2022 J2 H2 Mathematics Preliminary Examination P2 (Worked Solutions)

Qn	Solution	Notes
1(a) **(i)**	The graph of $y = \mathrm{f}(x)$ is decreasing when $\mathrm{f}'(x) < 0$. $x < -3$ or $0 < x < 4$	
(ii)	The graph of $y = \mathrm{f}(x)$ is increasing and concave downwards when $\mathrm{f}'(x) > 0$ and $\mathrm{f}'(x)$ is decreasing $-2 < x < 0$	
(b) **(i)**	$\frac{256}{3}\pi = \frac{4}{3}\pi r^3 \quad \Rightarrow \quad r = 4$ $\frac{\mathrm{d}V}{\mathrm{d}t} = 4\pi r^2 \frac{\mathrm{d}r}{\mathrm{d}t}$ When $r = 4$, $12\pi = 4\pi(4)^2 \frac{\mathrm{d}r}{\mathrm{d}t}$ $\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{3}{16}$ $A = 4\pi r^2 \quad \Rightarrow \quad \frac{\mathrm{d}A}{\mathrm{d}r} = 8\pi r$ $\frac{\mathrm{d}A}{\mathrm{d}t} = 8\pi r \frac{\mathrm{d}r}{\mathrm{d}t}$ $= 8\pi(4)\frac{3}{16}$ $= 6\pi$ cm^2/s	
(ii)	Since $\frac{\mathrm{d}V}{\mathrm{d}t} = 4\pi r^2 \frac{\mathrm{d}r}{\mathrm{d}t}$ $12\pi = A\frac{\mathrm{d}r}{\mathrm{d}t}$ $\frac{\mathrm{d}r}{\mathrm{d}t} = 12\pi\left(\frac{1}{A}\right)$ The rate of increase of balloon's radius is inversely proportional to the area of the balloon.	

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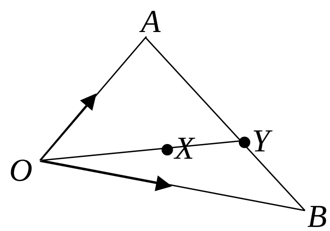
2(a)	O, X and Y are collinear,  $\overrightarrow{OY} = \alpha\overrightarrow{OX} = \alpha\left(\frac{1}{8}\mathbf{a} + \frac{3}{8}\mathbf{b}\right)$ --(1) $\overrightarrow{AY} = \beta\overrightarrow{AB}$ $\overrightarrow{OY} - \mathbf{a} = \beta(\mathbf{b} - \mathbf{a})$ $\overrightarrow{OY} = \beta\mathbf{b} + (1-\beta)\mathbf{a}$ ---(2) $\frac{\alpha}{8}\mathbf{a} + \frac{3\alpha}{8}\mathbf{b} = \beta\mathbf{b} + (1-\beta)\mathbf{a}$ Comparing, $\frac{\alpha}{8} = 1 - \beta$ ---(3) $\frac{3\alpha}{8} = \beta$ ---(4) Solving (3) and (4) $\alpha = 2, \quad \beta = \frac{3}{4}$ $\overrightarrow{OY} = 2\left(\frac{1}{8}\mathbf{a} + \frac{3}{8}\mathbf{b}\right) = \frac{1}{4}\mathbf{a} + \frac{3}{4}\mathbf{b}$ $\overrightarrow{AY} = \frac{3}{4}\overrightarrow{AB}$ $AY : YB = 3:1$	
(b)(i)	$(\mathbf{r} - \mathbf{p}) \times \mathbf{q} = \mathbf{0}$ $(\mathbf{r} - \mathbf{p}) // \mathbf{q}$ $(\mathbf{r} - \mathbf{p}) = k\mathbf{q}, k \in \mathbb{R}$ $\mathbf{r} = \mathbf{p} + k\mathbf{q}, k \in \mathbb{R}$ R represents the points on the line containing the point P and parallel to $\mathbf{q}$.	
(ii)	$(\mathbf{r} - \mathbf{p}) \bullet \mathbf{q} = 0$ $\mathbf{r} \bullet \mathbf{q} - \mathbf{p} \bullet \mathbf{q} = 0$ $\mathbf{r} \bullet \mathbf{q} = \mathbf{p} \bullet \mathbf{q}$ R represents the points on the plane that is perpendicular to $\mathbf{q}$ and containing the point P.	

