RAFFLES INSTITUTION
2022 YEAR 5 PROMOTION EXAMINATION

CANDIDATE NAME

CLASS 23

MATHEMATICS 9758

3 hours

Candidates answer on the Question Paper
Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name and class on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing calculator is expected, where appropriate.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

FOR EXAMINER'S USE

Q1	Q2	Q3	Q4	Q5	Q6	TOTAL
4	5	5	7	8	9	
Q7	**Q8**	**Q9**	**Q10**	**Q11**	**Q12**	
9	9	10	10	12	12	100

This document consists of **7** printed pages.

1 Elly runs a home bakery business and sells three different flavors of cakes. A 0.5 kg Strawberrylicious, Cheeseburst and Chocofanatic cake is priced at x, y and z dollars respectively. A 1 kg cake costs 40% more than a 0.5 kg cake of the same flavour.

Customer A bought one 0.5 kg Strawberrylicious cake, one 0.5 kg Cheeseburst cake and one 0.5 kg Chocofanatic cake. She paid a total of \$175.

Customer B bought one 1 kg Strawberrylicious cake, three 1 kg Cheeseburst cakes and two 0.5 kg Chocofanatic cakes. She paid a total of \$463.

During a 11.11 sale, Customer C bought one 1 kg Strawberrylicious cake, one 1 kg Cheeseburst cake and two 1 kg Chocofanatic cakes at 10% off the total bill. He paid a total of \$296.10 after the discount.

(i) Find the values of x, y and z. [3]

(ii) Find the total amount paid by Customer D, who bought three 0.5 kg Strawberrylicious cakes, two 0.5 kg Cheeseburst cakes and two 1 kg Chocofanatic cakes at 25% off the total bill during a 12.12 sale. [1]

2 Without using a calculator, solve the inequality $3x+1 \leq \dfrac{x^2+2}{x}$. [5]

3 The origin O and the points A, B and C lie in the same plane, where $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$.

(a) Show that if $\mathbf{c} = |\mathbf{a}|\mathbf{b} + |\mathbf{b}|\mathbf{a}$, then OC bisects the angle AOB. [3]

(b) Given that $\mathbf{a}+\mathbf{b}+\mathbf{c} = \mathbf{0}$, show that $\mathbf{a}\times\mathbf{b} = \mathbf{b}\times\mathbf{c} = \mathbf{c}\times\mathbf{a}$. [2]

4 **(i)** Find $\displaystyle\sum_{r=1}^{n} \frac{1}{(r+1)(r+3)}$, where $n \geq 3$. (There is no need to express your answer as a single algebraic fraction.) [5]

(ii) Explain why $\displaystyle\sum_{r=1}^{\infty} \frac{1}{(r+1)(r+3)}$ is a convergent series, and state the value of the sum to infinity. [2]

5 **Do not use a calculator in answering this question.**

(a) Solve the equation $z^3 - 4z^2 + 6z - 4 = 0$. [3]

(b) The complex numbers w and z are given by $-1+\mathrm{i}\sqrt{3}$ and $9\sin\alpha + 9\mathrm{i}\cos\alpha$, where $0 < \alpha < \dfrac{\pi}{2}$, respectively.

It is further given that $\arg(wz) = \dfrac{5}{6}\pi$.

Find z^* in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $0 \le \theta < 2\pi$. [5]

6 **(i)** Show that $\dfrac{1-\left(1-x^2\right)^{\frac{1}{2}}}{1+\left(1-x^2\right)^{\frac{1}{2}}} = \dfrac{1}{x^2}\left[1-\left(1-x^2\right)^{\frac{1}{2}}\right]^2$ for $0 < x < \dfrac{1}{2}$. [1]

(ii) Hence, show that the expansion of $\sqrt{\dfrac{1-\left(1-x^2\right)^{\frac{1}{2}}}{1+\left(1-x^2\right)^{\frac{1}{2}}}}$, up to and including the term in x^3, is $\dfrac{1}{2}x + \dfrac{1}{8}x^3$, for $0 < x < \dfrac{1}{2}$. [2]

(iii) By making a suitable substitution for x, deduce that

$$\tan\theta \approx \frac{1}{2}\sin 2\theta + \frac{1}{8}\sin^3 2\theta \quad \text{for } 0 < \theta < \frac{\pi}{12}.$$ [3]

(iv) Using standard series from the List of Formulae (MF26), write down the series expansion for $\sin 2\theta$, up to and including the term in θ^3. [1]

(v) Using the results in parts **(iii)** and **(iv)**, find the series expansion for $\tan\theta$, up to and including the term in θ^3. [2]

7 **(a)** Sketch on the same diagram the graphs of $y = |x-1|$ and $y = x^2 - 3$.

