RAFFLES INSTITUTION
2022 Year 6 H2 Mathematics Preliminary Examination Paper 2
Solutions

1(i)	$\dfrac{d}{dx}\left(\dfrac{1}{\cos^2 x}\right)$ $=\dfrac{d}{dx}(\cos x)^{-2}$ $=-2(\cos x)^{-3}(-\sin x)$ $=\dfrac{2\sin x}{\cos^3 x}$ (i.e. k = 2 and n = 3)
(ii)	$\int_0^{\frac{\pi}{4}} \sin^2 x \sec^3 x \, dx = \int_0^{\frac{\pi}{4}} \dfrac{\sin^2 x}{\cos^3 x} \, dx$ $=\int_0^{\frac{\pi}{4}} \dfrac{2\sin x}{\cos^3 x} \cdot \dfrac{\sin x}{2} \, dx$ $=\left[\dfrac{1}{\cos^2 x} \cdot \dfrac{\sin x}{2}\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \dfrac{1}{\cos^2 x} \cdot \dfrac{\cos x}{2} \, dx$ $=\dfrac{1}{2}\left[\dfrac{\sin x}{\cos^2 x}\right]_0^{\frac{\pi}{4}} - \dfrac{1}{2}\int_0^{\frac{\pi}{4}} \sec x \, dx$ $=\dfrac{1}{2}\left[\dfrac{\sin\frac{\pi}{4}}{\cos^2\frac{\pi}{4}} - 0\right] - \dfrac{1}{2}\left[\ln(\sec x + \tan x)\right]_0^{\frac{\pi}{4}}$ $=\dfrac{1}{2}\left[\dfrac{\frac{1}{\sqrt{2}}}{\frac{1}{2}}\right] - \dfrac{1}{2}\left[\ln(\sec\frac{\pi}{4} + \tan\frac{\pi}{4}) - \ln(1+0)\right]$ $=\dfrac{1}{\sqrt{2}} - \dfrac{1}{2}\ln(\sqrt{2}+1)$

2(a)	The student's claim is true. Reasons: 1. From $\alpha^2+\beta^2+\gamma^2+\delta^2<0$, it shows that at least one of the roots is complex. 2. As the coefficients of $\mathrm{f}(z)$ are real, we know that complex roots exist in conjugate pairs, so there is at least one pair of complex conjugate roots. 3. If there are 2 pairs of complex conjugate roots, then $\alpha\beta\gamma\delta>0$. However, given that $\alpha\beta\gamma\delta<0$, then there is only a pair of complex conjugate roots and 2 real roots of opposite signs. So with 1, 2 and 3, we can conclude that the equation $\mathrm{f}(z)=0$ has one positive real root, one negative real root and a pair of complex conjugate roots.
(b)	Since $\mathrm{f}(2\mathrm{i})=(2\mathrm{i})^4+(2\mathrm{i})^3-2(2\mathrm{i})^2+4(2\mathrm{i})-24$ $=16-8\mathrm{i}+8+8\mathrm{i}-24=0,$ so $z=2\mathrm{i}$ is a root of the equation $\mathrm{f}(z)=0$ (verified). As complex roots occur in conjugate pair, so $z=-2\mathrm{i}$ is the other complex root. Now $(z-2\mathrm{i})(z+2\mathrm{i})=z^2+4$. Hence $z^4+z^3-2z^2+4z-24=(z^2+4)(z^2+az-6)$. Comparing the coefficient of z^3: $a=1$. Thus $z^4+z^3-2z^2+4z-24=(z^2+4)(z^2+z-6)$ $=(z^2+4)(z+3)(z-2)$ So the other 3 roots of the equation $\mathrm{f}(z)=0$ are $-2\mathrm{i}$, 2 and -3.

