\left[1-\left(1-x^2\right)^{\frac{1}{2}}\right]^2}{x^2}$ (shown)	Do be reminded to show working for "show" question.

(ii) **[2]**	$\sqrt{\dfrac{1-\left(1-x^2\right)^{\frac{1}{2}}}{1+\left(1-x^2\right)^{\frac{1}{2}}}}=\sqrt{\dfrac{\left(1-\left(1-x^2\right)^{\frac{1}{2}}\right)^2}{x^2}}$ for $0<x<\dfrac{1}{2}$ $=\dfrac{1-\left(1-x^2\right)^{\frac{1}{2}}}{x}$ Since $\left(1-x^2\right)^{\frac{1}{2}}=1+\dfrac{1}{2}\left(-x^2\right)+\dfrac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}\left(-x^2\right)^2+\ldots$ $=1-\dfrac{1}{2}x^2-\dfrac{1}{8}x^4+\ldots,$ $\sqrt{\dfrac{1-\left(1-x^2\right)^{\frac{1}{2}}}{1+\left(1-x^2\right)^{\frac{1}{2}}}}=\dfrac{1-\left(1-\frac{1}{2}x^2-\frac{1}{8}x^4+\ldots\right)}{x}$ $=\dfrac{\frac{1}{2}x^2+\frac{1}{8}x^4+\ldots}{x}$ $=\dfrac{1}{2}x+\dfrac{1}{8}x^3+\ldots$ (shown)	Again, this is a "show" question. Do provide sufficient working, in particular, show the expansion of $\left(1-x^2\right)^{\frac{1}{2}}$ clearly.
(iii) **[3]**	Substitute x as $\sin 2\theta$ where $0<\theta<\dfrac{\pi}{12}$ into part **(ii)** $\sqrt{\dfrac{1-\left(1-x^2\right)^{\frac{1}{2}}}{1+\left(1-x^2\right)^{\frac{1}{2}}}}=\dfrac{1}{2}x+\dfrac{1}{8}x^3+\ldots$ $LHS=\sqrt{\dfrac{1-\left(1-x^2\right)^{\frac{1}{2}}}{1+\left(1-x^2\right)^{\frac{1}{2}}}}=\sqrt{\dfrac{1-\left(1-\sin^2 2\theta\right)^{\frac{1}{2}}}{1+\left(1-\sin^2 2\theta\right)^{\frac{1}{2}}}}$ $=\sqrt{\dfrac{1-\left(\cos^2 2\theta\right)^{\frac{1}{2}}}{1+\left(\cos^2 2\theta\right)^{\frac{1}{2}}}}$ $=\sqrt{\dfrac{1-\cos 2\theta}{1+\cos 2\theta}}$ $=\sqrt{\dfrac{1-\left(1-2\sin^2\theta\right)}{1+\left(2\cos^2\theta-1\right)}}$ $=\sqrt{\dfrac{2\sin^2\theta}{2\cos^2\theta}}=\tan\theta$ and $RHS=\dfrac{1}{2}\sin 2\theta+\dfrac{1}{8}\sin^3 2\theta$ $\therefore \tan\theta\approx\dfrac{1}{2}\sin 2\theta+\dfrac{1}{8}\sin^3 2\theta$ (Shown)	This is not a small angle approximation question. It is a trigo identity problem where we need to show that LHS is $\tan\theta$ after substituting x as $\sin 2\theta$.

	Alternatively, $$LHS=\sqrt{\frac{1-\left(1-x^2\right)^{\frac{1}{2}}}{1+\left(1-x^2\right)^{\frac{1}{2}}}}$$ $$=\frac{1-\left(1-x^2\right)^{\frac{1}{2}}}{x}$$ $$=\frac{1-\left(1-\sin^2 2\theta\right)^{\frac{1}{2}}}{\sin 2\theta}$$ $$=\frac{1-\cos 2\theta}{\sin 2\theta}$$ $$=\frac{1-\left(1-2\sin^2\theta\right)}{2\sin\theta\cos\theta}$$ $$=\frac{2\sin^2\theta}{2\sin\theta\cos\theta}$$ $$=\tan\theta$$	
(iv) **[1]**	From MF26, $$\sin(2\theta)=(2\theta)-\frac{(2\theta)^3}{3!}+\ldots$$ $$=2\theta-\frac{8}{6}\theta^3+\ldots$$ $$=2\theta-\frac{4}{3}\theta^3+\ldots$$	Do copy the formula from MF26 carefully.
(v) **[2]**	From (iii), $$\tan\theta\approx\frac{1}{2}\sin(2\theta)+\frac{1}{8}\sin^3(2\theta)$$ $$=\frac{1}{2}\left(2\theta-\frac{4}{3}\theta^3\right)+\frac{1}{8}\left(2\theta-\frac{4}{3}\theta^3\right)^3+\ldots$$ $$=\theta-\frac{2}{3}\theta^3+\frac{1}{8}\left(8\theta^3+\ldots\right)$$ $$=\theta+\frac{1}{3}\theta^3+\ldots$$	

