2022 J2 H2 Mathematics Preliminary Examination P1 (Worked Solutions)

<table>
<tr><th>Qn</th><th>Solution</th><th>Notes</th></tr>
<tr><td rowspan="3">1</td><td>$a+b+c+d=-19----(1)$
$a(-1)^3+b(-1)^2+c(-1)+d=13$
$-a+b-c+d=13----(2)$

$\mathrm{f}'(x)=3ax^2+2bx+c$
$0=3a(1)+2b(-1)+c$
$3a-2b+c=0----(3)$</td><td></td></tr>
<tr><td>$\int_{-1}^{0} ax^3+bx^2+cx+d \ \mathrm{d}x=9.5$
$\left[\frac{ax^4}{4}+\frac{bx^3}{3}+\frac{cx^2}{2}+dx\right]_{-1}^{0}=9.5$
$0-\left(\frac{1}{4}a-\frac{1}{3}b+\frac{1}{2}c-d\right)=9.5$
$-\frac{1}{4}a+\frac{1}{3}b-\frac{1}{2}c+d=9.5----(4)$</td><td></td></tr>
<tr><td>Using GC,
$a=2, \quad b=-6 \quad c=-18, \quad d=3$</td><td></td></tr>
</table>

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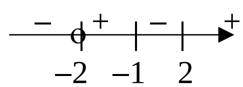
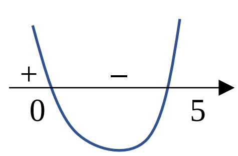
Qn	Solution	Notes
2(i)	$\dfrac{3x^2+9x+10}{x+2} \geq 2x+6$ $\dfrac{3x^2+9x+10-(2x+6)(x+2)}{x+2} \geq 0$ $\dfrac{3x^2+9x+10-(2x^2+10x+12)}{x+2} \geq 0$ $\dfrac{x^2-x-2}{x+2} \geq 0$ $\dfrac{(x-2)(x+1)}{x+2} \geq 0$  $-2 < x \leq -1$ or $x \geq 2$	
(ii)	$\mathrm{f}(x) = \dfrac{ax^2+3ax+10}{x+2}$ $\mathrm{f}'(x) = \dfrac{(x+2)(2ax+3a)-(ax^2+3ax+10)}{(x+2)^2}$ $= \dfrac{ax^2+4ax+6a-10}{(x+2)^2}$	
	$\dfrac{ax^2+4ax+6a-10}{(x+2)^2} > 0$ Since $(x+2)^2 > 0$ for all real values of *x*, $x \neq -2$ We need $ax^2+4ax+6a-10 > 0$ for all real values of *x*, $x \neq -2$ $a > 0$ & Discriminant < 0 $(4a)^2 - 4a(6a-10) < 0$ $16a^2 - 24a^2 + 40a < 0$ $8a^2 - 40a > 0$ $(a)(a-5) > 0$  $a < 0$ or $a > 5$ (N.A. since $a > 0$)	

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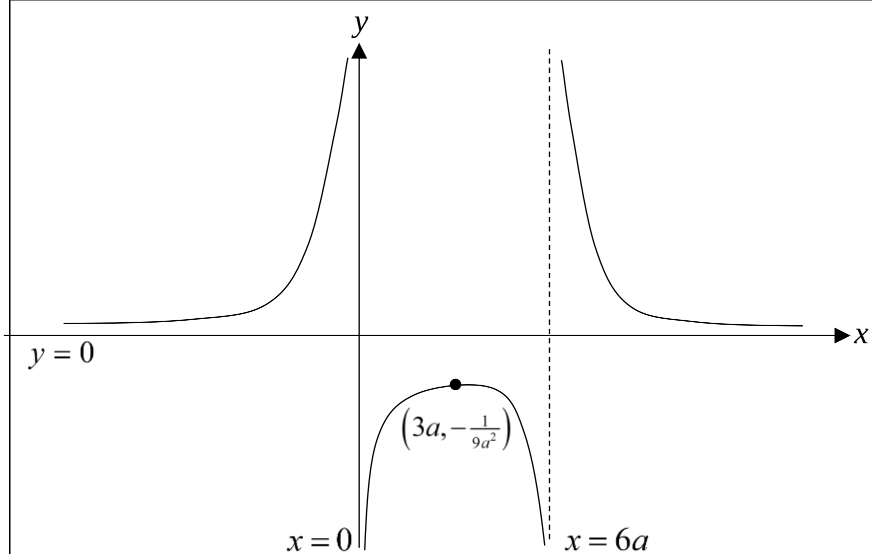
<table>
<tr><th>Qn</th><th>Solution</th><th>Notes</th></tr>
<tr><td rowspan="2">3(i)</td><td>Asymptotes: $y=0$, $x=0$, $x=6a$
$\frac{\mathrm{d}y}{\mathrm{d}x}=-\left(x^2-6ax\right)^{-2}\left(2x-6a\right)=-\frac{2x-6a}{\left(x^2-6ax\right)^2}=0$
$2x-6a=0$
$x=3a$
$y=-\frac{1}{9a^2}$</td><td></td></tr>
<tr><td></td><td></td></tr>
<tr><td rowspan="2">(ii)</td><td>$y=\frac{1}{x^2-6ax}=\frac{1}{(x-3a)^2-9a^2}$</td><td rowspan="2"></td></tr>
<tr><td>$y=\frac{1}{(x-3a)^2-9a^2}\rightarrow y=\frac{1}{x^2-9a^2}$
Replace x by $x+3a$
<u>Translation</u> of $y=\frac{1}{x^2-6ax}$ by <u>$3a$ units</u> in the <u>negative x-direction</u> to get $y=\frac{1}{x^2-9a^2}$.</td></tr>
</table>