3(i)	**Method 1** The z-coordinate of P is 0, since P lies on the xy-plane. Thus, putting z = 0 into L_1, $1-y=\frac{0-1}{2} \quad \Rightarrow \quad y=\frac{3}{2}$ The position vector of P is $\overrightarrow{OP}=\begin{pmatrix}2\\ \frac{3}{2}\\ 0\end{pmatrix}$. Hence, equation of L_2 is $\mathbf{r}=\lambda\begin{pmatrix}4\\3\\0\end{pmatrix}$, $\lambda\in\mathbb{R}$ **Method 2** Vector equation of line L_1: $\mathbf{r}=\begin{pmatrix}2\\1\\1\end{pmatrix}+\lambda\begin{pmatrix}0\\-1\\2\end{pmatrix}$, $\mu\in\mathbb{R}$ Equation of xy-plane : $\mathbf{r}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}=0$ To find P, consider $\left(\begin{pmatrix}2\\1\\1\end{pmatrix}+\mu\begin{pmatrix}0\\-1\\2\end{pmatrix}\right)\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}=0$ $1+2\mu=0$ $\mu=-\frac{1}{2}$ Hence, $\overrightarrow{OP}=\begin{pmatrix}2\\1\\1\end{pmatrix}+\left(-\frac{1}{2}\right)\begin{pmatrix}0\\-1\\2\end{pmatrix}=\frac{1}{2}\begin{pmatrix}4\\3\\0\end{pmatrix}$. Hence, equation of L_2 is $\mathbf{r}=\lambda\begin{pmatrix}4\\3\\0\end{pmatrix}$, $\lambda\in\mathbb{R}$
(ii)	$\begin{pmatrix}4\\3\\0\end{pmatrix}\times\begin{pmatrix}0\\-1\\2\end{pmatrix}=\begin{pmatrix}6\\-8\\-4\end{pmatrix}=-2\begin{pmatrix}-3\\4\\2\end{pmatrix}$ Hence, equation of π is $\mathbf{r}\cdot\begin{pmatrix}-3\\4\\2\end{pmatrix}=0$
(iii)	Let F be the foot of perpendicular from A to π.

Equation of the line AF is $\mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix}$.

Since F lies on the line AF, then

$$\overrightarrow{OF} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix}, \text{ for some } \lambda \in \mathbb{R}.$$

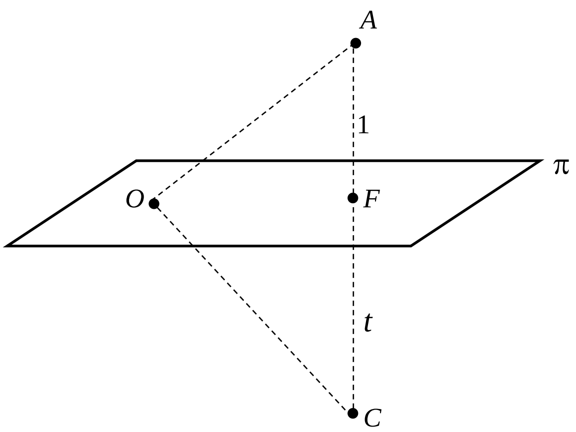
Since F is also on π, then

$$\left[\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} \right] \cdot \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} = 0$$

$$\left(-9-4+4\right) + \lambda(9+16+4) = 0$$

$$\lambda = \frac{9}{29}$$

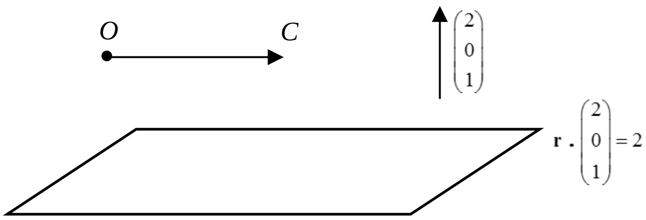
Hence, $\overrightarrow{OF} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \frac{9}{29} \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} = \frac{1}{29} \begin{pmatrix} 60 \\ 7 \\ 76 \end{pmatrix}$.