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Qn	Solutions	Notes
3(a)	Since all the coefficients of the polynomial are real, another root is $1-\mathrm{i}$ The quadratic factor is $\left[z-(1+\mathrm{i})\right]\left[z-(1-\mathrm{i})\right]$ $=\left[(z-1)-\mathrm{i}\right]\left[(z-1)+\mathrm{i}\right]$ $=z^2-2z+2$	
	$z^4+4z^2-8z+12=(z^2-2z+2)(z^2+2z+6)$ The other quadratic factor is z^2+2z+6 Let $z^2+2z+6=0$ $z=\dfrac{-2\pm\sqrt{4-24}}{2}$ $=-1\pm\sqrt{5}\mathrm{i}$	
	Hence the other roots are $-1+\sqrt{5}\mathrm{i},\ -1-\sqrt{5}\mathrm{i}$ and $1-\mathrm{i}$	
3 (b)	**Method 1:** $\dfrac{1+\mathrm{i}\cos\frac{3\pi}{8}+\sin\frac{3\pi}{8}}{1-\mathrm{i}\cos\frac{3\pi}{8}+\sin\frac{3\pi}{8}}=\dfrac{1+\mathrm{i}\left(\cos\frac{3\pi}{8}-\mathrm{i}\sin\frac{3\pi}{8}\right)}{1-\mathrm{i}\left(\cos\frac{3\pi}{8}+\mathrm{i}\sin\frac{3\pi}{8}\right)}$ $=\dfrac{1+\mathrm{i}\mathrm{e}^{-\frac{3\pi}{8}\mathrm{i}}}{1-\mathrm{i}\mathrm{e}^{\frac{3\pi}{8}\mathrm{i}}}$ $=\dfrac{1+\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}\mathrm{e}^{-\frac{3\pi}{8}\mathrm{i}}}{1+\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\mathrm{e}^{\frac{3\pi}{8}\mathrm{i}}}$ $=\dfrac{1+\mathrm{e}^{\frac{\pi}{8}\mathrm{i}}}{1+\mathrm{e}^{-\frac{\pi}{8}\mathrm{i}}}$ $=\dfrac{\left(1+\mathrm{e}^{\frac{\pi}{8}\mathrm{i}}\right)}{\mathrm{e}^{-\frac{\pi}{8}\mathrm{i}}\left(1+\mathrm{e}^{\frac{\pi}{8}\mathrm{i}}\right)}$ $=\mathrm{e}^{\frac{\pi}{8}\mathrm{i}}$ $=\cos\frac{\pi}{8}+\mathrm{i}\sin\frac{\pi}{8}$	

Qn	Solution	Notes
3 (b)	**Method 2:** $\dfrac{1+\sin\frac{3\pi}{8}+\mathrm{i}\cos\frac{3\pi}{8}}{1+\sin\frac{3\pi}{8}-\mathrm{i}\cos\frac{3\pi}{8}}=\dfrac{1+\cos\frac{\pi}{8}+\mathrm{i}\sin\frac{\pi}{8}}{1+\cos\frac{\pi}{8}-\mathrm{i}\sin\frac{\pi}{8}}$ $=\dfrac{1+\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{8}\right)}}{1+\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{8}\right)}}\bullet\dfrac{\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{8}\right)}}{\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{8}\right)}}$	
	$=\dfrac{\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{8}\right)}+\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{8}\right)}\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{8}\right)}}{\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{8}\right)}+1}$ $=\dfrac{\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{8}\right)}\left(1+\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{8}\right)}\right)}{\left(1+\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{8}\right)}\right)}$ $=\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{8}\right)}$ $=\cos\frac{\pi}{8}+\mathrm{i}\sin\frac{\pi}{8}$	
	$\left(\dfrac{1+\sin\frac{3\pi}{8}+\mathrm{i}\cos\frac{3\pi}{8}}{1+\sin\frac{3\pi}{8}-\mathrm{i}\cos\frac{3\pi}{8}}\right)^n=\cos\frac{n\pi}{8}+\mathrm{i}\sin\frac{n\pi}{8}=\mathrm{i}$ Comparing real and imaginary parts, $\cos\frac{n\pi}{8}=0$ and $\sin\frac{n\pi}{8}=1$	
	For $\cos\frac{n\pi}{8}=0$ $\frac{n\pi}{8}=\frac{\pi}{2},\frac{3\pi}{2},\frac{5\pi}{2},\frac{7\pi}{2},...$ n = 4, 12, 20, 28…	
	For $\sin\frac{n\pi}{8}=1$, When n = 4, $\sin\frac{\pi}{2}=1$ n = 12, $\sin\frac{3\pi}{2}=-1$ n =20, $\sin\frac{5\pi}{2}=1$ The two smallest positive integer values of n are 4 and 20.	

