NANYANG JUNIOR COLLEGE

JC2 PRELIMINARY EXAMINATION

Higher 2

CANDIDATE NAME	

CT CLASS	2	1			

Centre Number/ Index Number					/					

MATHEMATICS

9758/01

Paper 1

30 August 2022

3 hours

Candidates answer on the Question Paper.

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name and class on the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the Question Paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

For examiner's use only	
Question number	Marks
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
Total	

NANYANG JUNIOR COLLEGE
Internal Examinations

1 A function f is defined by $\mathrm{f}(x) = ax^3 + bx^2 + cx + d$. The graph of $y = \mathrm{f}(x)$ has a minimum point at $(-1, -1)$ and $\int_0^1 \mathrm{f}(x)\,\mathrm{d}x = \frac{9}{4}$. When $\mathrm{f}(x)$ is divided by $(x + 2)$, the remainder is 3. Find the values of a, b, c and d. [5]

2 **(i)** Sketch the graph of $y = \left|\left(\frac{1}{3}\right)^{2x} - 18\right|$, giving the exact values of any points where the curve meets the axes. [3]

(ii) Without using a calculator, and showing all your working, find the exact interval, or intervals, for which $\left|\left(\frac{1}{3}\right)^{2x} - 18\right| \le 9$. Give your answer in its simplest form. [3]

3 **Do not use a calculator in answering this question.**

A curve C has equation $x^3 + 2y^3 - 3xy - 40 = 0$.

(i) Find exact coordinates of the stationary points of C. [4]

(ii) For the stationary point with $x < 0$, determine whether it is a maximum or minimum point. [2]

4 It is given that $y = \cos\left(1 - \mathrm{e}^{-x}\right)$.

(i) Show that $\mathrm{e}^{2x}\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x}\right) + y = 0$. [2]

(ii) Find the first 3 non-zero terms of the Maclaurin series for $\cos\left(1 - \mathrm{e}^{-x}\right)$. [3]

(iii) Use your series from part **(ii)** to estimate $\int_0^{0.05} \cos\left(1 - \mathrm{e}^{-x}\right)\mathrm{d}x$, correct to 7 decimal places. [1]

(iv) It is now given that $\int_0^{0.05} \cos\left(1 - \mathrm{e}^{-x}\right)\mathrm{d}x \approx 0.0499799$. Comment on the accuracy of your estimation in part **(iii)**, justifying your answer. [1]

5 **(i)** It is given that $-1 + 2\mathrm{i}$ is a root of the equation $z^3 + 2(1+\mathrm{i})z^2 + (5 + 4\mathrm{i})z + 10\mathrm{i} = 0$. Explain why $-1 - 2\mathrm{i}$ may not be a root. [1]

(ii) Without using a calculator, solve the equation $z^3 + 2(1+\mathrm{i})z^2 + (5 + 4\mathrm{i})z + 10\mathrm{i} = 0$, giving your answers in the form $a + \mathrm{i}b$, where a and b are exact values. [5]

(iii) Hence solve $\mathrm{i}z^3 + 2(1+\mathrm{i})z^2 + (4 - 5\mathrm{i})z - 10\mathrm{i} = 0$. [2]

6 **(a)** A sequence $\{u_r\}$ is defined by $u_r = u_{r-1} + \frac{2}{r^3 - r} - \left(\frac{4}{5}\right)^{r-1}$, $r \ge 2$ and $u_1 = 3$.

(i) By expressing $\frac{2}{r^3-r}$ as $\frac{A}{r-1}+\frac{B}{r}+\frac{C}{r+1}$, where A, B and C are constants to be determined, find an expression for $\sum_{r=2}^{N}\left(\frac{2}{r^3-r}-\left(\frac{4}{5}\right)^{r-1}\right)$. [4]

(ii) Hence, by considering $\sum_{r=2}^{N}\left(u_r-u_{r-1}\right)$, express u_N in terms of N. [2]

(b) Another sequence $\{b_n\}$ is such that $1-\left(\frac{1}{2}\right)^n < b_n < 1+\left(\frac{1}{3}\right)^n$ for all positive integers $n \geq 1$.

The squeeze theorem states that if $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences of real numbers such that $a_n \leq b_n \leq c_n$ for all positive integers $n \geq 1$ and that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then b_n is convergent and $\lim_{n\to\infty} b_n = L$. Use the squeeze theorem to show that b_n is convergent and state its limit. [3]

7 A curve is defined parametrically by

$$x=\frac{at}{t+1},\ y=\frac{at^2}{t+1},$$

where a is a non-zero constant and $t \in \mathbb{R}, t \neq -1$.