By Ratio Theorem, $\overrightarrow{OF} = \dfrac{t\overrightarrow{OA} + \overrightarrow{OC}}{(t+1)}$

$$\overrightarrow{OC} = (t+1)\overrightarrow{OF} - t\overrightarrow{OA} = \frac{(t+1)}{29} \begin{pmatrix} 60 \\ 7 \\ 76 \end{pmatrix} - t \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = \frac{1}{29} \begin{pmatrix} 60-27t \\ 7+36t \\ 76+18t \end{pmatrix}$$

(iv)



OC must be perpendicular to the normal vector of the plane.

$$\overrightarrow{OC}\cdot\begin{pmatrix}2\\0\\1\end{pmatrix}=0$$

$$\frac{1}{29}\begin{pmatrix}60-27t\\7+36t\\76+18t\end{pmatrix}\cdot\begin{pmatrix}2\\0\\1\end{pmatrix}=0$$

$$2(60-27t)+(76+18t)=0$$

$$-36t+196=0$$

$$t=\frac{49}{9}$$

4(a)

Given a, b, 2 is GP: $\frac{b}{a}=\frac{2}{b} \quad \Rightarrow \quad b^2 = 2a \quad$ - (1)

Given 2, a, b is AP: $a-2=b-a \quad \Rightarrow \quad 2a-b=2 \quad$ - (2)

Solving the 2 equations:

$b^2-b-2=0$

$(b-2)(b+1)=0$

$b=2$ or $b=-1$

If $b=2$, then $a=2$, $\Rightarrow$ common diff, $b-a=0$ (reject)

If $b=-1$, then $a=\frac{1}{2}$, $\Rightarrow$ common diff, $b-a\neq 0$

$\therefore a=\frac{1}{2},\ b=-1$

(b)

$p,\ p,\ p+d,\ pr,\ p+2d,\ pr^2 \ldots,\ p+(2n-1)d,\ pr^{2n-1}$

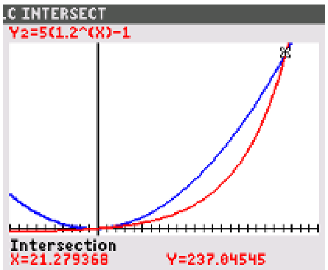
Sum of odd terms $=\frac{n}{2}\left[2p+(n-1)p\right]=\frac{pn}{2}(n+1)$

Sum of even terms $=\frac{p\left(1.2^n-1\right)}{1.2-1}=5p\left(1.2^n-1\right)$

Given $\frac{pn}{2}(n+1)<5p\left(1.2^n-1\right)\Rightarrow\frac{n}{2}(n+1)<5\left(1.2^n-1\right)$ **(since $p > 0$)**

Using GC, $n>21.3$

Least value of n is 22.



__Alternatively__,

From G.C.,

n = 21, $\frac{n}{2}(n+1)-5\left(1.2^n-1\right)=5.9744>0$

n = 22, $\frac{n}{2}(n+1)-5\left(1.2^n-1\right)=-18.03<0$

Hence, the least value of n is 22.

(c)

Sum of 1st k terms $=\frac{q\left(r^k-1\right)}{r-1}$

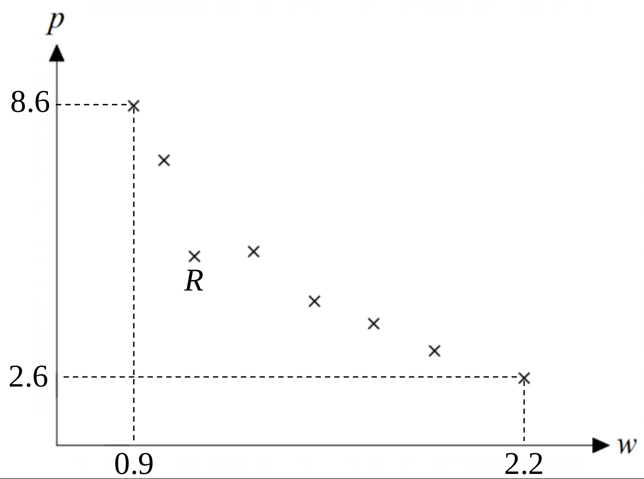
Sum of last k terms $=\frac{q\left(r^n-1\right)}{r-1}-\frac{q\left(r^{n-k}-1\right)}{r-1}$

$=\frac{q}{r-1}\left(r^n-1-r^{n-k}+1\right)$

$=\frac{q}{r-1}\left(r^n-r^{n-k}\right)=\frac{q}{r-1}r^{n-k}\left(r^k-1\right)$

	Difference required $=\left\lvert \dfrac{q\left(r^{k}-1\right)}{r-1}-\dfrac{q}{r-1}r^{n-k}\left(r^{k}-1\right)\right\rvert$ $=\left\lvert \dfrac{q}{r-1}\left[\left(r^{k}-1\right)-r^{n-k}\left(r^{k}-1\right)\right]\right\rvert$ $=\left\lvert \dfrac{q}{r-1}\left(r^{k}-1\right)\left(1-r^{n-k}\right)\right\rvert$