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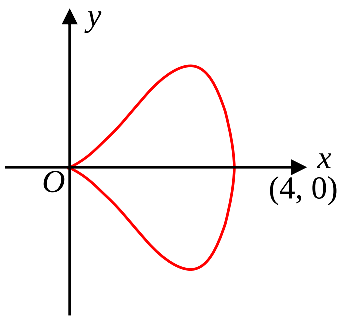
<table>
<tr><th>Qn</th><th>Solution</th><th>Notes</th></tr>
<tr><td>4(i)</td><td>
</td><td></td></tr>
<tr><td>(ii)</td><td>$x=2+2\sin\theta,\qquad y=2\cos\theta+\sin 2\theta$
$\dfrac{\mathrm{d}x}{\mathrm{d}\theta}=2\cos\theta$
when $x = 0$, $\sin\theta=-1 \quad\Rightarrow\quad \theta=-\dfrac{\pi}{2}$
when $x = 4$, $\sin\theta=1 \quad\Rightarrow\quad \theta=\dfrac{\pi}{2}$</td><td></td></tr>
<tr><td></td><td>Area bounded by the curve
$=2\int_0^4 y\,\mathrm{d}x$
$=2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(2\cos\theta+\sin 2\theta)(2\cos\theta)\,\mathrm{d}\theta$</td><td></td></tr>
<tr><td></td><td><u>Method 1</u>
$=2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(2\cos 2\theta+2+4\sin\theta\cos^2\theta)\,\mathrm{d}\theta$
$=2\left[\sin 2\theta+2\theta-\dfrac{4\cos^3\theta}{3}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$
$=4\pi$</td><td></td></tr>
<tr><td></td><td><u>Method 2</u>
$=2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(2\cos 2\theta+2+\sin 3\theta+\sin\theta)\,\mathrm{d}\theta$
$=2\left[\sin 2\theta+2\theta-\dfrac{1}{3}\cos 3\theta-\cos\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$
$=4\pi$</td><td></td></tr>
</table>

Qn	Solution	Notes
4(iii)	$\text{RHS} = (2+2\sin\theta)\cos\theta$ $= 2\cos\theta + 2\sin\theta\cos\theta$ $= 2\cos\theta + \sin 2\theta = y = \text{LHS}$	
	From $x = 2+2\sin\theta$ $\Rightarrow \sin\theta = \dfrac{x-2}{2}$ $y = x\cos\theta \quad \Rightarrow \cos\theta = \dfrac{y}{x}$ Since $\sin^2\theta + \cos^2\theta = 1$ $\left(\dfrac{x-2}{2}\right)^2 + \left(\dfrac{y}{x}\right)^2 = 1$ $y^2 = x^2\left[1-\left(\dfrac{x-2}{2}\right)^2\right]$ $y^2 = x^2\left[\dfrac{4-(x-2)^2}{4}\right]$ $4y^2 = x^2\left[4-(x^2-4x+4)\right]$ $4y^2 = 4x^3 - x^4$	
(iv)	Volume of solid generated $= \pi\int_0^4 \left(x^3 - \dfrac{1}{4}x^4\right) \mathrm{d}x$ $= \dfrac{64\pi}{5}$ or 40.2	

<table>
<tr><th>Qn</th><th>Solution</th><th>Notes</th></tr>
<tr><td>5(i)</td><td>P(W = 2) = P(1st die picked & ‘1’ obtained) + P(2nd die picked & ‘2’ obtained)
$= \frac{1}{2} \times \frac{2}{4} + \frac{1}{2} \times \frac{1}{4}$
$= \frac{3}{8}$ (shown)</td><td></td></tr>
<tr><td></td><td>P(W = 1) = P(2nd die picked & ‘1’ obtained)
$= \frac{1}{2} \times \frac{1}{4}$
$= \frac{1}{8}$

