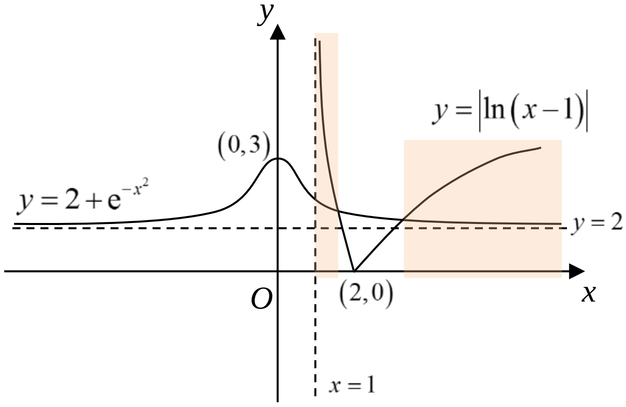
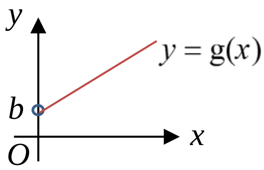
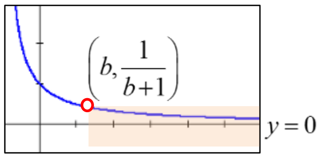
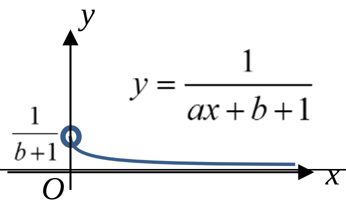
2022 PU3 H2 MATHEMATICS Preliminary Examination Paper 1

Qn	Solution
1 **[4]**	
1 **[2]**	x-coordinates of points of intersection: 1.10 and 8.39 For $2 + e^{-x^2} \le \lvert\ln(x-1)\rvert$, $1 < x \le 1.10$ or $x \ge 8.39$ (3 s.f.).

Qn	Solution
2(i) **[3]**	$w = x^2 y$ Differentiate with respect to x: $\frac{dw}{dx} = 2xy + x^2\left(\frac{dy}{dx}\right)$ Presentation 1 $2y + x\frac{dy}{dx} = -\frac{5}{x^4}$ Multiply x throughout: $2xy + x^2\frac{dy}{dx} = -\frac{5}{x^3}$ $\frac{dw}{dx} = -\frac{5}{x^3}$ Presentation 2 $x^2\frac{dy}{dx} = \frac{dw}{dx} - 2xy$ $\frac{dy}{dx} = \left(\frac{1}{x^2}\right)\frac{dw}{dx} - \frac{2y}{x}$ $2y + x\left[\left(\frac{1}{x^2}\right)\frac{dw}{dx} - \frac{2y}{x}\right] = -\frac{5}{x^4}$ $\frac{1}{x}\left(\frac{dw}{dx}\right) = -\frac{5}{x^4}$ $\frac{dw}{dx} = -\frac{5}{x^3}$

Qn	Solution
2(ii) **[4]**	$\frac{\mathrm{d}w}{\mathrm{d}x}=-\frac{5}{x^3}$ $\int \frac{\mathrm{d}w}{\mathrm{d}x}\,\mathrm{d}x=\int -\frac{5}{x^3}\,\mathrm{d}x$ $w=-5\left(\frac{x^{-2}}{-2}\right)+c$ $w=\frac{5}{2x^2}+c$ $x^2y=\frac{5}{2x^2}+c$ $y=\frac{5}{2x^4}+\frac{c}{x^2}$ when $x=\frac{1}{2},\ \ y=3$ $3=\frac{5}{2\left(\frac{1}{2}\right)^4}+\frac{c}{\left(\frac{1}{2}\right)^2}$ $c=-\frac{37}{4}$ $y=\frac{5}{2x^4}-\frac{37}{4x^2}$

Qn	Solution
3(i) **[2]**	$\mathrm{f}\left(x\right)=\frac{1}{x+1}$ $\mathrm{g}\left(x\right)=ax+b$ Note: $\mathrm{D_g}=\left(0,\infty\right)$ $\mathrm{R_g}=\left(b,\infty\right)$ and $\mathrm{D_f}=\left(-\infty,-1\right)\cup\left(-1,\infty\right)=\mathbb{R}\backslash\left\{-1\right\}$ Since $b>0$, $\mathrm{R_g}\subseteq\mathrm{D_f}$. Therefore, fg exists. 
3(ii) **[2]**	$\mathrm{fg}\left(x\right)=\mathrm{f}\left(\mathrm{g}\left(x\right)\right)$ $=\mathrm{f}\left(ax+b\right)$ $=\frac{1}{ax+b+1}$ $\mathrm{D_{fg}}=\mathrm{D_g}=\left(0,\infty\right)$