5(i)	2 from each of *P* and *Q*, 1 from *R*: ${}^5C_2 \times {}^4C_2 \times {}^3C_1 = 180$
(ii)	2 from Class *R* & 3 from *P* or *Q*: ${}^3C_2 \times {}^9C_3 = 252$ 3 from Class *R* & 2 from *P* or *Q*: ${}^3C_3 \times {}^9C_2 = 36$ Total number of ways is 288 **Alternatively,** Consider complement cases of none from *R* or one from *R*: ${}^{12}C_5 - {}^9C_5 - {}^3C_1 \times {}^9C_4 = 288$
(iii)	Class *P* students are all separated : $7! \times {}^8C_5 \times 5! = 33868800$ separated in groups of 2, 1, 1, 1 : $7! \times {}^8C_1 \times {}^7C_3 \times 5! = 169344000$ (or $7! \times {}^8C_4 \times 4 \times 5!$) separated in groups of 2, 2, 1 : $7! \times {}^8C_2 \times {}^6C_1 \times 5! = 101606400$ (or $7! \times {}^8C_3 \times 3 \times 5!$) Total number of ways is 304 819 200 **Alternatively,** consider the complement method: separated in groups of 3, 1, 1 : $7! \times {}^8C_3 \times 3 \times 5! = 101\ 606\ 400$ separated in groups of 3, 2 : $7! \times {}^8C_2 \times 2! \times 5! = 33\ 868\ 800$ separated in groups of 4, 1 : $7! \times {}^8C_2 \times 2! \times 5! = 33\ 868\ 800$ 5 members of P are seated in a group : $8! \times 5! = 4\ 838\ 400$ Total number of ways = $12! - 101606400 - 33868800 - 33868800 - 4838400 = 304819200$

6(i)	Since $r = -0.92821 = -0.928$ (3.s.f) is close to -1, it suggests a strong negative linear correlation, a linear model seems appropriate.
(ii)	
(iii)	With the point R removed, the values of p decreases as w increases, but by decreasing amounts. Hence it is consistent with a model of the form $\ln p = a + bw$.
(iv)	From GC, $\ln p = 2.910569 - 0.916387w$ … (1) a = 2.91 (3 s.f.) b = –0.916 (3 s.f.)`
(v)	Substitute w = 1.1 into (1) : $\ln p = 2.910569 - 0.916387(1.1)$ $p = 6.70$ (to 3 s.f.)

7(i)	<u>Condition 1</u> The event that a student infected by the disease is independent of any other student infected by the disease. This might not be true. Since the disease is contagious, it can easily be passed from one student to another. <u>Condition 2</u> The probability of a student infected by the disease is constant. This might not be true as the probability will depend on each individual's lifestyle or exposure, some will have a higher chance of catching the disease.
(ii)	$X \sim \mathrm{B}(20, 0.11)$ $\mathrm{P}(1 \le X < 10) = \mathrm{P}(1 \le X \le 9)$ $= \mathrm{P}(X \le 9) - \mathrm{P}(X = 0)$ $= 0.999983 - 0.097230$ $= 0.902753$ $= 0.903$
(iii)	Let Y denote the number of students who has once been infected by the disease before out of 19. $Y \sim \mathrm{B}(19.0.11)$ Required probability = $\mathrm{P}(Y = 4) \times 0.11 = 0.0109$ (to 3s.f.) <u>Alternative solution</u> Required probability = $\dfrac{{}^{19}C_4}{{}^{20}C_5} \mathrm{P}(X = 5) = 0.0109$ (to 3 s.f.)
(iv)	Lower. The event described in part **(iii)** is a subset of having exactly 5 students in the class who have been infected by the disease before. For example, it is possible that the first 5 students approached are the only 5 students in the class who has once been infected. This event would have been considered under "exactly 5 students" in the class have once been infected, but not considered in **(iii).**