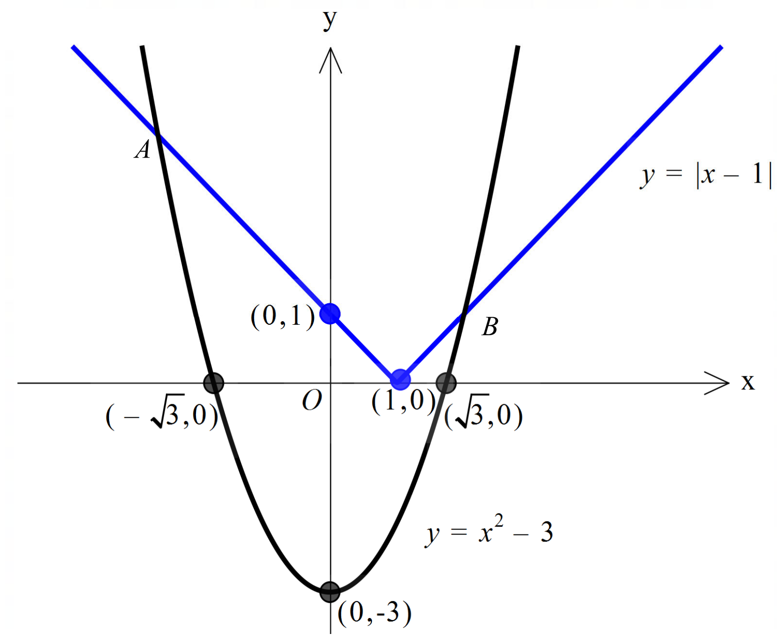
7 **(a)** Sketch on the same diagram the graphs of $y=|x-1|$ and $y=x^2-3$.

Hence solve the inequality $|x-1|<x^2-3$. [3]

(b) A curve C has equation $y=\dfrac{2x^2+x+a}{x}$.

(i) For $a>1$, sketch C, labelling clearly the equations of the asymptotes and the coordinates of the stationary points in terms of a where appropriate. [4]

(ii) Given that the solution set of the inequality $\dfrac{2x^2+x+a}{x}\le 0$ is $(-\infty,0)$, deduce the set of values of a. [2]

	Solution	**Comments**
7 (a) [3]	 From GC, x coordinates of A and B are -2.5616 (5sf) and 2 respectively. $\therefore$ For $\lvert x-1\rvert<x^2-3,\quad x<-2.56$ or $x>2$	Students should indicate all critical features which in this case are x and y intercepts. GC could be utilized here. Question was generally well done.
(bi) [4]	$y=\dfrac{2x^2+x+a}{x}=2x+1+\dfrac{a}{x}$ $\Rightarrow$Asymptotes are $y=2x+1$ and $x=0$.	Most students were able to identify the asymptotes and shape of rational function.

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$\frac{\mathrm{d}y}{\mathrm{d}x} = 2 - \frac{a}{x^2}$

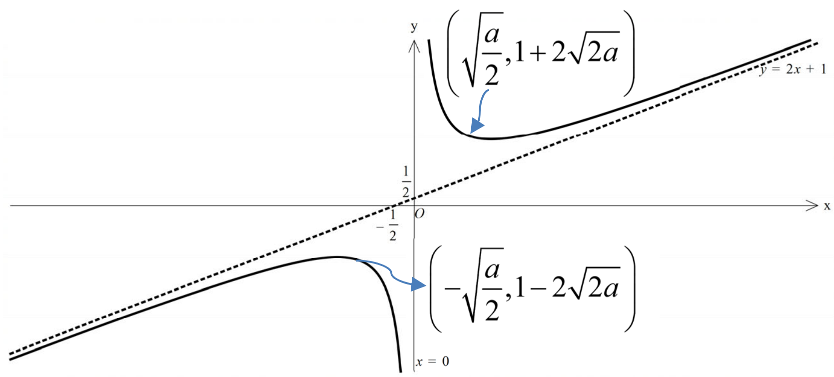
$\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Leftrightarrow 2 - \frac{a}{x^2} = 0 \qquad \Leftrightarrow x^2 = \frac{a}{2}$

Since $a > 1 > 0$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ has 2 distinct solutions, hence the graph has 2 distinct stationary points.

When $x = \pm\sqrt{\frac{a}{2}}$, $y = 2\left(\pm\sqrt{\frac{a}{2}}\right) + 1 \pm \frac{a}{\sqrt{\frac{a}{2}}} = 1 \pm 2\sqrt{2a}$

Coordinates of stationary points are

$\left(\sqrt{\frac{a}{2}}, 1+2\sqrt{2a}\right)$ and $\left(-\sqrt{\frac{a}{2}}, 1-2\sqrt{2a}\right)$.



</td>
<td>There were slips in finding the y-coordinates of the stationary points.</td>
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<td>(bii)
[2]</td>
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For solution set of $\frac{2x^2+x+a}{x} \le 0$ to be $(-\infty, 0)$,

$2x^2 + x + a \ge 0$ for all $x \in (-\infty, 0)$.

Require $2x^2 + x + a = 2\left(x + \frac{1}{4}\right)^2 + a - \frac{1}{8} \ge 0$

$\Rightarrow a \ge \frac{1}{8}$

Alternatively, using graph we require $1 - 2\sqrt{2a} \le 0$

$\Rightarrow \sqrt{a} \ge \frac{1}{2\sqrt{2}} \Rightarrow a \ge \frac{1}{8}$

Set of values of $a = \left[\frac{1}{8}, \infty\right)$.

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<td>This part proved challenging for many students.