(i) Show that the equation of the normal to the curve at the point T with parameter t is given by

$$t(t+1)(t+2)y+(t+1)x=at(t^3+2t^2+1)\ . \qquad [4]$$

(ii) The normal to the curve at the point $P\left(-a,\frac{1}{2}a\right)$ meets the curve again at point Q. Find the exact coordinates of Q in terms of a. [4]

(iii) Let M be the mid-point of PQ. Find a cartesian equation of the curve traced by M if a varies. [2]

8 A curve C has parametric equations

$$x=2\sin 2t+3,\ \ y=\cos 2t\ , \quad \text{for } \frac{\pi}{2} \leq t \leq \pi\ .$$

(i) Sketch the graph of C. Give the coordinates of the point(s) where C meets the x-axis and the end points on the curve. [2]

(ii) Find by integration, the exact area of the region R bounded by C, the line $y = 1$ and the axes. [4]

(iii) Show that the cartesian equation of C is $y^2 = 1-\frac{(x-3)^2}{4}$. [1]

(iv) Write down the cartesian equation of the curve if C is translated in the direction of the negative y-axis by 1 unit. Hence find the volume of the solid of revolution when R is rotated about the line $y = 1$. [4]

9 An arithmetic progression has first term a and common difference d, and a geometric progression has first term b and common ratio r, where a, b and d are non-zero. The second, fourth and sixth term of the arithmetic progression are equal to the first, fourth and eighth term of the geometric progression respectively.

(i) Show that $r^7 - 2r^3 + 1 = 0$. [2]

It is given that the geometric progression has positive terms.

(ii) Find the value of r and justify that this is the only answer. Deduce whether the geometric progression is convergent. [3]

(iii) Another arithmetic progression has first term k and common difference $3k$, where $k > 0$. The difference between the sum of the first $2n$ terms of this arithmetic progression and the nth term of the geometric progression with the common ratio found in part **(ii)** is at most $1000k$. Given that $b = 4k$, write down an inequality satisfied by n, and hence find the largest possible value of n. [5]

(iv) Let u_n denote the nth term of the geometric progression. Show that a new sequence with nth term $\ln\left(\frac{1}{u_n}\right)$ is an arithmetic progression. [2]

10 Two charged particles, U and V, are confined to the planes F_1 and F_2 with position vectors given by

$$(3+6p)\mathbf{i}+(1+4p+q)\mathbf{j}+(6+2p-4q)\mathbf{k} \text{ and } (-9+3p)\mathbf{i}+(1+p-2q)\mathbf{j}+(3-p+8q)\mathbf{k}$$

respectively, where $p, q \in \mathbb{R}$.

(i) Obtain the equation of F_1 in scalar product form. [2]

(ii) Find the acute angle between F_1 and F_2. [3]

The forces of the two particles U and V allow another charged particle W to remain suspended between them, such that $2UW = WV$.

(iii) As the positions of U and V vary, show that the set of points described by the path of W is a line l, whose vector equation is to be determined. [3]

(iv) An uncharged particle A is fired along a path described by a line $\mathbf{r} = s\mathbf{k}$, where $s \in \mathbb{R}$ and crosses F_1 at some instant in time. Find the shortest distance of A from l at this instant. [4]

11 In a chemical reaction, two substances A and B are combined to form a new substance C. The initial masses of A, B and C are 8, 6 and 0 units respectively. After time t seconds, the masses of A and B are each reduced by x units, and the mass of C increases by $2x$ units. The rate of change of the mass of C with respect to t is proportional to the product of the masses of A and B at any time t.

(i) Write down a differential equation relating x and t. [1]

(ii) It is observed that when $t = 5$, $x = 5$. By solving the differential equation in part **(i)**, show that

$$x = \frac{24\left(\frac{9}{4}\right)^{-\frac{t}{5}} - 24}{3\left(\frac{9}{4}\right)^{-\frac{t}{5}} - 4}.$$ [7]

(iii) Sketch the graph of x against t. [2]

In another experiment with the same initial masses of substances A and B as before, the increase in the mass of C is modelled by the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=-8\mathrm{e}^{-2t}-\frac{16}{9}(t+1)^{-\frac{7}{3}}.$$

Given that x eventually stabilises at 6 units after a long time, find x in terms of t. [4]

BLANK PAGE