Qn	Solution
	<u>Method 1 (Mapping):</u> $\mathrm{D_g}=(0,\infty)\xrightarrow{\mathrm{g}}\mathrm{R_g}=(b,\infty)\xrightarrow{\mathrm{f}}\mathrm{R_{fg}}=\left(0,\dfrac{1}{b+1}\right)$ $\mathrm{R_{fg}}=\left(0,\dfrac{1}{b+1}\right)$  **<u>Method 2</u>** Using graph of $y=\mathrm{fg}(x)=\dfrac{1}{ax+b+1}$ under $\mathrm{D_{fg}}=\mathrm{D_g}=(0,\infty)$, $\mathrm{R_{fg}}=\left(0,\dfrac{1}{b+1}\right)$ 
3(iii) **[2]**	**<u>Possible Sequence 1:</u>** $y=\dfrac{1}{x+1}\xrightarrow{(1)}y=\dfrac{1}{x+b+1}\xrightarrow{(2)}y=\dfrac{1}{ax+b+1}$ (1) **Translate** b units in the negative x-direction (2) **Scale** parallel to the x-axis by scale factor $\dfrac{1}{a}$ **<u>Possible Sequence 2:</u>** $y=\dfrac{1}{x+1}\xrightarrow{(1)}y=\dfrac{1}{ax+1}\xrightarrow{(2)}y=\dfrac{1}{a\left(x+\dfrac{b}{a}\right)+1}$ $=\dfrac{1}{ax+b+1}$ (1) **Scale** parallel to the x-axis by scale factor $\dfrac{1}{a}$ (2) **Translate** $\dfrac{b}{a}$ units in the negative x-direction

Qn	Solution
4(i) **[1]**	$x>0$ and $y>0$
4(ii) **[2]**	$\dfrac{1}{\sqrt{x}}+\dfrac{1}{\sqrt{y}}=\dfrac{1}{\sqrt{a}}$ $\Rightarrow x^{-\frac{1}{2}}+y^{-\frac{1}{2}}=\dfrac{1}{\sqrt{a}}$ Differentiate with respect to x:

Qn	Solution
	$-\frac{1}{2}x^{-\frac{1}{2}-1}-\frac{1}{2}y^{-\frac{1}{2}-1}\frac{\mathrm{d}y}{\mathrm{d}x}=0$ $-\frac{1}{2}x^{-\frac{3}{2}}-\frac{1}{2}y^{-\frac{3}{2}}\frac{\mathrm{d}y}{\mathrm{d}x}=0$ $x^{-\frac{3}{2}}+y^{-\frac{3}{2}}\frac{\mathrm{d}y}{\mathrm{d}x}=0$ $y^{-\frac{3}{2}}\frac{\mathrm{d}y}{\mathrm{d}x}=-x^{-\frac{3}{2}}$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-x^{-\frac{3}{2}}}{y^{-\frac{3}{2}}}$ $=-\left(\frac{x}{y}\right)^{-\frac{3}{2}}=-\left(\frac{y}{x}\right)^{\frac{3}{2}}$
4(iii) **[5]**	When $x=9a$, $\frac{1}{\sqrt{9a}}+\frac{1}{\sqrt{y}}=\frac{1}{\sqrt{a}}$ $\frac{1}{\sqrt{9}\sqrt{a}}+\frac{1}{\sqrt{y}}=\frac{1}{\sqrt{a}}$ $\frac{1}{3\sqrt{a}}+\frac{1}{\sqrt{y}}=\frac{1}{\sqrt{a}}$ $\frac{1}{\sqrt{y}}=\frac{1}{\sqrt{a}}-\frac{1}{3\sqrt{a}}$ $\frac{1}{\sqrt{y}}=\frac{2}{3\sqrt{a}}$ $\sqrt{y}=\frac{3\sqrt{a}}{2}$ $y=\left(\frac{3\sqrt{a}}{2}\right)^2=\frac{9a}{4}$ At $x=9a$ and $y=\frac{9a}{4}$: $\frac{\mathrm{d}y}{\mathrm{d}x}=-\left(\frac{\frac{9a}{4}}{9a}\right)^{\frac{3}{2}}=-\left(\frac{1}{4}\right)^{\frac{3}{2}}=-\frac{1}{8}$

Qn	Solution
	[or $\frac{\mathrm{d}y}{\mathrm{d}x}=-\left(\frac{9a}{\frac{9a}{4}}\right)^{-\frac{3}{2}}=-(4)^{-\frac{3}{2}}=-\frac{1}{8}$] Therefore, the gradient of the normal at $x=9a$ is 8. When $x=9a$, $y=\frac{9a}{4}$ and gradient of normal = 8: <table><tr><th>Use $y-y_1=m(x-x_1)$</th><th>Use $y=mx+c$</th></tr><tr><td>$y-\frac{9a}{4}=8(x-9a)$ $y-\frac{9a}{4}=8x-72a$ $4y-9a=32x-288a$ $4y-32x=-279a$</td><td>$\frac{9a}{4}=8(9a)+c$ $\Rightarrow c=\frac{9a}{4}-72a=-\frac{279a}{4}$ Therefore, $y=8x-\frac{279a}{4}$ $\Rightarrow 4y-32x=-279a$</td></tr></table>Therefore, the equation of the normal to the curve at the point where $x=9a$ is $4y-32x=-279a$, where $p=4$, $q=-32$ and $r=-279$