Among those who applied the correct method a number did not consider case for equality.</td>
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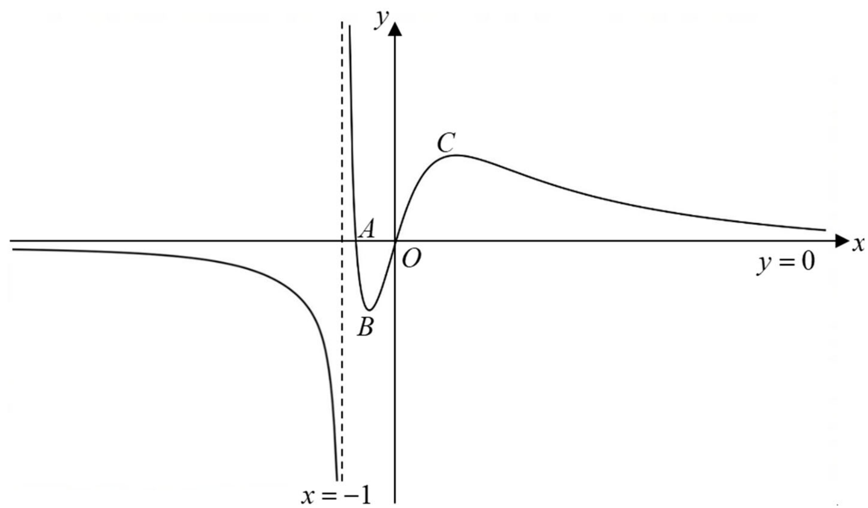


The diagram shows the curve of $y = \mathrm{f}(x)$, where $x = -1$ and $y = 0$ are asymptotes. The curve crosses the x-axis at the point A and the origin, and has turning points at B and C. The coordinates of A, B and C are $\left(-\frac{3}{4}, 0\right), \left(-\frac{1}{2}, -\frac{3}{2}\right)$ and $(1, 2)$ respectively. It is given that $\mathrm{f}'(0) = 2$ is the maximum gradient of the curve.

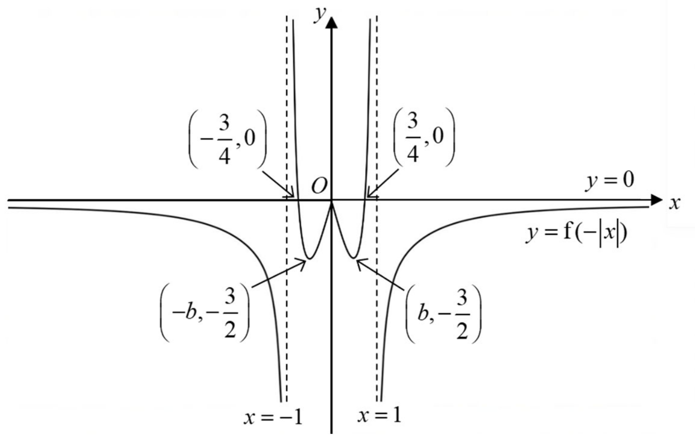
(a) Describe fully a sequence of two transformations which would transform the curve of $y = \mathrm{f}(x)$ onto the curve $y = \mathrm{f}(2x-1)$, and state the coordinates of the point on $y = \mathrm{f}(2x-1)$ corresponding to A. [3]

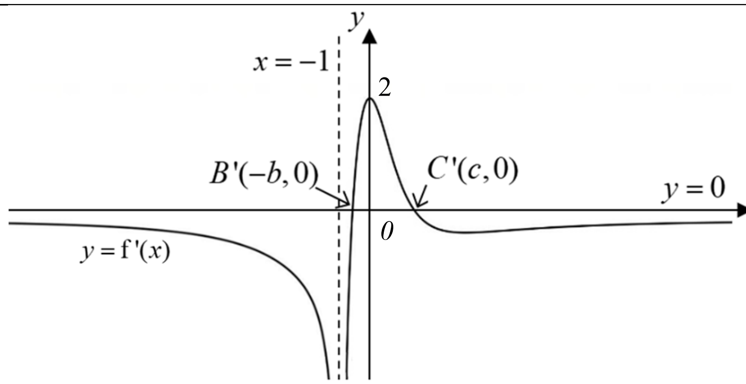
By showing clearly the equations of asymptotes and the coordinates of the points where the curve crosses the axes and turning points where appropriate, sketch, on separate diagrams, the curves of

(b) $y = \mathrm{f}(-|x|)$, [3]

(c) $y = \mathrm{f}'(x)$. [3]

	Solution	Comments
8(a) **[3]**	To obtain $y = \mathrm{f}(2x-1)$: Translate $y = \mathrm{f}(x)$ by 1 unit in the positive x-direction, then scale by factor ½ parallel to the x-axis. OR Scale $y = \mathrm{f}(x)$ by factor ½ parallel to the x-axis, then translate ½ unit in the positive x-direction.	Cambridge markers have commented that mathematical terms should be used to describe the transformations as well as the directions involved. Descriptions such as ‘along the x-axis’,

	Coordinates of point on $y=\mathrm{f}\left(2x-1\right)$ corresponding to A is $\left(\frac{1}{8},0\right)$ **<u>Note:</u>** For students who choose to do the scaling first, you will first obtain the graph of $y=\mathrm{f}\left(2x\right)$. To perform correctly the subsequent translation, you can first replace x by $x-c$ if you are not sure. Then the resultant graph is that of $y=\mathrm{f}\left(2(x-c)\right)=\mathrm{f}\left(2x-1\right)\Rightarrow c=\frac{1}{2}$. Hence you should translate by ½ unit in the positive x direction, and not 1 (which is a common mistake).	‘in the x-axis’ or ‘on the x-axis’ are NOT the same as in the x direction. The mathematical terms to use to describe transformations in the syllabus are **translate, stretch/scale** or **reflect**. When describing a **stretch/scale**, it is always accompanied by a **stretch/scale factor <u>parallel</u>** to an axis of choice.
(b) **[3]**	$y=\mathrm{f}\left(x\right)\xrightarrow{\text{replace } x \text{ by } -x} y=\mathrm{f}\left(-x\right)\xrightarrow{\text{replace } x \text{ by } \lvert x\rvert} y=\mathrm{f}\left(-\lvert x\rvert\right)$  In the above sketch, $b=\frac{1}{2}$.	Alternative: $y=\mathrm{f}(-\lvert x\rvert)$ $=\begin{cases}\mathrm{f}(-x) & \text{if } x\geq 0\\ \mathrm{f}(x) & \text{if } x<0\end{cases}$ Hence to obtain the desired graph, we keep the portion of the graph for $x<0$, and for positive x, we reflect the portion where $x<0$ about the y-axis. All coordinates of the reflected points need to be clearly indicated.