	Alternatively, $y=\dfrac{1}{x^2-6ax}=\dfrac{1}{x(x-6a)}$ $y=\dfrac{1}{x^2-9a^2}=\dfrac{1}{(x+3a)(x-3a)}$	
	$y=\dfrac{1}{x(x-6a)}\to y=\dfrac{1}{(x+3a)(x-3a)}$ Replace x by $x+3a$ Translation of $y=\dfrac{1}{x^2-6ax}$ by $3a$ units in the negative x-direction to get $y=\dfrac{1}{x^2-9a^2}$.	

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Qn	Solutions	Notes
4(i)	$\frac{1}{(2r+1)(2r+3)}=\frac{A}{2r+1}+\frac{B}{2r+3}=\frac{A(2r+3)+B(2r+1)}{(2r+1)(2r+3)}$ Subs $r=-\frac{3}{2}$ $\quad B=-\frac{1}{2}$ $r=-\frac{1}{2}$ $\quad A=\frac{1}{2}$ $\frac{1}{(2r+1)(2r+3)}=\frac{\frac{1}{2}}{(2r+1)}-\frac{\frac{1}{2}}{(2r+3)}$ $=\frac{1}{2(2r+1)}-\frac{1}{2(2r+3)}$ $=\frac{1}{2}\left[\frac{1}{(2r+1)}-\frac{1}{(2r+3)}\right]$	
(ii)	$\sum_{r=1}^{n}\frac{1}{(2r+1)(2r+3)}=\frac{1}{2}\sum_{r=1}^{n}\left[\frac{1}{2r+1}-\frac{1}{2r+3}\right]$ $=\frac{1}{2}\begin{bmatrix} & \frac{1}{3} & - & \cancel{\frac{1}{5}} \\ + & \cancel{\frac{1}{5}} & - & \cancel{\frac{1}{7}} \\ + & \cancel{\frac{1}{7}} & - & \cancel{\frac{1}{9}} \\ & & \cdot & \\ & & \cdot & \\ & & \cdot & \\ + & \cancel{\frac{1}{2n-1}} & - & \cancel{\frac{1}{2n+1}} \\ + & \cancel{\frac{1}{2n+1}} & - & \frac{1}{2n+3} \end{bmatrix}$ $=\frac{1}{2}\left[\frac{1}{3}-\frac{1}{2n+3}\right]$ $=\frac{1}{6}-\frac{1}{2}\left(\frac{1}{2n+3}\right)$	