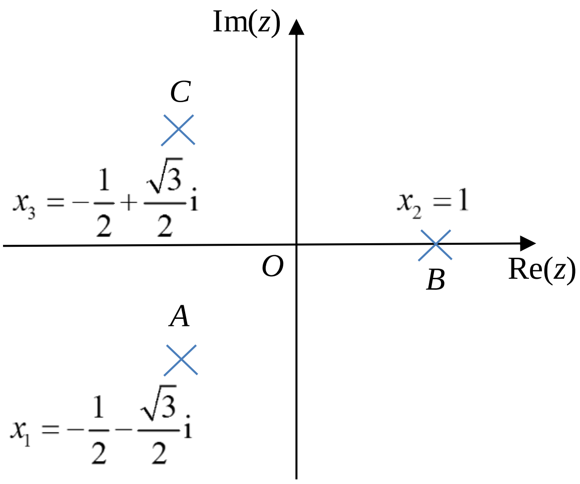
Qn	Solution
5(i) **[3]**	Observe that $\mathrm{f}(1)=1^3-1=0$. Hence $x=1$ is a root. $\mathrm{f}(x)=x^3-1$ $\quad=(x-1)(ax^2+bx+c)$ Comparing x^3: $a=1$ Comparing constant: $c=-1$ Comparing x: $0=-b+1\Rightarrow b=1$ $\mathrm{f}(x)=(x-1)(x^2+x+1)$ Solving for $(x-1)(x^2+x+1)=0$ $x=1$ or $x^2+x+1=0$ <table><tr><th><u>Method 1 (complete the square)</u></th><th><u>Method 2 (apply quadratic formula)</u></th></tr></table>

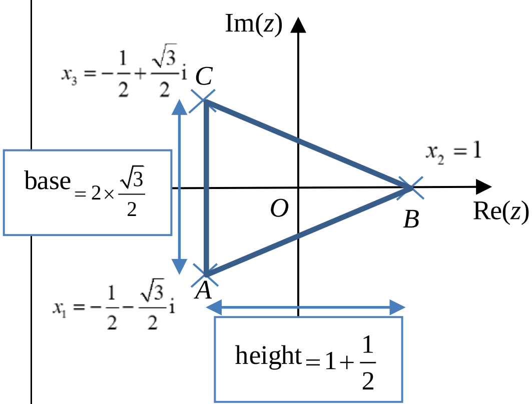
Method 1	Method 2
$x^2+x+1=0$ $\left(x+\frac{1}{2}\right)^2+\frac{3}{4}=0$ $x+\frac{1}{2}=\pm\sqrt{-\frac{3}{4}}$ $x=-\frac{1}{2}\pm\frac{\sqrt{3}}{2}\mathrm{i}$	$x^2+x+1=0$ $x=\frac{-1\pm\sqrt{1^2-4(1)(1)}}{2(1)}$ $=\frac{-1\pm\sqrt{-3}}{2}$ $=-\frac{1}{2}\pm\frac{\sqrt{3}}{2}\mathrm{i}$

Since $-\pi<\arg(x_1)<\arg(x_2)<\arg(x_3)<\pi$,

$x_1=-\frac{1}{2}-\frac{\sqrt{3}}{2}\mathrm{i}$, $\quad x_2=1 \quad$ and $\quad x_3=-\frac{1}{2}+\frac{\sqrt{3}}{2}\mathrm{i}$.

5(ii)
[3]



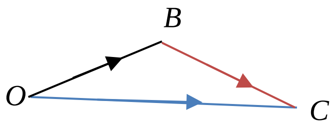
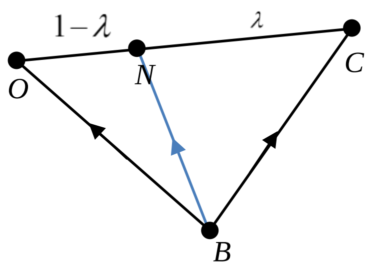


The area of triangle ABC

$$=\frac{1}{2}\left(2\times\frac{\sqrt{3}}{2}\right)\left(1+\frac{1}{2}\right)$$

$$=\frac{3\sqrt{3}}{4}\text{ units}^2$$

5(iii) **[2]**	$x_1^{\,2}=\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}\mathrm{i}\right)^2$ $=\frac{1}{4}-\frac{3}{4}+\frac{\sqrt{3}}{2}\mathrm{i}$ $=-\frac{1}{2}+\frac{\sqrt{3}}{2}\mathrm{i}=x_3$

Qn	Solution
6(i) **[3]**	Since B lies on AC, $\overrightarrow{BC}$ is parallel to $\overrightarrow{AB}$, i.e. $\overrightarrow{BC}=\beta\left(\overrightarrow{AB}\right)$ $3\mathbf{b}-k\mathbf{a}=\beta(\mathbf{b}-\mathbf{a})$ By comparing **b**, $\beta=3$ $\therefore\overrightarrow{BC}=3\mathbf{b}-3\mathbf{a}$ So, $k=3$ $\overrightarrow{OC}=\overrightarrow{OB}+\overrightarrow{BC}$ $\overrightarrow{OC}=\mathbf{b}+3\mathbf{b}-3\mathbf{a}$ $=4\mathbf{b}-3\mathbf{a}$ 
6(ii) **[4]**	To find $\overrightarrow{BN}$, **<u>Method 1</u>:** Based on diagram on the right,  $\overrightarrow{ON}=(1-\lambda)\overrightarrow{OC}$ $=(1-\lambda)(4\mathrm{b}-3\mathrm{a})$ $\overrightarrow{BN}=\overrightarrow{ON}-\overrightarrow{OB}$ $\overrightarrow{BN}=(1-\lambda)(4\mathbf{b}-3\mathbf{a})-\mathbf{b}$ $=(3-4\lambda)\mathbf{b}-3(1-\lambda)\mathbf{a}$ --- **<u>Method 2: By ratio theorem:</u>** $\overrightarrow{BN}=(1-\lambda)\overrightarrow{BC}+\lambda\overrightarrow{BO}$ $=(1-\lambda)(3\mathbf{b}-3\mathbf{a})+\lambda(-\mathbf{b})$ $=(3-4\lambda)\mathbf{b}-3(1-\lambda)\mathbf{a}$  Since $\overrightarrow{OA}$ is perpendicular to $\overrightarrow{OB}$, $\mathbf{b}\cdot\mathbf{a}=\mathbf{a}\cdot\mathbf{b}=0$