(c) **[3]**	 In the above sketch, $b=\frac{1}{2}, c=1$.	Students should take note that when a graph has asymptotes, the sketch should be shown to approach them. The 2 x-intercepts are not B and C, and should not be labelled as them as it would be incorrect. Do also take note that the graph should be smooth, except for the discontinuity at $x=-1$, where it is undefined. A very common mistake is omitting the information '$\mathrm{f}'(0)=2$ is the maximum gradient of the curve' given in the question. Thus the maximum point on the graph of $y=\mathrm{f}'(x)$ should be (0, 2).

9 The cartesian equation of the plane π_1 is given by $4x-5y+z=5$.

(i) Find the perpendicular distance from the origin O to π_1. [1]

(ii) Find the acute angle between π_1 and the y-axis. [2]

The cartesian equation of the plane π_2 is given by $2x+5y+3z=5$.

(iii) Find a vector equation of l, the line of intersection of π_1 and π_2. [2]

The cartesian equation of the plane π_3 is given by $3x-3y+z=0$.

(iv) Describe the geometrical relationship between l and π_3, justifying your answer. [3]

(v) Find an equation of the plane p which contains the point $(3,-5,1)$ and is perpendicular to π_1, π_2 and π_3. [2]

	Solution	**Comments**
9(i) [1]	$\pi_1: 4x-5y+z=5 \Leftrightarrow \mathbf{r}.\begin{pmatrix}4\\-5\\1\end{pmatrix}=5 \Leftrightarrow \mathbf{r}.\frac{1}{\sqrt{42}}\begin{pmatrix}4\\-5\\1\end{pmatrix}=\frac{5}{\sqrt{42}}$ The perpendicular distance from the origin O to π_1 is $\frac{5}{\sqrt{42}}$.	Many students made the same mistake of taking perpendicular distance as 5 units or $\sqrt{42}$ units instead of $\frac{5}{\sqrt{42}}$.
(ii) [2]	The acute angle between π_1 and the y-axis $=\sin^{-1}\frac{\left\|\begin{pmatrix}4\\-5\\1\end{pmatrix}.\begin{pmatrix}0\\1\\0\end{pmatrix}\right\|}{\sqrt{4^2+(-5)^2+1^2}}=50.5°\ \text{(1d.p.)}$ Alternatively, it can be found by $90^\circ-\cos^{-1}\frac{\left\|\begin{pmatrix}4\\-5\\1\end{pmatrix}.\begin{pmatrix}0\\1\\0\end{pmatrix}\right\|}{\sqrt{4^2+(-5)^2+1^2}}$	Many students found the angle with vertical using cosine inverse but did not proceed to subtract it from $90°$ to get the acute angle. Another common mistake is the wrong interpretation of the vector parallel to y-axis. Some students who found their angles in radian wrote degree instead.

(iii) **[2]**	Solve $\begin{cases} 4x-5y+z=5 \\ 2x+5y+3z=5 \end{cases}$ using GC to get $\begin{cases} x=\frac{5}{3}-\frac{2}{3}z \\ y=\frac{1}{3}-\frac{1}{3}z \\ z=z \end{cases}$ l: $\mathbf{r}=\begin{pmatrix} \frac{5}{3}-\frac{2}{3}z \\ \frac{1}{3}-\frac{1}{3}z \\ z \end{pmatrix}=\begin{pmatrix} \frac{5}{3} \\ \frac{1}{3} \\ 0 \end{pmatrix}+z\begin{pmatrix} -\frac{2}{3} \\ -\frac{1}{3} \\ 1 \end{pmatrix}=\begin{pmatrix} \frac{5}{3} \\ \frac{1}{3} \\ 0 \end{pmatrix}+\mu\begin{pmatrix} -2 \\ -1 \\ 3 \end{pmatrix}$, where $\mu\in\mathbb{R}$ Alternative Method: $\begin{pmatrix} 4 \\ -5 \\ 1 \end{pmatrix}\times\begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix}=\begin{pmatrix} -15-5 \\ 2-12 \\ 20+10 \end{pmatrix}=\begin{pmatrix} -20 \\ -10 \\ 30 \end{pmatrix}//\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$ By observation, $(1,0,1)$ lies on both π_1 and π_2, therefore l: $\mathbf{r}=\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}+\lambda\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$, where $\lambda\in\mathbb{R}$.	This part of the question is generally well done with the utilization of the GC. Some common mistakes include translating the wrong position/direction vector from the GC. Students who did not present their vector equation with "**r** =" will not get the full credit as it is an incomplete equation.
(iv) **[3]**	l: $\mathbf{r}=\begin{pmatrix} \frac{5}{3} \\ \frac{1}{3} \\ 0 \end{pmatrix}+\mu\begin{pmatrix} -2 \\ -1 \\ 3 \end{pmatrix}$, where $\mu\in\mathbb{R}$ $\pi_3: 3x-3y+z=0 \Leftrightarrow \mathbf{r}.\begin{pmatrix} 3 \\ -3 \\ 1 \end{pmatrix}=0$ $\begin{pmatrix} -2 \\ -1 \\ 3 \end{pmatrix}.\begin{pmatrix} 3 \\ -3 \\ 1 \end{pmatrix}=-6+3+3=0$ *l* is perpendicular to the normal of π_3, therefore *l* is parallel to π_3.	Students were generally able to carry out the dot product of the position/direction vector of the line with the plane. A common misconception is concluding that the plane is perpendicular to the line, however it is the **normal** of the plane that is perpendicular to the line, which makes the plane parallel to the line.