(iii)	$\frac{1}{(3)(5)}+\frac{1}{(5)(7)}+\frac{1}{(7)(9)}+\frac{1}{(9)(11)}+\frac{1}{(11)(13)}+\ldots$ $=\sum_{r=1}^{\infty}\frac{1}{(2r+1)(2r+3)}$ As $n\to\infty$, $\frac{1}{2n+3}\to 0$ $\sum_{r=1}^{n}\frac{1}{(2r+1)(2r+3)}\to\frac{1}{6}$ (finite number) Hence the series converges. $\therefore S_\infty=\frac{1}{6}$	
(iv)	$\frac{1}{(10)(14)}+\frac{1}{(14)(18)}+\frac{1}{(18)(22)}+\frac{1}{(22)(26)}+\ldots$ $=\frac{1}{4}\left[\frac{1}{(5)(7)}+\frac{1}{(7)(9)}+\frac{1}{(9)(11)}+\frac{1}{(11)(13)}\ldots\right]$ $=\frac{1}{4}\sum_{r=2}^{\infty}\frac{1}{(2r+1)(2r+3)}$ $=\frac{1}{4}\left[S_\infty-S_1\right]$ $=\frac{1}{4}\left[\frac{1}{6}-\frac{1}{(3)(5)}\right]$ $=\frac{1}{40}$	

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Qn	Solution	Notes
5(a)	$\int 3\sin x\cos 3x \text{ d}x = \int 3\cos 3x\sin x \text{ d}x$ $= \frac{3}{2}\int(\sin 4x - \sin 2x) \text{ d}x$ $= -\frac{3}{8}\cos 4x + \frac{3}{4}\cos 2x + c$	
(b)	$\theta = \sqrt{x} \Rightarrow \theta^2 = x$ $2\theta\frac{\text{d}\theta}{\text{d}x} = 1$ when $\theta = \sqrt{\frac{\pi}{4}}$, $x = \frac{\pi}{4}$; when $\theta = \sqrt{\pi}$, $x = \pi$	
	$\int_{\sqrt{\frac{\pi}{4}}}^{\sqrt{\pi}} \theta^3 \sin(\theta^2) \text{ d}\theta$ $= \frac{1}{2}\int_{\sqrt{\frac{\pi}{4}}}^{\sqrt{\pi}} 2\theta(\theta^2)\sin(\theta^2) \text{ d}\theta$ $= \frac{1}{2}\int_{\frac{\pi}{4}}^{\pi} x\sin x \text{ d}x$ $= \frac{1}{2}\left[[-x\cos x]_{\frac{\pi}{4}}^{\pi} - \int_{\frac{\pi}{4}}^{\pi}(-\cos x) \text{ d}x\right]$ $= \frac{1}{2}\left(\pi + \frac{\pi}{4\sqrt{2}} + [\sin x]_{\frac{\pi}{4}}^{\pi}\right)$ $= \frac{1}{2}\left(\pi + \frac{\pi}{4\sqrt{2}} - \frac{1}{\sqrt{2}}\right)$ $u = x$, $\frac{\text{d}v}{\text{d}x} = \sin x$ $\Rightarrow \frac{\text{d}u}{\text{d}x} = 1 \Rightarrow v = -\cos x$	

Qn	Solution	Notes
6(i)	$u = xy$ $\dfrac{\mathrm{d}u}{\mathrm{d}x} = x\dfrac{\mathrm{d}y}{\mathrm{d}x} + y$ Substituting into $x\dfrac{\mathrm{d}y}{\mathrm{d}x} + y - 2\left(xy\right)^2 = 0$ $\dfrac{\mathrm{d}u}{\mathrm{d}x} - 2u^2 = 0 \quad \Rightarrow \quad \dfrac{\mathrm{d}u}{\mathrm{d}x} = 2u^2$	
(ii)	$\dfrac{\mathrm{d}u}{\mathrm{d}x} = 2u^2$ $\displaystyle\int \frac{1}{u^2}\,\mathrm{d}u = \int 2\,\mathrm{d}x$ $\Rightarrow -\dfrac{1}{u} = 2x + c$, where c is an arbitrary constant $y = -\dfrac{1}{x(2x+c)}$	
(iii)	When $x = 1$, $y = \dfrac{1}{2}$, $c = -4$ $y = -\dfrac{1}{x(2x-4)} = -\dfrac{1}{2x(x-2)}$	
(iv)	Curve has no stationary point when $c = 0$ i.e. $y = -\dfrac{1}{2x^2}$	