	$\overrightarrow{BN} \cdot \overrightarrow{OC}$ $=\left[(3-4\lambda)\mathbf{b}-3(1-\lambda)\mathbf{a}\right] \cdot (4\mathbf{b}-3\mathbf{a})$ $=4(3-4\lambda)\mathbf{b}\cdot\mathbf{b}-3(3-4\lambda)\underbrace{\mathbf{b}\cdot\mathbf{a}}_{0}-12(1-\lambda)\underbrace{\mathbf{a}\cdot\mathbf{b}}_{0}+9(1-\lambda)\mathbf{a}\cdot\mathbf{a}$ $=9(1-\lambda)\lvert\mathbf{a}\rvert^2+4(3-4\lambda)\lvert\mathbf{b}\rvert^2$ where $p=9(1-\lambda),\ q=4(3-4\lambda)$
6(iii) **[2]**	Given $\overrightarrow{BN} \cdot \overrightarrow{OC}=0$ and $\lvert\mathbf{a}\rvert=\frac{2}{3}\lvert\mathbf{b}\rvert$, Using the result in part **(ii)**: $9(1-\lambda)\left(\frac{2}{3}\right)^2\lvert\mathbf{b}\rvert^2+4(3-4\lambda)\lvert\mathbf{b}\rvert^2=0$ $4-4\lambda+12-16\lambda=0$ $20\lambda=16$ $\lambda=\frac{4}{5}$

Qn	Solution
7(a) **[2]**	$\int \frac{e^{\tan^{-1}2x}}{1+4x^2}\,dx$ $=\int \frac{1}{1+4x^2}e^{\tan^{-1}2x}\,dx$ $=\frac{1}{2}\int \frac{2}{1+4x^2}e^{\tan^{-1}2x}\,dx$ $=\frac{1}{2}\underbrace{\int \frac{2}{1+4x^2}e^{\tan^{-1}2x}\,dx}$ $=\frac{1}{2}e^{\tan^{-1}2x}+c$

Qn	Solution
7(b) **[4]**	$\int (\sec x + 3\tan x)^2 \, dx$ $= \int (\sec x)^2 + 2(\sec x)(3\tan x) + (3\tan x)^2 \, dx$ $= \int (\sec x)^2 + 2(\sec x)(3\tan x) + 3^2 (\tan x)^2 \, dx$ $= \int \sec^2 x + 6\sec x \tan x + 9\tan^2 x \, dx$ $= \int \sec^2 x + 6\sec x \tan x + 9(\sec^2 x - 1) \, dx$ $= \int \sec^2 x + 6\sec x \tan x + 9\sec^2 x - 9 \, dx$ $= \int 10\sec^2 x + 6\sec x \tan x - 9 \, dx$ $= 10\tan x + 6\sec x - 9x + c$
7(c) **[5]**	$\int \frac{x^2+x+1}{x^2-x+1} \, dx$ $= \int \left(1 + \frac{2x}{x^2-x+1}\right) dx$ $= \int \left(1 + \frac{2x-1}{x^2-x+1} + \frac{1}{x^2-x+1}\right) dx$ $= \int \left(1 + \frac{2x-1}{x^2-x+1} + \frac{1}{\left(x-\frac{1}{2}\right)^2 + \frac{3}{4}}\right) dx$ $= \int \left(1 + \frac{2x-1}{x^2-x+1} + \frac{1}{\left(x-\frac{1}{2}\right)^2 + \left(\sqrt{\frac{3}{4}}\right)^2}\right) dx$

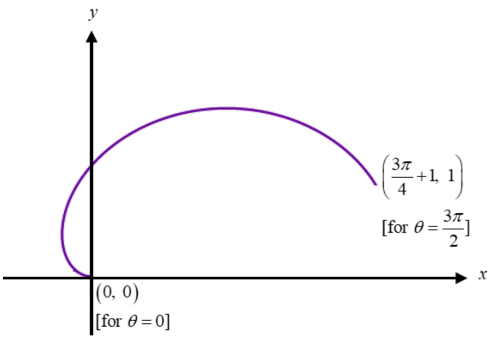
Qn	Solution
	$= x + \ln\left\|x^2 - x + 1\right\| + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x - \frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + c$ $= x + \ln\left\|x^2 - x + 1\right\| + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{\frac{1}{2}(2x-1)}{\frac{1}{2}\left(\sqrt{3}\right)}\right) + c$ $= x + \ln\left\|x^2 - x + 1\right\| + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + c$ $= x + \ln\left(x^2 - x + 1\right) + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + c$ since $x^2 - x + 1 > 0$ for all $x \in \mathbb{R}$.