	$\begin{pmatrix} \frac{5}{3} \\ \frac{1}{3} \\ 0 \end{pmatrix} . \begin{pmatrix} 3 \\ -3 \\ 1 \end{pmatrix} = 5-1+0=4 \neq 0$, so we can further conclude that l does not lie on π_3. l is parallel to π_3 but does not lie on π_3. <u>Alternative Method:</u> $\left[\begin{pmatrix} \frac{5}{3} \\ \frac{1}{3} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ -1 \\ 3 \end{pmatrix} \right] . \begin{pmatrix} 3 \\ -3 \\ 1 \end{pmatrix}$ $= \begin{pmatrix} \frac{5}{3} \\ \frac{1}{3} \\ 0 \end{pmatrix} . \begin{pmatrix} 3 \\ -3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ -1 \\ 3 \end{pmatrix} . \begin{pmatrix} 3 \\ -3 \\ 1 \end{pmatrix}$ $= (5-1) + \lambda(-6+3+3)$ $= 4 \neq 0 \quad$ for all $\lambda \in \mathbb{R}$ Each point on l does not lie on π_3 $\Rightarrow l$ is parallel to π_3 and does not lie on π_3	Students have to justify clearly and substantiate with working on why the line does not lie on the plane.
(v) **[2]**	Since $\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$ is parallel to π_1, π_2 and π_3, it can be the normal of p. $\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ -5 \\ 1 \end{pmatrix} = 6-5-3=-2$, therefore an equation of p is $\mathbf{r} \bullet \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} = -2$	This part of the question proves challenging as many were unable to identify the correct normal vector of p.

	<u>Alternative Method</u> Since p is perpendicular to π_1, π_2 and π_3, p is parallel to the normal of these planes. $\therefore \mathbf{r}=\begin{pmatrix}3\\-5\\1\end{pmatrix}+\alpha\begin{pmatrix}4\\-5\\1\end{pmatrix}+\beta\begin{pmatrix}2\\5\\3\end{pmatrix}, \quad \alpha,\beta\in\mathbb{R}$ or $\mathbf{r}=\begin{pmatrix}3\\-5\\1\end{pmatrix}+\alpha'\begin{pmatrix}4\\-5\\1\end{pmatrix}+\beta'\begin{pmatrix}3\\-3\\1\end{pmatrix}, \quad \alpha',\beta'\in\mathbb{R}$	Take note that if the final answer is given as a vector equation, $\begin{pmatrix}2\\1\\-3\end{pmatrix}$ cannot be one of the direction vectors as it is the normal vector of p.

10 The function f is defined by

$$\mathrm{f}: x \mapsto \ln\left(x^2-6x+10\right)-5, \quad x \in \mathbb{R}, \quad x \leq 3.$$

(i) Find $\mathrm{f}^{-1}(x)$ and state the domain of f^{-1}. [3]

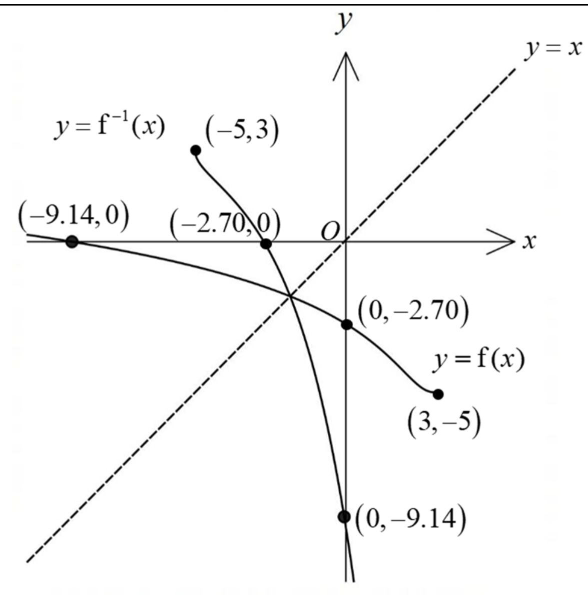
(ii) Sketch on the same diagram the graphs of $y=\mathrm{f}(x)$ and $y=\mathrm{f}^{-1}(x)$, giving the coordinates of the axial intercepts. [3]

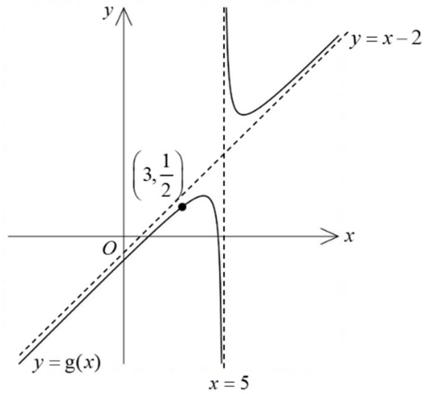
The function g is defined by

$$\mathrm{g}: x \mapsto \frac{1}{x-5}+x-2, \quad x \in \mathbb{R}, \; x \neq 5.$$