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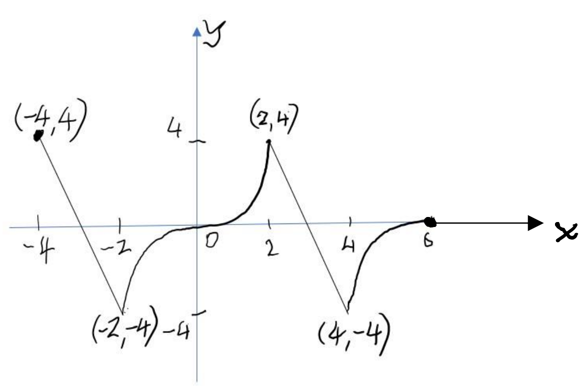
Qn	Solution	Notes
7(i)	$2x^3+y^3-3xy=k$ Differentiate w.r.t. x, $6x^2+3y^2\frac{\mathrm{d}y}{\mathrm{d}x}-3x\frac{\mathrm{d}y}{\mathrm{d}x}-3y=0$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{3y-6x^2}{3y^2-3x}=\frac{y-2x^2}{y^2-x}$	
(ii)	When tangent is parallel to the y-axis, $y^2-x=0$	
	Sub $x=y^2$ into $2x^3+y^3-3xy=k$ $2(y^2)^3+y^3-3(y^2)y=k$ $2y^6-2y^3-k=0$ i.e. a = 2, b = –2	
(iii)	line x = 1 is a tangent to the curve C $y^2=1$ $y=\pm 1$	
	When $y=1$, $2-2-k=0$ $k=0$	
	When $y=-1$, $2+2-k=0$ $k=4$	

Qn	Solution	Notes
8	$y=\dfrac{\ln\sqrt{1-x}}{2+x} \quad\Rightarrow\quad y(2+x)=\dfrac{1}{2}\ln(1-x)$ Differentiate w.r.t. x $y+(2+x)\dfrac{\mathrm{d}y}{\mathrm{d}x}=-\dfrac{1}{2(1-x)}$ $2y+2(2+x)\dfrac{\mathrm{d}y}{\mathrm{d}x}+\dfrac{1}{1-x}=0$	
(i)	Differentiate w.r.t. x $2\dfrac{\mathrm{d}y}{\mathrm{d}x}+2\dfrac{\mathrm{d}y}{\mathrm{d}x}+2(2+x)\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+\dfrac{1}{(1-x)^2}=0$ $4\dfrac{\mathrm{d}y}{\mathrm{d}x}+2(2+x)\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+\dfrac{1}{(1-x)^2}=0$	
	Differentiate w.r.t. x $4\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+2\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+2(2+x)\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}+\dfrac{2}{(1-x)^3}=0$	
	when $x=0$, $\mathrm{f}(0)=0$, $\mathrm{f}'(0)=-\dfrac{1}{4}$, $\mathrm{f}''(0)=0$, $\mathrm{f}'''(0)=-\dfrac{1}{2}$	
	By Maclaurin's series, $y=0+\left(-\dfrac{1}{4}\right)x+(0)\dfrac{x^2}{2!}+\left(-\dfrac{1}{2}\right)\dfrac{x^3}{3!}+\ldots$ $\approx-\dfrac{1}{4}x-\dfrac{1}{12}x^3$	
(ii)	$y=\dfrac{\ln\sqrt{1-x}}{2+x}=\dfrac{1}{2}\ln(1-x)\cdot\dfrac{1}{2}\left(1+\dfrac{x}{2}\right)^{-1}$ $=\dfrac{1}{2}\ln(1-x)\cdot\dfrac{1}{2}\left(1+\dfrac{x}{2}\right)^{-1}$ $=\dfrac{1}{4}\left(-x-\dfrac{x^2}{2}-\dfrac{x^3}{3}+\ldots.\right)\left(1-\dfrac{x}{2}+\dfrac{x^2}{4}-\dfrac{x^3}{8}+\ldots\right)$ $=\dfrac{1}{4}\left(-x+\dfrac{x^2}{2}-\dfrac{x^3}{4}-\dfrac{x^2}{2}+\dfrac{x^3}{4}-\dfrac{x^3}{3}+\ldots.\right)$ $\approx-\dfrac{1}{4}x-\dfrac{1}{12}x^3$	

Qn	Solution	
(iii)	When $x = -1$, LHS $= \ln\sqrt{2} = \frac{1}{2}\ln 2$ RHS $= -\frac{1}{4}(-1) - \frac{1}{12}(-1)^3 = \frac{1}{3}$ $= 0.333333$ $\therefore$ $\ln 2 = 2(0.333333) \approx 0.6667$ (4 d.p.)	