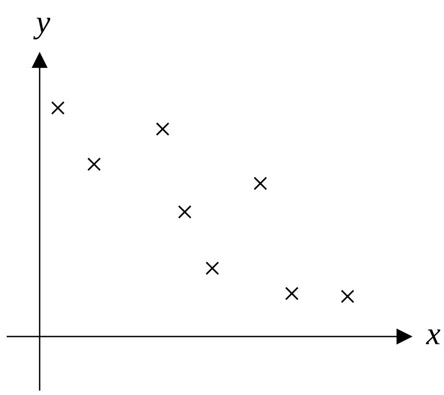
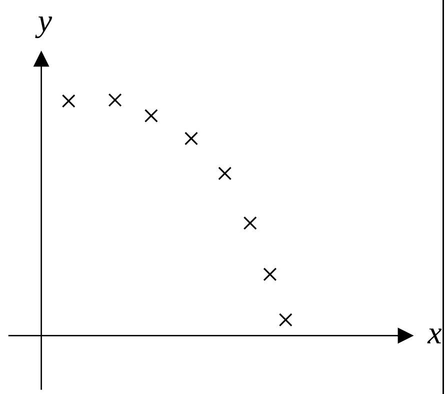
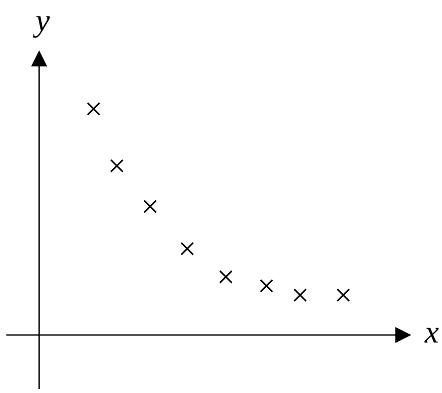
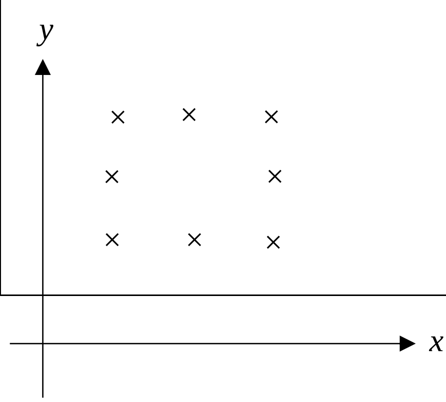
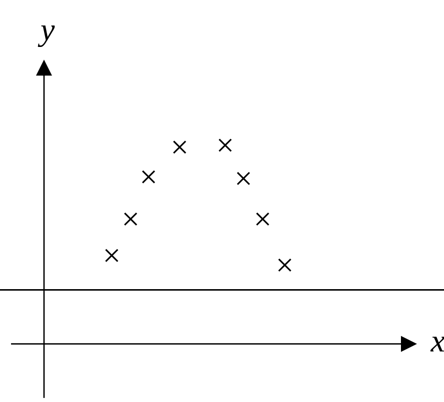
P(W = 3) = P (2nd die picked & ‘3’ obtained)
$= \frac{1}{2} \times \frac{2}{4}$
$= \frac{1}{4}$
P(W = 4) = P(1st die picked & ‘2’ obtained)
$= \frac{1}{2} \times \frac{1}{4}$
$= \frac{1}{8}$

P(W = 6) = P(1st die picked & ‘3’ obtained)
$= = \frac{1}{2} \times \frac{1}{4}$
$= \frac{1}{8}$
The probability distribution of W is:
<table><tr><td>w</td><td>1</td><td>2</td><td>3</td><td>4</td><td>6</td></tr><tr><td>$P(W = w)$</td><td>$\frac{1}{8}$</td><td>$\frac{3}{8}$</td><td>$\frac{1}{4}$</td><td>$\frac{1}{8}$</td><td>$\frac{1}{8}$</td></tr></table></td><td></td></tr>
</table>

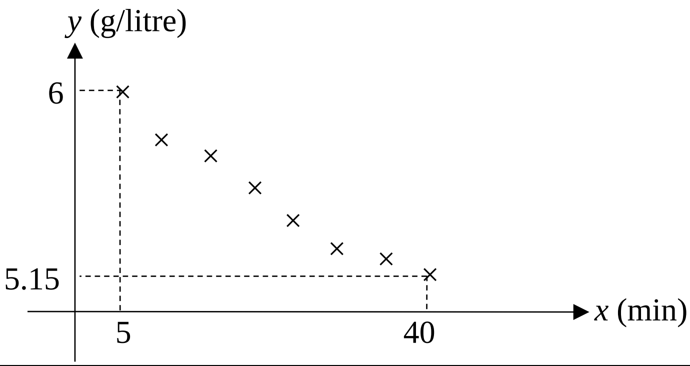
(ii)	E(W) $= 1\left(\frac{1}{8}\right) + 2\left(\frac{3}{8}\right) + 3\left(\frac{1}{4}\right) + 4\left(\frac{1}{8}\right) + 6\left(\frac{1}{8}\right)$ $= \frac{23}{8} = 2\frac{7}{8}$	
	E(W^2) $= 1^2 \times \frac{1}{8} + 2^2 \times \frac{3}{8} + 3^2 \times \frac{1}{4} + 4^2 \times \frac{1}{8} + 6^2 \times \frac{1}{8}$ $= \frac{83}{8} = 10\frac{3}{8}$ Var(W) = E(W^2) – [E(W)]^2 $= \frac{83}{8} - \left(\frac{23}{8}\right)^2$ $= \frac{135}{64} = 2\frac{7}{64}$	