Qn	Solution
8(i) **[2]**	$u_0 = k$ and $u_{n+1} = u_n - 3n^2 + 2^n$ $u_1 = k - 3(0)^2 + 2^0$ $= k + 1$ $u_2 = k - 3(0)^2 + 2^0 - 3(1)^2 + 2^1$ $= k$ $u_3 = k - 3(0)^2 + 2^0 - 3(1)^2 + 2^1 - 3(2)^2 + 2^2$ $= k - 8$
8(ii) **[5]**	**<u>Method 1: Considering $\sum_{r=1}^{n}(u_{r+1} - u_r)$</u>** Given: $u_{n+1} = u_n - 3n^2 + 2^n$ $\Rightarrow u_{n+1} - u_n = -3n^2 + 2^n$ $\sum_{r=1}^{n}(u_{r+1} - u_r) = \sum_{r=1}^{n}\left(-3r^2 + 2^r\right)$ $\begin{bmatrix} \cancel{u_2} - u_1 \\ + \cancel{u_3} - \cancel{u_2} \\ + \cancel{u_4} - \cancel{u_5} \\ + \ldots \\ + \cancel{u_{n-1}} - \cancel{u_{n-2}} \\ + \cancel{u_n} - \cancel{u_{n-1}} \\ + u_{n+1} - \cancel{u_n} \end{bmatrix} = -3\sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} 2^r$ sum of GP: $a=2$, $r=2$, n terms

Qn	Solution
	$u_{n+1}-u_1=-3\left[\frac{1}{6}n(n+1)(2n+1)\right]+\frac{2\left(1-2^n\right)}{1-2}$ $u_{n+1}-(k+1)=-\frac{1}{2}n(n+1)(2n+1)-2\left(1-2^n\right)$ $u_{n+1}=k+1-\frac{1}{2}n(n+1)(2n+1)-2+2^{n+1}$ **Either:** replace n with $(n-1)$: $u_{(n-1)+1}=k-\frac{1}{2}(n-1)(n-1+1)(2(n-1)+1)-1+2^n$ $u_n=k-\frac{1}{2}n(n-1)(2n-1)-1+2^n$ (shown) **Or:** $u_n-3n^2+2^n=k+1-\frac{1}{2}n(n+1)(2n+1)-2+2^{n+1}$ $u_n=k+1-\frac{1}{2}n(n+1)(2n+1)-2+2^{n+1}+3n^2-2^n$ $=k-\frac{1}{2}\left[n\left(2n^2+3n+1\right)\right]-1+2\left(2^n\right)+3n^2-2^n$ $=k-\frac{1}{2}\left(2n^3+3n^2+n\right)+3n^2-1+2^n$ $=k-\frac{1}{2}\left(2n^3-3n^2+n\right)-1+2^n$ $=k-\frac{1}{2}n\left(2n^2-3n+1\right)-1+2^n$ $=k-\frac{1}{2}n(n-1)(2n-1)-1+2^n$ (shown) **<u>Method 2: otherwise</u>** $u_n=k-3(0)^2+2^0-3(1)^2+2^1-3(2)^2+2^2+\ldots-3(n-1)^2+2^{n-1}$ $=k-3\left(1^2+2^2+\ldots+(n-1)^2\right)+\underbrace{2^0+2^1+\ldots++2^{n-1}}_{\text{Sum of GP: } a=2^0, r=2, \text{ number of terms: } n}$ $=k-3\sum_{r=1}^{n-1}r^2+\frac{2^0\left(1-2^n\right)}{1-2}$ $=k-3\left[\frac{1}{6}(n-1)(n)(2(n-1)+1)\right]-\left(1-2^n\right)$ $=k-\frac{1}{2}n(n-1)(2n-1)-1+2^n$ (shown)

Qn	Solution
8(iii) **[2]**	$u_n = 3 - \frac{1}{2}n(n-1)(2n-1) - 1 + 2^n$ $= 2 - \frac{1}{2}n(n-1)(2n-1) + 2^n$ $2 - \frac{1}{2}n(n-1)(2n-1) + 2^n > 3000$ Using GC, (see table below) **Minimum value** of *n* is 13.

n	$2 - \frac{1}{2}n(n-1)(2n-1) + 2^n$
12	2580
13	6244
14	13929

Qn	Solution
9(i) **[2]**	

Let $\theta = 0$;

$x = \frac{0}{2} - \sin 0 = 0$

$y = 1 - \cos 0 = 0$

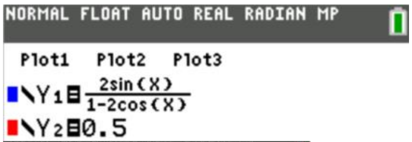
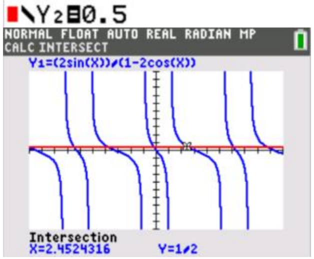
Therefore, when $\theta = 0$, the corresponding coordinates is $(0,\ 0)$.