8(i)	Let X be the score obtained in one game. $\mathrm{P}\left(X=4\right)=\mathrm{P}\left((\mathrm{H},4),(\mathrm{T},1+3),(\mathrm{T},2+2)\right)$ $=\frac{1}{2}\cdot\frac{1}{4}+\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{4}\cdot3=\frac{7}{32}$
(ii)	$\mathrm{P}\left(X=1\right)=\mathrm{P}(\mathrm{H},1)=\frac{1}{2}\cdot\frac{1}{4}=\frac{1}{8}$ $\mathrm{P}\left(X=2\right)=\mathrm{P}\left((\mathrm{H},2),(\mathrm{T},1+1)\right)=\frac{1}{2}\cdot\frac{1}{4}+\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{4}=\frac{5}{32}$ $\mathrm{P}\left(X=3\right)=\mathrm{P}\left((\mathrm{H},3),(\mathrm{T},1+2)\right)=\frac{1}{2}\cdot\frac{1}{4}+\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{4}\cdot2=\frac{3}{16}$ $\mathrm{P}\left(X=5\right)=\mathrm{P}\left((\mathrm{T},1+4),(\mathrm{T},2+3)\right)=\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{4}\cdot4=\frac{1}{8}$ $\mathrm{P}\left(X=6\right)=\mathrm{P}\left((\mathrm{T},2+4),(\mathrm{T},3+3)\right)=\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{4}\cdot3=\frac{3}{32}$ $\mathrm{P}(X=7)=\mathrm{P}(\mathrm{T},3+4)=\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{4}\cdot2=\frac{1}{16}$ $\mathrm{P}\left(X=8\right)=\mathrm{P}\left(\mathrm{T},4+4\right)=\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{4}=\frac{1}{32}$ $\mathrm{E}\left(X\right)=\sum_{r=1}^{8}r\,\mathrm{P}\left(X=r\right)=\frac{15}{4}$ (or 3.75) $\mathrm{E}\left(X^2\right)=\sum_{r=1}^{8}r^2\,\mathrm{P}\left(X=r\right)=\frac{35}{2}$ (or 17.5) $\mathrm{Var}\,X=\mathrm{E}\left(X^2\right)-\left[\mathrm{E}\left(X\right)\right]^2=\frac{35}{2}-\left(\frac{15}{4}\right)^2=\frac{55}{16}$
(iii)	$\mathrm{P}\left(\frac{X_1+\cdots+X_{35}}{35}\geq 4\,\middle\vert\, X_1=3 \text{ and } X_2=4\right)$ $=\frac{\mathrm{P}\left(\frac{X_1+\cdots+X_{35}}{35}\geq 4\cap\left(X_1=3\text{ and }X_2=4\right)\right)}{\mathrm{P}\left(X_1=3\text{ and }X_2=4\right)}$ $=\frac{\mathrm{P}\left(\frac{3+4+X_3+\cdots+X_{35}}{35}\geq 4\right)\mathrm{P}\left(X_1=3\right)\mathrm{P}\left(X_2=4\right)}{\mathrm{P}\left(X_1=3\right)\mathrm{P}\left(X_2=4\right)}$ $=\mathrm{P}\left(X_3+\cdots+X_{35}\geq 4\times35-3-4\right)$ $=\mathrm{P}\left(X_3+\cdots+X_{35}\geq 133\right)$ The sample size 33 is large. By Central Limit Theorem,

	$X_3 + \cdots + X_{35} \sim \mathrm{N}\left(33 \times \frac{15}{4} = \frac{495}{4},\ 33 \times \frac{55}{16} = \frac{1815}{16}\right)$ approximately. $\mathrm{P}\left(X_3 + \cdots + X_{35} \geq 133\right) = 0.1925638 = 0.193$ (3sf)

