CANDIDATE NAME

CLASS

ADMISSION NUMBER

2022 Preliminary Exams
Pre-University 3

MATHEMATICS 9758/01

Paper 1

13 September 2022

3 hours

Candidates answer on the Question Paper.

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your admission number, name and class on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give your answers in the spaces provided in the Question Paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

Qn No.	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	*	Total
Score													
Max Score	**6**	**7**	**6**	**8**	**8**	**9**	**11**	**9**	**11**	**12**	**13**		**100**

1 On the same axes, sketch the graphs of $y=2+\mathrm{e}^{-x^2}$ and $y=\left|\ln(x-1)\right|$. Label clearly the equations of any asymptotes and any points of intersection with the axes. [4]

Hence solve the inequality $2+\mathrm{e}^{-x^2}\le\left|\ln(x-1)\right|$. [2]

2 **(i)** Show, by means of the substitution $w=x^2y$, that the differential equation

$$2y+x\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{5}{x^4}$$

can be reduced to the form

$$\frac{\mathrm{d}w}{\mathrm{d}x}=-\frac{5}{x^3}.$$ [3]

(ii) Hence, given that $y=3$ when $x=\frac{1}{2}$, solve the differential equation

$$2y+x\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{5}{x^4},$$

to find y in terms of x. [4]

3 The functions f and g are defined by

$$\mathrm{f}:x\mapsto\frac{1}{x+1},\qquad x\in\mathbb{R},\ x\neq -1,$$

$$\mathrm{g}:x\mapsto ax+b,\qquad x\in\mathbb{R}\text{ and }x>0,$$

where a and b are positive constants.

(i) Explain clearly why the composite function fg exists. [2]

(ii) Find an expression for $\mathrm{fg}(x)$ and the range of fg. [2]

(iii) Describe a sequence of transformations which transforms the graph of $y=\mathrm{f}(x)$ on to the graph of $y=\mathrm{fg}(x)$. [2]

4 A curve has equation $\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{y}}=\frac{1}{\sqrt{a}}$, where a is a positive constant.

(i) State the range of values of x and y for the curve to be defined. [1]

(ii) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y. [2]

(iii) Find the equation of the normal to the curve at the point where $x=9a$, leaving your answer in the form of $py+qx=ra$, where p, q and r are integers to be determined. [5]

5 **Do not use a calculator in answering this question.**

The cubic polynomial $\mathrm{f}(x)$ is given by x^3-1.

(i) The roots to the equation $\mathrm{f}(x)=0$ are denoted by x_1, x_2 and x_3 respectively, where $-\pi<\arg(x_1)<\arg(x_2)<\arg(x_3)<\pi$. Find exactly x_1, x_2 and x_3. [3]

(ii) Show x_1, x_2 and x_3 on a single Argand diagram. The points A, B and C represent x_1, x_2 and x_3 respectively. Find the exact area of triangle ABC. [3]

(iii) Show that $x_1^{\,2}=x_3$. [2]

6 With respect to the origin O, the points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively, where $\mathbf{a}$ and $\mathbf{b}$ are non-zero and not parallel.

(i) It is given that B lies on the line segment AC, such that $\overrightarrow{BC}=3\mathbf{b}-k\mathbf{a}$, where k is a constant. State, with a reason, the value of k. Hence find $\overrightarrow{OC}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. [3]

The point N divides the line OC in the ratio $1-\lambda:\lambda$.

(ii) Given that $\overrightarrow{OA}$ is perpendicular to $\overrightarrow{OB}$, Show that $\overrightarrow{BN}\boldsymbol{.}\overrightarrow{OC}$ can be written as $p|\mathbf{a}|^2+q|\mathbf{b}|^2$, where p and q are constants to be found in terms of λ. [4]

(iii) Given that $\overrightarrow{BN}\boldsymbol{.}\overrightarrow{OC}=0$ and $|\mathbf{a}|=\frac{2}{3}|\mathbf{b}|$, find the value of λ. [2]

7 **(a)** Find $\displaystyle\int \frac{\mathrm{e}^{\tan^{-1}2x}}{1+4x^2}\ \mathrm{d}x$. [2]

(b) Find $\displaystyle\int (\sec x+3\tan x)^2\ \mathrm{d}x$. [4]

(c) Find $\displaystyle\int \frac{x^2+x+1}{x^2-x+1}\ \mathrm{d}x$. [5]

8 It is given that $\displaystyle\sum_{r=1}^{n} r^2=\frac{1}{6}n(n+1)(2n+1)$.

A sequence $u_0,\ u_1,\ u_2,\ldots$ is defined by $u_0=k$, where k is a constant, and

$$u_{n+1}=u_n-3n^2+2^n \quad \text{for} \quad n\geq 0.$$

(i) Find u_1, u_2 and u_3 in terms of k. [2]

(ii) By considering $\displaystyle\sum_{r=1}^{n}(u_{r+1}-u_r)$, or otherwise, show that

$$u_n=k-\frac{1}{2}n(n-1)(2n-1)-1+2^n.$$ [5]

(iii) Given that $k=3$, find the minimum value of n such that $u_n>3000$. [2]

9 The curve C is defined by the parametric equations

$$x=\frac{\theta}{2}-\sin\theta, \qquad y=1-\cos\theta,$$

where $0 \le \theta \le \frac{3\pi}{2}$.