Qn	Solutions	Notes
9(a)(i)	$y = \frac{x-2}{x+2}$ Making x the subject, $x = \frac{2+2y}{1-y}$ $\mathrm{f}^{-1}(x) = \frac{2+2x}{1-x}, \quad x \neq 1$	
(ii)	$(\mathrm{fg})(x) = \mathrm{f}\left(\mathrm{g}(x)\right)$ $= \mathrm{f}\,(-x^2)$ $= \frac{-x^2-2}{-x^2+2}$	
	Method 1: Let $(\mathrm{fg})^{-1}(3) = p$ $\therefore (\mathrm{fg})(p) = 3$ $\frac{-p^2-2}{-p^2+2} = 3$ $-p^2 - 2 = -3p^2 + 6$	
	$p^2 = 4$ $p = \pm 2$ Reject $p = 2$ Hence $(\mathrm{fg})^{-1}(3) = -2$	

	Method 2: Let $y=\dfrac{-x^2-2}{-x^2+2}$ $-yx^2+2y=-x^2-2$ $x^2(1-y)=-2(y+1)$ $x^2=\dfrac{2(y+1)}{y-1}$ $x=\pm\sqrt{\dfrac{2(y+1)}{y-1}}$, reject $\sqrt{\dfrac{2(y+1)}{y-1}}$ since $x<-\sqrt{2}$ $(\text{fg})^{-1}(x)=-\sqrt{\dfrac{2(x+1)}{x-1}}$ $(\text{fg})^{-1}(3)=-2$	
	Let $y=\dfrac{-x^2-2}{-x^2+2}$ $-yx^2+2y=-x^2-2$ $x^2(1-y)=-2(y+1)$ $x^2=\dfrac{2(y+1)}{y-1}$ $x=\pm\sqrt{\dfrac{2(y+1)}{y-1}}$, reject $\sqrt{\dfrac{2(y+1)}{y-1}}$ since $x<-\sqrt{2}$ $(\text{fg})^{-1}(x)=-\sqrt{\dfrac{2(y+1)}{y-1}}$ $(\text{fg})^{-1}(3)=-2$	
(b)(i)	$\text{h}(-4) = -4(-4)-12=4$ h(12) = h(6) = h(0) = 0	
(ii)	 The line $y = 0$ cuts $y=\text{h}(x)$ at two or more points. Hence h does not have an inverse.	

<table>
<tr><th>Qn</th><th>Solutions</th><th>Notes</th></tr>
<tr><td>10(i)</td><td>At the start of month 1, customer owes $15 000
At mid month 1, customer owes $15 000 − x
At the end of month 1, he owes 1.04 (15000 − x)

$15000 = 1.04\,(15000 - x)$

$x = 576.923 \approx \$576.92$</td><td></td></tr>
<tr><td>(ii)</td><td>
<table>
<tr><th>Month</th><th>Amount owed in the middle of the month</th><th>Amount owed on the last day of the month</th></tr>
<tr><td>1</td><td>$15000 - x$</td><td>$1.04\,(15000 - x)$</td></tr>
<tr><td>2</td><td>$1.04\,(15000 - x) - x$</td><td>$1.04[1.04\,(15000 - x) - x]$
$= (1.04)^2(15000) - (1.04)^2x - 1.04x$</td></tr>
<tr><td>3</td><td>$(1.04)^2(15000) - (1.04)^2x - 1.04x - x$</td><td>$(1.04)^3(15000) - (1.04)^3x - (1.04)^2x - 1.04x$</td></tr>
</table>

At the nth month, the customer owes

$(1.04)^n(15000) - (1.04)^nx - (1.04)^{n-1}x - \ldots(1.04)x$
$= (1.04)^n(15000) - x[1.04 + 1.04^2 + \ldots + 1.04^n]$

$= (1.04)^n(15000) - \dfrac{x(1.04)[(1.04)^n - 1]}{0.04}$
$= (1.04)^n(15000) - 26\,x\,(1.04^n - 1)$

$k = 26$</td><td></td></tr>
<tr><td>(iii)</td><td>From
$(1.04)^n(15000) - 26\,x\,(1.04^n - 1)$

Substituting n = 12
$(1.04)^{12}(15000) - 26\,(1000)\,(1.04^{12} - 1)$
= $8388.65</td><td></td></tr>
</table>