(iii) Find the range of gf^{-1}. [2]

(iv) Solve the inequality $\mathrm{f}(x)<\mathrm{g}(x)$. [2]

	Solution	Comments
10(i) **[3]**	Let $y=\ln\left(x^2-6x+10\right)-5$ $e^{y+5}=\left(x-3\right)^2+1$ $x=3\pm\sqrt{e^{y+5}-1}$ Since $x \leq 3$, $\mathrm{f}^{-1}\left(x\right)=3-\sqrt{e^{x+5}-1}$ $D_{\mathrm{f}^{-1}}=R_{\mathrm{f}}=\left[-5,\infty\right)$	Students need to complete the square or use quadratic formula to find the inverse function. Students need to justify the choice of negative square root. GC could be utilized to find the range of f, which is the domain of f^{-1}.
(ii) **[3]**		Students should indicate all critical features which in this case are x and y intercepts and end points The intercepts should be labelled in the coordinate form as requested by the question. The two graphs should be symmetrical about the line $y=x$, meaning each pair of corresponding points should be equidistant to the line $y=x$. The graph should be of appropriate proportion, and the same scale should be used for both axes.

<table>
<tr>
<td>(iii)
[2]</td>
<td>$R_{f^{-1}} = D_f = (-\infty, 3]$


Using the graph of g,
$D_{f^{-1}} \xrightarrow{f^{-1}} (-\infty, 3] \xrightarrow{g} \left(-\infty, \frac{1}{2}\right]$
$R_{gf^{-1}} = \left(-\infty, \frac{1}{2}\right]$</td>
<td>Students need to map the correct domain into function f^{-1} followed by function g in the correct order.

GC could be utilized here to use the graph of g to work out the correct mapping.

Please take note that the solution set is inclusive of the end value $\frac{1}{2}$.

It is not necessary to find the rule of gf^{-1} in this question.</td>
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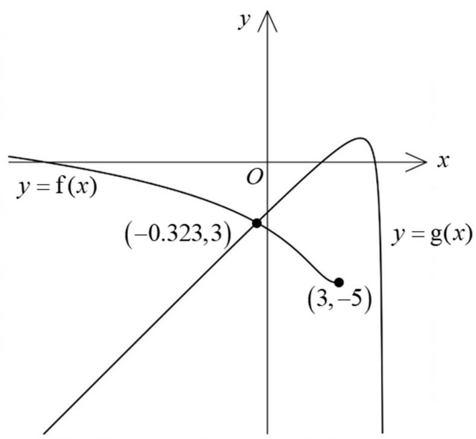
<tr>
<td>(iv)
[2]</td>
<td>From GC, when $f(x) = g(x)$, $x = -0.323$


$\therefore$ For $f(x) < g(x)$, $-0.323 < x \le 3$</td>
<td>GC should be utilized here to find the intersection point of f and g.

Students need to consider the domain of function f here, which is $(-\infty, 3]$. Therefore 3 is the upper limit of the solution set.</td>
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11 [It is given that a right circular cone of radius r and height h has total surface area $\pi r^2 + \pi r\sqrt{r^2 + h^2}$ and volume $\frac{1}{3}\pi r^2 h$.]

An ice-cream company plans to launch a new product called Tasty Cone into the market. Each Tasty Cone is in the shape of a right circular cone with a base radius r cm and height h cm as shown in Fig. 1.

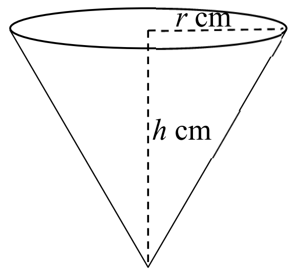


Fig. 1

(i) The volume of each Tasty Cone is set by the company to be 100 ml and the total surface area, S cm^2, should be as small as possible to reduce the cost of packaging. It is given that $1 \text{ ml} = 1 \text{ cm}^3$.

Show that $S = \pi r^2 + \frac{1}{r}\sqrt{\pi^2 r^6 + 90000}$. [2]

(ii) Sketch the graph of S for $r > 0$. Hence write down the value of r and find the corresponding value of h which will give the smallest S, giving your answers correct to 3 significant figures. [3]

A special edition of the Tasty Cone will include a mystery flavour in the shape of a sphere inscribed in the cone as shown in Fig. 2. The sphere has centre O and radius 2 cm. The cross-section of the cone is shown in Fig. 3, where A is a point of contact of the sphere with the cone.