(i) Sketch C, indicating clearly the exact coordinates of the endpoints. [2]

(ii) Find $\frac{dy}{dx}$. Hence find the coordinates of the point on C such that the tangent to C at this point is parallel to the line $2y - x = 4$, leaving your answer to 3 decimal places. [5]

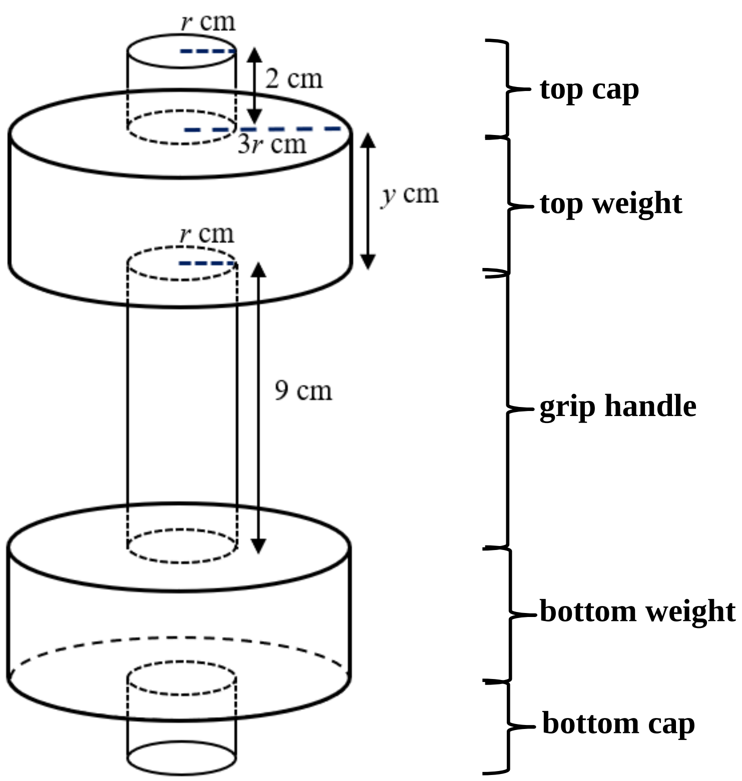
(iii) Find the exact area of the region bounded by C, the x-axis and the line $x = \frac{3\pi}{4} + 1$. [4]

10 One of the benefits of a water filled dumbbell is that it can be brought anywhere and can be used anytime for physical workouts by just filling it with water.

A particular model of a water filled dumbbell is made up of five parts and they are all cylindrical in shape.

- The top and bottom caps have radius r cm and height 2 cm.
- The top and bottom weights have radius $3r$ cm and height y cm.
- The grip handle has radius r cm and height 9 cm.

The five parts are joined together as shown in the diagram below. The dumbbell is made with plastic material of negligible thickness.



It is known that the external surface area of the dumbbell, A cm^2 is $34\pi r^2 + 26\pi r + 12\pi ry$.

(i) Given that the volume of the dumbbell is a fixed value k cm^3, show that

$$A = 34\pi r^2 + \frac{52\pi r}{3} + \frac{2k}{3r}.$$ [3]

(ii) The manufacturer wants to minimise the external surface area of the dumbbell. Using differentiation, show that r satisfies the equation $102\pi r^3 + 26\pi r^2 - k = 0$. [2]

It is now known that the volume of the dumbbell is 1500 cm^3.

(iii) Find the minimum external surface area of the dumbbell, proving that it is a minimum. [3]

Assume that $r = 2$ and $y = 7$ for the rest of the question.

(iv) Water is poured into the empty dumbbell through the top cap at a rate of 15 cm^3 s^{-1}. State, with justification, which part of the dumbbell would the water level be at after 1 minute. [2]

(v) Hence find the exact rate at which the depth of the water is increasing after 1 minute. [2]

11 Mrs Tay decides to open some savings accounts for her child.

(a) In a Child Development Account (CDA), the government will co-match dollar-for-dollar for each dollar that she deposits in the CDA. If $$x$ is deposited at the start of a month, the government will deposit $$x$ into the CDA too.

Assume that the government co-matches the amount immediately once it is deposited. The interest rate is 1.2% per month, so that at the end of each month, the total amount of money in the CDA is increased by 1.2%.

Mrs Tay opens a CDA for her child and deposits $100 on the first day of each month, starting from 1 January 2022.

(i) Write down how much money is in the CDA at the end of January 2022. [1]

(ii) Find the minimum number of months needed for the account to have a total amount of at least \$6000. State the month and year during which this occurs. [5]

Instead, suppose Mrs Tay deposits \$3000 on 1 January 2022. The government co-matches that amount and there is no further co-matching provided by the government thereafter.

(iii) Find the minimum amount of money that she needs to deposit at the start of each month from 1 February 2022 onwards if she wishes to have at least \$10 000 in the CDA by 31 December 2022. [3]

11 **(b)** In another savings account, there is no co-matching provided by the government. At the end of each month, the account earns a fixed interest based on the total amount of money deposited into the account. For each \$200 in the total amount of money deposited in the account, an interest of \$$x$ is added into the account at the end of the month. The accumulated interests do not earn any further interest.

Mrs Tay deposits \$200 at the start of every month in this savings account. Find, in terms of x, the total amount of money in the savings account at the end of 2 years. [4]

End of Paper