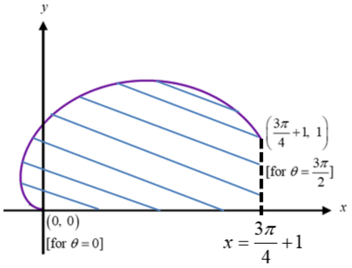
Let $\theta = \frac{3\pi}{2}$;

$x = \frac{\frac{3\pi}{2}}{2} - \sin\frac{3\pi}{2} = \frac{3\pi}{4} - (-1) = \frac{3\pi}{4} + 1$

$y = 1 - \cos\frac{3\pi}{2} = 1 - 0 = 1$

Therefore, when $\theta = \frac{3\pi}{2}$, the corresponding coordinates is $\left(\frac{3\pi}{4}+1,\ 1\right)$.

Qn	Solution
9(ii) **[5]**	$x=\dfrac{\theta}{2}-\sin\theta \Rightarrow \dfrac{\mathrm{d}x}{\mathrm{d}\theta}=\dfrac{1}{2}-\cos\theta$ $y=1-\cos\theta \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}\theta}=\sin\theta$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\frac{\mathrm{d}y}{\mathrm{d}\theta}}{\frac{\mathrm{d}x}{\mathrm{d}\theta}}=\dfrac{\sin\theta}{\frac{1}{2}-\cos\theta}$ $=\dfrac{\sin\theta}{\frac{1-2\cos\theta}{2}}$ $=\sin\theta\times\dfrac{2}{1-2\cos\theta}$ $=\dfrac{2\sin\theta}{1-2\cos\theta}$ Since tangent to C at the required point is parallel to the line $2y-x=4$ (i.e. $y=\dfrac{1}{2}x+2$), this means $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{1}{2}$. $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{2\sin\theta}{1-2\cos\theta}=\dfrac{1}{2}$ Using graphical approach to solve $\dfrac{2\sin\theta}{1-2\cos\theta}=\dfrac{1}{2}$:   $\theta=2.4524$ [$\theta=2.4524$ is the **only solution**, as $0\le\theta\le\dfrac{3\pi}{2}$] Substitute $\theta=2.4524$ into $x=\dfrac{\theta}{2}-\sin\theta$ and $y=1-\cos\theta$: $x=\dfrac{2.4524}{2}-\sin 2.4524=0.59029=0.590$ (to 3 d.p.) $y=1-\cos 2.4524=1.77176=1.772$ (to 3 d.p.) Therefore, the required coordinates is $(0.590,\ 1.772)$.

Qn	Solution
9(iii) [4]	 Required area $=\int_0^{\frac{3\pi}{2}}(1-\cos\theta)\left(\frac{1}{2}-\cos\theta\right)\mathrm{d}\theta$ Required area $=\int_0^{\frac{3\pi}{2}}(1-\cos\theta)\left(\frac{1}{2}-\cos\theta\right)\mathrm{d}\theta$ $=\int_0^{\frac{3\pi}{2}}\left(\frac{1}{2}-\cos\theta-\frac{1}{2}\cos\theta+\cos^2\theta\right)\mathrm{d}\theta$ $=\int_0^{\frac{3\pi}{2}}\left(\frac{1}{2}-\frac{3}{2}\cos\theta+\cos^2\theta\right)\mathrm{d}\theta$ $=\int_0^{\frac{3\pi}{2}}\left(\frac{1}{2}-\frac{3}{2}\cos\theta+\frac{1+\cos 2\theta}{2}\right)\mathrm{d}\theta$ $=\int_0^{\frac{3\pi}{2}}\left(\frac{1}{2}-\frac{3}{2}\cos\theta+\frac{1}{2}+\frac{1}{2}\cos 2\theta\right)\mathrm{d}\theta$ $=\int_0^{\frac{3\pi}{2}}\left(1-\frac{3}{2}\cos\theta+\frac{1}{2}\cos 2\theta\right)\mathrm{d}\theta$ $=\left[\theta-\frac{3}{2}\sin\theta+\frac{1}{2}\left(\frac{\sin 2\theta}{2}\right)\right]_0^{\frac{3\pi}{2}}$ $=\left[\theta-\frac{3}{2}\sin\theta+\frac{1}{4}\sin 2\theta\right]_0^{\frac{3\pi}{2}}$ $=\left[\left(\frac{3\pi}{2}\right)-\frac{3}{2}\sin\frac{3\pi}{2}+\frac{1}{4}\sin 2\left(\frac{3\pi}{2}\right)\right]$ $-\left[(0)-\frac{3}{2}\sin 0+\frac{1}{4}\sin 2(0)\right]$ $=\left[\frac{3\pi}{2}-\frac{3}{2}(-1)+\frac{1}{4}(0)\right]-0$ $=\frac{3\pi}{2}+\frac{3}{2}$ units^2