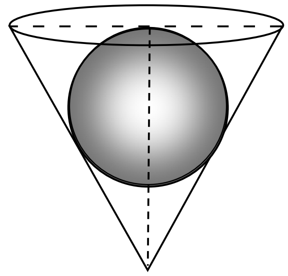


Fig. 2

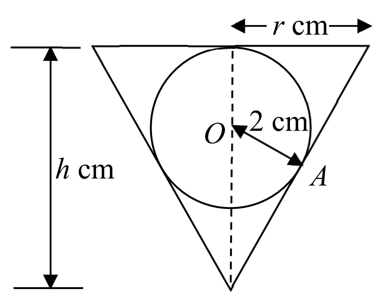
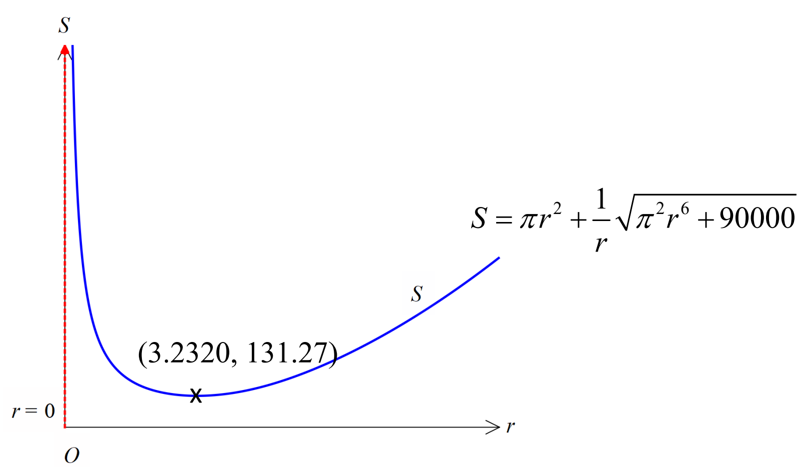
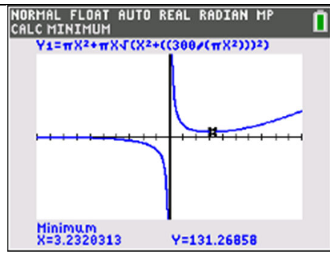
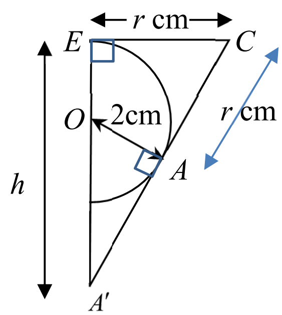


Fig. 3

(iii) Show that $r^2 = \dfrac{4h}{h-4}$. [2]

(iv) Use differentiation to find the exact values of h and r which will give the smallest volume of the cone in the special edition. [5]

	Solution	**Comments**
11 (i) [2]	$\frac{1}{3}\pi r^2 h = 100 \Rightarrow h = \frac{300}{\pi r^2}$ $S = \pi r^2 + \pi r\sqrt{r^2+h^2}$ $= \pi r^2 + \pi r\sqrt{r^2 + \left(\frac{300}{\pi r^2}\right)^2}$ $= \pi r^2 + \pi r\sqrt{\frac{\pi^2 r^6 + 300^2}{\left(\pi r^2\right)^2}}$ $\left(\text{Note: } \sqrt{\frac{1}{\left(\pi r^2\right)^2}} = \frac{1}{\pi r^2} \text{ as } \pi r^2 > 0\right)$ $= \pi r^2 + \frac{1}{r}\sqrt{\pi^2 r^6 + 90000}$ (Shown)	Clear working in dealing with surds and indices needs to be presented. In general, consciously put in more steps for a 'Show' question.
(ii) [3]	 By GC, the minimum point happens at $r \approx 3.2320 = 3.23$ cm (3 s.f.) The corresponding value of h is $\frac{300}{\pi(3.2320)^2} = 9.14$ cm (3 s.f.)	You should plot S against r, and present only the graph for $r > 0$. Note the vertical asymptote $r = 0$. With the existence of a *smallest* S, there should be a minimum point on the curve. Use GC to calculate the value of r at minimum S. For h accurate to 3 sf, we need at least 5 sf for r in the intermediate working.

	 (under ZoomFit)	
(iii) **[2]**	**Method 1 : Similar triangles** $\Delta A'AO$ and $\Delta A'EC$ are similar triangles. $\frac{r}{2}=\frac{\sqrt{r^2+h^2}}{h-2}=\frac{h}{\sqrt{(h-2)^2-2^2}}$ $\left(\text{or} =\frac{h}{\sqrt{r^2+h^2}-r}\right)$ $\Rightarrow r(h-2)=2\sqrt{r^2+h^2}$ $\Rightarrow r^2(h-2)^2=4(r^2+h^2)$ $\Rightarrow r^2h^2-4r^2h+4r^2=4r^2+4h^2$ $\Rightarrow r^2(h-4)=4h$ $\Rightarrow r^2=\frac{4h}{h-4}$ (Shown)  **Method 2 : Area of triangle** Consider Area of $\Delta OA'C$ (or Area of $\Delta A'EC$). $\frac{1}{2}(r)(h-2)=\frac{1}{2}(2)\sqrt{r^2+h^2}$	Note that $CD=r$ by symmetry (tangent lines from external point to circumference of circle) A more popular approach: $\frac{r}{2}=\frac{h}{\sqrt{(h-2)^2-2^2}}$ $\frac{r^2}{4}=\frac{h^2}{h^2-4h}$ $=\frac{h}{h-4}$
(iv) **[5]**	Let V cm^3 denote the volume of the cone. $V=\frac{1}{3}\pi\left(\frac{4h}{h-4}\right)h=\frac{4\pi h^2}{3(h-4)}$ $\frac{dV}{dh}=\frac{4\pi}{3}\left(\frac{(h-4)(2h)-h^2}{(h-4)^2}\right)=\frac{4\pi h(h-8)}{3(h-4)^2}$ For stationary values of V, let $\frac{dV}{dh}=0$ $\Rightarrow\frac{dV}{dh}=\frac{4\pi h(h-8)}{3(h-4)^2}=0$ $\Rightarrow h=0$ (rej) or $h=8$	With $V=\frac{1}{3}\pi r^2h$, it is conceptually wrong to conclude $\frac{dV}{dh}=\frac{1}{3}\pi r^2$ or $\frac{dV}{dr}=\frac{2}{3}\pi rh$, since r and h are both variables. $h>0$ for the cone to exist physically. In fact, for the cone to contain the sphere, $h>4$.