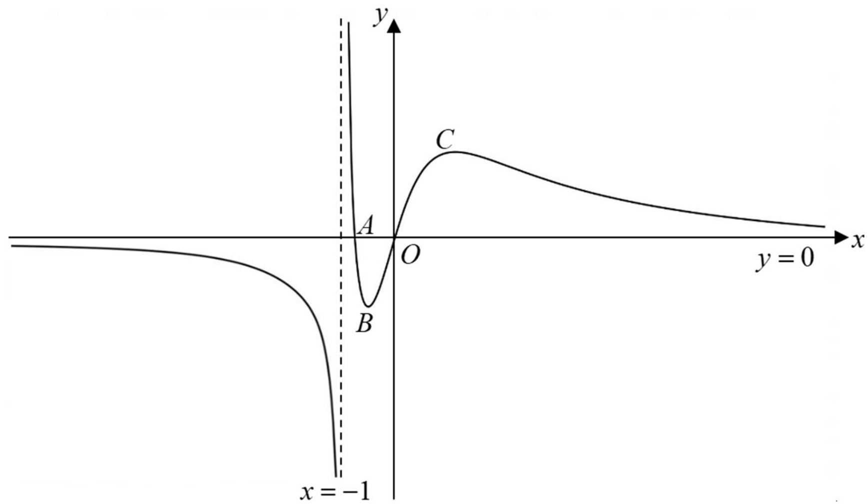
Hence solve the inequality $|x-1| < x^2 - 3$. [3]

(b) A curve C has equation $y = \dfrac{2x^2 + x + a}{x}$.

(i) For $a > 1$, sketch C**,** labelling clearly the equations of the asymptotes and the coordinates of the stationary points in terms of a where appropriate. [4]

(ii) Given that the solution set of the inequality $\dfrac{2x^2 + x + a}{x} \le 0$ is $(-\infty, 0)$, deduce the set of values of a. [2]

8



The diagram shows the curve of $y=\mathrm{f}(x)$, where $x=-1$ and $y=0$ are asymptotes. The curve crosses the x-axis at the point A and the origin, and has turning points at B and C. The coordinates of A, B and C are $\left(-\frac{3}{4},0\right)$, $\left(-\frac{1}{2},-\frac{3}{2}\right)$ and $(1,2)$ respectively. It is given that $\mathrm{f}'(0)=2$ is the maximum gradient of the curve.

(a) Describe fully a sequence of two transformations which would transform the curve of $y=\mathrm{f}(x)$ onto the curve $y=\mathrm{f}(2x-1)$, and state the coordinates of the point on $y=\mathrm{f}(2x-1)$ corresponding to A. [3]

By showing clearly the equations of asymptotes and the coordinates of the points where the curve crosses the axes and turning points where appropriate, sketch, on separate diagrams, the curves of

(b) $y=\mathrm{f}(-|x|)$, [3]

(c) $y=\mathrm{f}'(x)$. [3]

9 The cartesian equation of the plane π_1 is given by $4x-5y+z=5$.

(i) Find the perpendicular distance from the origin O to π_1. [1]

(ii) Find the acute angle between π_1 and the y-axis. [2]

The cartesian equation of the plane π_2 is given by $2x+5y+3z=5$.

(iii) Find a vector equation of l, the line of intersection of π_1 and π_2. [2]

The cartesian equation of the plane π_3 is given by $3x-3y+z=0$.