9(i)	Unbiased estimate of population mean is $\frac{8.71}{200} = 0.04355$ thousand hrs = 43.55 hours Unbiased estimate of population variance is $\frac{1}{199}\left(0.505 - \frac{8.71^2}{200}\right) = 6.31555\times10^{-4}$ (thousand hours)^2 = 632 hours^2 (3sf)
(ii)	Let μ be the population mean amount of time, in thousand hours, an employee spends at work per week. Null Hypothesis $\mathrm{H_0}: \quad \mu = 0.041$ Alternative Hypothesis $\mathrm{H_1}: \quad \mu > 0.041$ Perform a 1-tail test at 5% significance level. Under $\mathrm{H_0}$, since sample size 200 is large, $\overline{X} \sim \mathrm{N}\left(0.041, \frac{6.31555\times10^{-4}}{200}\right)$ approximately by Central Limit Theorem. From the sample, $\overline{x} = 0.04355$ Using a z-test, p-value $= \mathrm{P}\left(\overline{X} \geq 0.04355\right) = 0.0756 > 0.05$ We do not reject $\mathrm{H_0}$. There is insufficient evidence, at the 5% significance level, that the average working hours of an employee per week is more than 41.
(iii) (a)	Assume that the number of working hours per week of an employee follows a normal distribution. Let Y be the number working hours per week of an employee after the restructuring. $\mathrm{H_0}: \quad \mu = 40$ $\mathrm{H_1}: \quad \mu \neq 40$ Under $\mathrm{H_0}$, $\overline{Y} \sim \mathrm{N}\left(40, \frac{k}{12}\right)$ If $\mathrm{H_0}$ is not rejected at the 5% significance level, $\mathrm{P}\left(\overline{Y} \geq 40.1\right) > 0.025$ $\mathrm{P}\left(Z \geq \frac{40.1-40}{\sqrt{\frac{k}{12}}}\right) > 0.025$ $\frac{0.1}{\sqrt{\frac{k}{12}}} < 1.9600$ $\frac{k}{12} > \left(\frac{0.1}{1.9600}\right)^2$ $k > 0.0312 \quad$ (3sf)
(b)	The Central Limit Theorem does not apply here as the sample size 12 is small.

10(a)	Let X denote the lug width (in mm) of their customers' watches. $X \sim \mathrm{N}(\mathrm{E}(X), \sigma^2)$ $\mathrm{E}(X) = 20$ $\mathrm{P}(X < 21) = 0.85$ $\mathrm{P}\left(Z < \frac{21-20}{\sigma}\right) = 0.85$ From G.C. $\frac{1}{\sigma} = 1.03643$ $\sigma = 0.96484 = 0.965$ (to 3s.f.)
(b)	Let W denote the strap width (in mm) made by the company. $W \sim \mathrm{N}(19.6, 1.1^2)$ $\mathrm{P}(W > 20.2) = 0.29272$ Let Y denote the number of straps of width more than 20.2mm out of 40. $Y \sim \mathrm{B}(40, 0.29272)$. $\mathrm{E}(Y) = 40 \times 0.29272 = 11.708 = 11.7$ (to 3s.f.)
(c)	$\mathrm{P}(19.8 < W \le 20) = 0.069798$ Required probability = $0.069798^2 = 0.0048717 = 0.00487$ (to 3s.f.)
(d)	Let α be the probability that a pair of straps is usable for the 18.5mm-watch, and β be the probability that a pair of straps is usable for the 20mm-watch. Clearly, $\alpha = \left[\mathrm{P}(18.3 < W \le 18.5)\right]^2 = 0.0016013$ (to 7 d.p.) . From part **(c)**, $\beta = \left[\mathrm{P}(19.8 < W \le 20)\right]^2 = 0.0048717$ (to 7d.p.) **<u>Method 1</u>** : Probability that a pair of straps is not usable for both watches $= 1 - \alpha - \beta$ Hence, the required probability = 1 – P(both pairs of straps are not usable for both watches) $= 1 - (1-\alpha-\beta)^2 \quad \ldots (*)$ = 0.0129041904 = 0.01290 (5 d.p.)

	Method 2 : Let Q be the number of pairs of straps (out of 2) that are usable (for either the 18.5mm-watch or the 20mm-watch. $$Q \sim \mathrm{B}(2,\ \alpha+\beta)$$ Hence, the required probability $= \mathrm{P}(Q \geq 1) = 1 - \mathrm{P}(Q = 0)$ $= 1 - (1-(\alpha+\beta))^2 = 1 - (1-\alpha-\beta)^2$ (same as (*)) $= 0.0129041904 = 0.01290$ (5 d.p.)