(iv)	Using the amount owed at the end of the nth month, $(1.04)^n(15000)-26x\left[(1.04)^n-1\right]\le 0$ $(1.04)^n(15000)-26(800)\left[(1.04)^n-1\right]\le 0$ $n\ge 32.56$ Least number of months = 33	

Qn	Solution	Notes
11(i)	$\overrightarrow{AB}=\begin{pmatrix}-3\\-2\\2\end{pmatrix}-\begin{pmatrix}2\\-2\\10\end{pmatrix}=\begin{pmatrix}-5\\0\\-8\end{pmatrix}=-\begin{pmatrix}5\\0\\8\end{pmatrix}$ $\overrightarrow{BE}=\begin{pmatrix}-3\\2\\2\end{pmatrix}-\begin{pmatrix}-3\\-2\\2\end{pmatrix}=\begin{pmatrix}0\\4\\0\end{pmatrix}=4\begin{pmatrix}0\\1\\0\end{pmatrix}$ A normal to the plane $\begin{pmatrix}5\\0\\8\end{pmatrix}\times\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}-8\\0\\5\end{pmatrix}$	
	Plane *ABED* $\mathbf{r}\bullet\begin{pmatrix}-8\\0\\5\end{pmatrix}=\begin{pmatrix}2\\-2\\10\end{pmatrix}\bullet\begin{pmatrix}-8\\0\\5\end{pmatrix}=34$ $\mathbf{r}\bullet\begin{pmatrix}-8\\0\\5\end{pmatrix}=34$	

(ii)	$\overrightarrow{ST}=\begin{pmatrix}-0.5\\0\\6\end{pmatrix}-\begin{pmatrix}-8\\0\\4\end{pmatrix}=\begin{pmatrix}7.5\\0\\2\end{pmatrix}$ $\cos\theta=\dfrac{\left\lvert\begin{pmatrix}7.5\\0\\2\end{pmatrix}\bullet\begin{pmatrix}-8\\0\\5\end{pmatrix}\right\rvert}{\sqrt{7.5^2+2^2}\sqrt{8^2+5^2}}$ $=\dfrac{50}{\sqrt{\frac{241}{4}}\sqrt{89}}=0.682880$ $\theta=46.9^\circ$	
(iii)	$\overrightarrow{BC}=\begin{pmatrix}7\\-2\\2\end{pmatrix}-\begin{pmatrix}-3\\-2\\2\end{pmatrix}=\begin{pmatrix}10\\0\\0\end{pmatrix}=10\begin{pmatrix}1\\0\\0\end{pmatrix}$ Line *TU* $\mathbf{r}=\begin{pmatrix}-0.5\\0\\6\end{pmatrix}+\lambda\begin{pmatrix}1\\0\\0\end{pmatrix},\lambda\in\mathbb{R}$	
	Plane *ADFC* is given by $\mathbf{r}\bullet\begin{pmatrix}8\\0\\5\end{pmatrix}=66$. Coordinates of *U*: $\left[\begin{pmatrix}-0.5\\0\\6\end{pmatrix}+\lambda\begin{pmatrix}1\\0\\0\end{pmatrix}\right]\bullet\begin{pmatrix}8\\0\\5\end{pmatrix}=66$ $-4+8\lambda+30=66$ $\lambda=5$ $\overrightarrow{OU}=\begin{pmatrix}-0.5\\0\\6\end{pmatrix}+5\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}4.5\\0\\6\end{pmatrix}$ $U\left(4.5,0,6\right)$	

(iv)	Plane *ADFC* is given by $\mathbf{r} \bullet \begin{pmatrix} 8 \\ 0 \\ 5 \end{pmatrix} = 66$. $\overrightarrow{TC} = \overrightarrow{OC} - \overrightarrow{OT} = \begin{pmatrix} 7 \\ -2 \\ 2 \end{pmatrix} - \begin{pmatrix} -0.5 \\ 0 \\ 6 \end{pmatrix} = \begin{pmatrix} 15/2 \\ -2 \\ -4 \end{pmatrix}$ Shortest distance $= \left\lvert \overrightarrow{TC} \bullet \mathbf{n} \right\rvert = \dfrac{\left\lvert \begin{pmatrix} 15/2 \\ -2 \\ -4 \end{pmatrix} \bullet \begin{pmatrix} 8 \\ 0 \\ 5 \end{pmatrix} \right\rvert}{\sqrt{8^2 + 5^2}}$ $= \dfrac{40}{\sqrt{89}}$	