(iv) Describe the geometrical relationship between l and π_3, justifying your answer. [3]

(v) Find an equation of the plane p which contains the point $(3,-5,1)$ and is perpendicular to π_1, π_2 and π_3. [2]

10 The function f is defined by

$$\mathrm{f}: x \mapsto \ln\left(x^2-6x+10\right)-5, \quad x \in \mathbb{R}, \quad x \leq 3.$$

(i) Find $\mathrm{f}^{-1}(x)$ and state the domain of f^{-1}. [3]

(ii) Sketch on the same diagram the graphs of $y=\mathrm{f}(x)$ and $y=\mathrm{f}^{-1}(x)$, giving the coordinates of the axial intercepts. [3]

The function g is defined by

$$\mathrm{g}: x \mapsto \frac{1}{x-5}+x-2, \quad x \in \mathbb{R},\ x \neq 5.$$

(iii) Find the range of gf^{-1}. [2]

(iv) Solve the inequality $\mathrm{f}(x)<\mathrm{g}(x)$. [2]

11 [It is given that a right circular cone of radius r and height h has total surface area $\pi r^2 + \pi r\sqrt{r^2+h^2}$ and volume $\frac{1}{3}\pi r^2 h$.]

An ice-cream company plans to launch a new product called Tasty Cone into the market. Each Tasty Cone is in the shape of a right circular cone with a base radius r cm and height h cm as shown in Fig. 1.

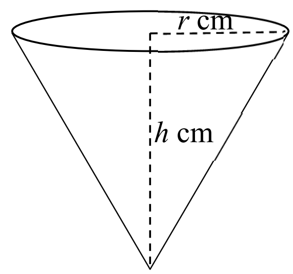


Fig. 1

(i) The volume of each Tasty Cone is set by the company to be 100 ml and the total surface area, S cm^2, should be as small as possible to reduce the cost of packaging. It is given that 1 ml $=1$ cm^3.

Show that $S = \pi r^2 + \frac{1}{r}\sqrt{\pi^2 r^6 + 90000}$. [2]

(ii) Sketch the graph of S for $r > 0$. Hence write down the value of r and find the corresponding value of h which will give the smallest S, giving your answers correct to 3 significant figures. [3]

A special edition of the Tasty Cone will include a mystery flavour in the shape of a sphere inscribed in the cone as shown in Fig. 2. The sphere has centre O and radius 2 cm. The cross-section of the cone is shown in Fig. 3, where A is a point of contact of the sphere with the cone.

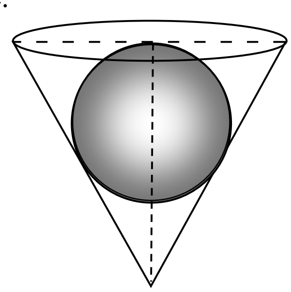


Fig. 2

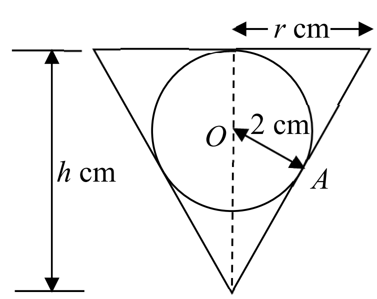


Fig. 3

(iii) Show that $r^2 = \frac{4h}{h-4}$. [2]

(iv) Use differentiation to find the exact values of h and r which will give the smallest volume of the cone in the special edition. [5]

12 On 1 Jan 2023, Sam takes an interest-free study loan of \$28 000 from his parents for his university fees. He starts to repay \$$m$ to his parents on 1 Feb 2023 and progressively increases \$50 in his repayment amount at the start of each subsequent month.

(i) Given that $m = 500$, on which date will Sam finish repaying his parents? [4]

(ii) If Sam wants to finish repaying his parents by the end of 2024, find the minimum value of m, giving your answer to the nearest dollars. [3]

On 1 Jan 2023, Sam's friend, Joshua, takes a study loan of \$28 000 from UAB Bank. The bank charges an interest rate of 0.2% per month at the end of each month from the start of the loan period.

(iii) Given that he repays \$500 to the bank on the first day of every month beginning from 1 Feb 2023 onwards, on which date will Joshua finish repaying his study loan? [5]