Qn	Solution
10(i) **[3]**	Volume of the water dumbbell $=2\left[\pi r^2(2)\right]+2\left[\pi(3r)^2(y)\right]+\pi r^2(9)$ Since volume of the water dumbbell is k cm^3; $\Rightarrow 2\left[\pi r^2(2)\right]+2\left[\pi(3r)^2(y)\right]+\pi r^2(9)=k$ $4\pi r^2+2\pi\left(9r^2\right)(y)+9\pi r^2=k$ $13\pi r^2+18\pi yr^2=k$ $\Rightarrow 18\pi yr^2=k-13\pi r^2$ $\therefore y=\dfrac{k-13\pi r^2}{18\pi r^2}$ $A=34\pi r^2+26\pi r+12\pi ry$ $=34\pi r^2+26\pi r+12\pi r\left[\dfrac{k-13\pi r^2}{18\pi r^2}\right]$ $=34\pi r^2+26\pi r+\dfrac{2}{3r}\left(k-13\pi r^2\right)$ $=34\pi r^2+26\pi r+\dfrac{2k}{3r}-\dfrac{26\pi r^2}{3r}$ $=34\pi r^2+26\pi r+\dfrac{2k}{3r}-\dfrac{26\pi r}{3}$ $=34\pi r^2+\dfrac{52\pi r}{3}+\dfrac{2k}{3r}$ (Shown)
10(ii) **[2]**	$A=34\pi r^2+\dfrac{52\pi r}{3}+\dfrac{2k}{3r}$ $=34\pi r^2+\dfrac{52\pi}{3}r+\dfrac{2k}{3}r^{-1}$ $\dfrac{\mathrm{d}A}{\mathrm{d}r}=2(34)\pi r+\dfrac{52\pi}{3}+(-1)\dfrac{2k}{3}r^{-2}$ $=68\pi r+\dfrac{52\pi}{3}-\dfrac{2k}{3r^2}$ To minimise the external surface area, A: Let $\dfrac{\mathrm{d}A}{\mathrm{d}r}=0$ $\Rightarrow 68\pi r+\dfrac{52\pi}{3}-\dfrac{2k}{3r^2}=0$ Multiply throughout by $3r^2$: $204\pi r^3+52\pi r^2-2k=0$ Divide throughout by 2: $102\pi r^3+26\pi r^2-k=0$ (Shown)

<table>
<tr><th>Qn</th><th>Solution</th></tr>
<tr><td>10(iii)
[3]</td><td>
The volume of the dumbbell is 1500 cm^3:
From (ii), $102\pi r^3 + 26\pi r^2 - 1500 = 0$, let $k = 1500$.

$102\pi r^3 + 26\pi r^2 - 1500 = 0$
Using GC: $r = 1.5920$

Therefore, A is minimised when $r = 1.5920$.

Minimum value of A
$= 34\pi(1.5920)^2 + \dfrac{52\pi(1.5920)}{3} + \dfrac{2(1500)}{3(1.5920)}$
$= 985.55$
$= 986\text{ cm}^2$ (to 3 s.f.)

Using First Derivative Test:
<table>
<tr><td>r</td><td>1.5920^-
$[r = 1.5919]$</td><td>1.5920</td><td>1.5920^+
$[r = 1.5921]$</td></tr>
<tr><td>$\dfrac{\mathrm{d}A}{\mathrm{d}r}$</td><td>$< 0$
$[\frac{\mathrm{d}A}{\mathrm{d}r} = -0.081152]$</td><td>$0$</td><td>$> 0$
$[\frac{\mathrm{d}A}{\mathrm{d}r} = 0.060710]$</td></tr>
<tr><td>Slope</td><td>\</td><td>—</td><td>/</td></tr>
</table>
Therefore, when $r = 1.5920$, A is minimum.

Or using Second Derivative Test (via GC):
$\dfrac{\mathrm{d}^2A}{\mathrm{d}r^2} = \dfrac{\mathrm{d}}{\mathrm{d}r}\left(68\pi r + \dfrac{52\pi}{3} - \dfrac{2(1500)}{3r^2}\right)\Bigg|_{r=1.5920}$
$= 709.31 = 709$
Since $\dfrac{\mathrm{d}^2A}{\mathrm{d}r^2} = 709 > 0$, A is minimum when $r = 1.5920$.

Or using Second Derivative Test (via algebraic approach) [not recommended as it is more tedious]:
From earlier part:
</td></tr>
</table>

<table>
<tr><th>Qn</th><th>Solution</th></tr>
<tr><td></td><td>$\frac{\mathrm{d}A}{\mathrm{d}r}=68\pi r+\frac{52\pi}{3}-\frac{2k}{3r^2}$
$=68\pi r+\frac{52\pi}{3}-\frac{2(1500)}{3r^2}$
$=68\pi r+\frac{52\pi}{3}-1000r^{-2}$
$\frac{\mathrm{d}^2A}{\mathrm{d}r^2}=68\pi-(-2)1000r^{-3}$
$=68\pi+2000r^{-3}$
When $r=1.5920$,
$\frac{\mathrm{d}^2A}{\mathrm{d}r^2}=68\pi+2000(1.5920)^{-3}$
$=709.308=709$
Since $\frac{\mathrm{d}^2A}{\mathrm{d}r^2}=709>0$, A is minimum when $r=1.5920$.</td></tr>
<tr><td>10(iv)
[2]</td><td>Volume of water poured after 1 min
= 15 (60) = 900 cm^3