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<u>1st derivative test</u>

With $h>0$ and $(h-4)^2>0$ for h near 8, the sign of $\frac{\mathrm{d}V}{\mathrm{d}h}$ is determined by the factor $(h-8)$.

h	8^-	8	8^+
$(h-8)$	-ve	0	+ve
$\frac{\mathrm{d}V}{\mathrm{d}h}$	-ve	0	+ve

<u>2nd derivative test</u>

$$\left.\frac{\mathrm{d}^2V}{\mathrm{d}h^2}\right|_{h=8}=2.09>0$$

Thus V is minimum when $h=8$.

$$r^2=\frac{4(8)}{8-4}=8\Rightarrow r=2\sqrt{2}$$

$h=8$ and $r=2\sqrt{2}$ will give the smallest volume of the cone in the special edition.

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The first derivative test is to determine the sign of the gradient of the curve $V=\mathrm{f}(h)$ before and after $h=8$.

The value 2.09 needs to be stated to substantiate why $\left.\frac{\mathrm{d}^2V}{\mathrm{d}h^2}\right|_{h=8}$ is positive.

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12 On 1 Jan 2023, Sam takes an interest-free study loan of \$28 000 from his parents for his university fees. He starts to repay \$$m$ to his parents on 1 Feb 2023 and progressively increases \$50 in his repayment amount at the start of each subsequent month.

(i) Given that $m = 500$, on which date will Sam finish repaying his parents? [4]

(ii) If Sam wants to finish repaying his parents by the end of 2024, find the minimum value of m, giving your answer to the nearest dollars. [3]

On 1 Jan 2023, Sam's friend, Joshua, takes a study loan of \$28 000 from UAB Bank. The bank charges an interest rate of 0.2% per month at the end of each month from the start of the loan period.

(iii) Given that he repays \$500 to the bank on the first day of every month beginning from 1 Feb 2023 onwards, on which date will Joshua finish repaying his study loan? [5]

	Solution	**Comments**
12(i) **[4]**	$S_n = \frac{n}{2}(2(500)+(n-1)(50))$ $= 500n + \frac{n}{2}(50)(n-1)$ To finish repaying his parents, $S_n \geq 28000$ Using GC, when $n = 25$, $S_n = 27500 < 28000$ when $n = 26$, $S_n = 29250 > 28000$ OR $S_n \geq 28000 \Leftrightarrow n^2 + 19n - 1120 \geq 0$ $n \leq -44.289$ or $n \geq 25.289$ Hence min $n = 26$ He will finish repaying his parents on the 26$^{\text{th}}$ month, i.e. on 1 Mar 2025.	Students should approach this question using inequalities. A handful of students made mistakes in the counting of the date.
(ii) **[3]**	To finish by the end of 2024, i.e. the 23$^{\text{rd}}$ month, $n = 23$ $S_{23} = 23m + \frac{23}{2}(50)(23-1)$ $S_{23} \geq 28000$ $23m + \frac{23}{2}(50)(23-1) \geq 28000$ $m \geq 667.3913$ Minimum value of m is \$667 or \$668 (nearest dollars)	The common mistake is getting to the correct value of n. Again, this question should be solved using inequalities.

(iii)
[5]

n	Month	Amount owed at the START of the month ($)	Amount owed at the END of the month ($)
1	Jan 2023	28 000	1.002(28000)
2	Feb 2023	1.002(28000) – 500	1.002[1.002(28000) – 500] $=1.002^2(28000)–1.002(500)$
3	Mar 2023	$1.002^2(28000)$ –1.002(500) – 500	$1.002[1.002^2(28000)$ –1.002(500) – 500] $= 1.002^3(28000)$ $– 1.002^2(500) – 1.002(500)$
4	Apr 2023	$1.002^3(28000)$ $– 1.002^2(500)$ – 1.002(500) – 500	$1.002[1.002^3(28000)$ $– 1.002^2(500) – 1.002(500)$ – 500] $= 1.002^4(28000)$ $– 1.002^3(500)$ $–1.002^2(500)$ – 1.002(500)
…		…	…
n (for $n\geq3$)		$1.002^{n–1}(28000)$ $– 1.002^{n–2}(500)$ $– 1.002^{n–3}(500)$ – … – 1.002(500) – 500	

For Joshua to finish repaying, the amount owed at the beginning of the nth month,

$$1.002^{n-1}(28000)-500\left(1.002^{n-2}+1.002^{n-3}+\cdots+1.002+1\right)\leq 0$$

$$1.002^{n-1}(28000)-500\left(\frac{1.002^{n-1}-1}{1.002-1}\right)\leq 0$$

Let $\mathbf{f}(n)=1.002^{n-1}(28000)-500\left(\frac{1.002^{n-1}-1}{1.002-1}\right)$

Using GC, when $n=60,\ \mathrm{f}(n)=225.24>0$

when $n=61,\ \mathrm{f}(n)=-274.31<0$

Hence min $n=61$.

Joshua will finish repaying the bank in 61st month, ie, on 1 Jan 2028.

<u>Alternatively</u>, from the graph of $y=\mathrm{f}(n)$, $n>60.451$. Hence min $n=61$ and Joshua will finish repaying the bank in 61st month, ie, on 1 Jan 2028.

Setting up a table would be very useful for this part of the question.

Students should be careful to be able to identify the starting value of n and its corresponding month.

Students should write the sequence for the nth month before attempting to state the sum of GP formula straightaway.

Students should note that there are a total of n–1 terms in the GP. .

Students should show either table or graphical method in obtaining n to justify their answer.