Volume of bottom cap
$=\pi(2)^2(2)=8\pi$ cm^3
Volume of bottom weight
$=\pi[3(2)]^2\times 7=252\pi$ cm^3
Volume of grip handle
$=\pi(2)^2\times 9=36\pi$ cm^3

Volume of bottom cap and bottom weight
$=8\pi+252\pi=816.814$ cm^3
Volume of bottom cap and bottom weight and grip handle
$=8\pi+252\pi+36\pi=929.911$ cm^3

Since $900\text{ cm}^3<929.91\text{ cm}^3$, the water level is <u>at the grip handle</u> after 1 minute.</td></tr>
<tr><td>10(v)
[2]</td><td>Let W = Volume of water in the grip handle and
h = depth of water from the base of the grip handle</td></tr>
</table>

Qn	Solution
	$W=\pi(2)^2 h=4\pi h$ $\Rightarrow \frac{\mathrm{d}W}{\mathrm{d}h}=4\pi$ Since water is poured into the empty dumbbell at a rate of 15 $\mathrm{cm}^3\ \mathrm{s}^{-1}$, this suggests that $\frac{\mathrm{d}W}{\mathrm{d}t}=15$. **Method 1**: $\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{\mathrm{d}W}{\mathrm{d}t}\times\frac{\mathrm{d}h}{\mathrm{d}W}$ $=15\times\frac{1}{4\pi}$ $=\frac{15}{4\pi}$ **Method 2**: $\frac{\mathrm{d}W}{\mathrm{d}t}=\frac{\mathrm{d}W}{\mathrm{d}h}\times\frac{\mathrm{d}h}{\mathrm{d}t}$ $15=4\pi\times\frac{\mathrm{d}h}{\mathrm{d}t}$ $\Rightarrow\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{15}{4\pi}$ Therefore, the depth of the water is increasing at a rate of $\frac{15}{4\pi}$ cm s^{-1} after 1 minute.

Qn	Solution
11a (i) [1]	Amount of money in the CDA at the end of 1st month $= 2(100)(1.012)=\$202.40$
11a (ii) [5]	(see table below) $200\left(1.012+1.012^2+\ldots+1.012^n\right)\geq 6000$ $200\left[\frac{1.012(1-1.012^n)}{1-1.012}\right]\geq 6000$

Month	Amount of money at start of month	Amount of money at end of month
1	$2(100)=200$	$200(1.012)$
2	$200+200(1.012)$	$\left(200+200(1.012)\right)(1.012)$ $=200\left(1.012+1.012^2\right)$
3	$200+200\left(1.012+1.012^2\right)$	$\left(200+200\left(1.012+1.012^2\right)\right)(1.012)$ $=200\left(1.012+1.012^2+1.012^3\right)$
..		
n	..	$200\left(1.012+1.012^2+\ldots+1.012^n\right)$

	Method 1

n	$200\left[\dfrac{1.012(1-1.012^n)}{1-1.012}\right]$
25	5860.3
26	6133.1
27	6409.1

Minimum $n = 26$

It occurs on February 2024. [n = 26 means 2 years 2 mths]

Method 2

$$\frac{1.012(1-1.012^n)}{1-1.012} \geq 30$$

$$1-1.012^n \leq -0.35573$$

$$n \geq \frac{\ln 1.35573}{\ln 1.012}$$

$$n \geq 25.51$$

Minimum $n = 26$

It occurs on February 2024. [n = 26 means 2 years 2 mths]

11a (iii) [3]

Let \$$m$ be the amount of money deposited from 1 Feb 22.

Month	Amount of money at the start of month	Amount of money at end of month
1	3000 + 3000 **Note**: no more co-matching thereafter	$6000(1.012)$
2	$m+6000(1.012)$	$(m+6000(1.012))(1.012)$ $= m(1.012)+6000(1.012)^2$
3	$m+m(1.012)$ $+6000(1.012)^2$	$\left[\begin{array}{l} m+m(1.012)+ \\ \quad 6000(1.012)^2 \end{array}\right](1.012)$ $= m(1.012+1.012^2)+6000(1.012)^3$
..		
12	..	$m(1.012+1.012^2+...+1.012^{11})$ $+6000(1.012)^{12}$

Note: total of 12 months

$$m(1.012+1.012^2+1.012^{11})+6000(1.012)^{12} \geq 10000$$

$$m\left(\frac{1.012(1-1.012^{11})}{1-1.012}\right)+6000(1.012)^{12} \geq 10000$$

$$m \geq 260.19017$$

Minimum amount is \$260.20

11(b)
[4]

mth	Amount of money in the account at start of month (**excludes** interest)	Amount of interest earned at end of month
1	200	x
2	200 + 200 = 200(2)	$2x$
3	$200(3)$	$3x$
..	..	..
24	$200(24)$	$24x$

Total amount of money in bank

$$= 200(24) + x + 2x + 3x + ... + 24x$$

$$= 4800 + x(1 + 2 + 3 + ... + 24)$$

$$= 4800 + x\left(\frac{24}{2}\right)(1 + 24)$$

Alternatively,

$$x\left(\frac{24}{2}\right)\left[2(1) + (24-1)1\right]$$

$$= 4800 + 300x$$

Answer: $(4800 + 300$x$)