Qn	Solution	Notes
6 (i)	Without any restriction, no. of codes generated $= 26\times26\times26\times26\times10\times10 = 26^4\times10^2 = 45697600$ No. of codes with four different letters and two different digits $= \left({}^{26}C_4 \times 4!\right)\left({}^{10}C_2 \times 2!\right) = 32292000$ Required probability = $\frac{32292000}{45697600} = \frac{3105}{4394}$	
(ii)	No. of codes with two different consonants, two different vowels where the consonants and vowels alternate $= \left({}^{21}C_2 \times 2! \times {}^{5}C_2 \times 2!\right) \times 2 \times 10^2 = 1680000$ Required probability = $\frac{1680000}{26^4 \times 10^2} = 0.0368$(3 sf)	

(iii)	No of codes with 2 letters the same , 2 letters different and 2 digits the same $=\left({}^{26}\mathrm{C}_3\times 3\times\dfrac{4!}{2!}\right)\times(10\times 1)=936000$ P(2 letters the same , 2 letters different and 2 digits the same) $=\dfrac{936000}{(26)^4(10)^2}$ $=\dfrac{45}{2197}$	

Qn	Solution	Notes
7(a)(i)	Some possible scatter plots  r between -0.8 and -0.5	
(ii)	Some possible scatter plots 	

	$r = 0$	

(b) (i)		
(ii)	No, a linear model is not suitable. From the scatter diagram, as time increases, the concentration decreases by decreasing amounts.	
(iii)	From the GC $\frac{1}{y}=0.16537+0.00074414x$ $\frac{1}{y}=0.165+0.000744x$ $a=0.165,\quad b=0.000744$ $r=0.988$	
(iv)	$\frac{1}{5.4}=0.16537+0.00074414x$ $x=26.628\approx 26.6$ Since x is the independent (controlled) variable, we should use the equation $\frac{1}{y}=ax+b$ and not $x=\frac{c}{y}+d$ to find the estimate.	

Qn	Solution	Notes
8	The event that a student completes the online Mathematics homework by the deadline is independent of any other student. The probability of each student completing the online Mathematics homework by the deadline is constant.	
(i)	$M \sim \mathrm{B}(25, 0.72)$ $\mathrm{P}(M < 15) = \mathrm{P}(M \le 14)$ $= 0.063632$ ≈ 0.0636	
(ii)	Required probability = $\left[\mathrm{P}(M < 15)\right]^3 \left[\mathrm{P}(M \ge 15)\right]^7 \left(\dfrac{10!}{3!7!}\right)$ ≈ 0.0195	
	Alternative method: X= Number of classes with fewer than 15 students who complete the online Mathematics homework by the deadline, out of 10 $X \sim \mathrm{B}(10, 0.063632)$ $\mathrm{P}(X = 3) \approx 0.0195$	
(iii)	$\mathrm{E}(M) = np = 18$, $\mathrm{Var}(M) = np(1-p) = 5.04$ Since number of classes (Sample size) = 30 is large, by Central Limit Theorem, $\overline{M} \sim N\left(18, \dfrac{5.04}{30}\right)$ approximately $\mathrm{P}\left(\overline{M} > 19\right) \approx 0.00735$	
(b)	Let $M \sim \mathrm{B}(n, 0.72)$ $\mathrm{P}(M \ge 20) > 0.7$ $1 - \mathrm{P}(M \le 19) > 0.7$ $\mathrm{P}(M \le 19) < 0.3$	
	From GC, when $n = 28$, $\mathrm{P}(M \le 19) = 0.37986 > 0.3$ when $n = 29$, $\mathrm{P}(M \le 19) = 0.27745 < 0.3$ when $n = 30$, $\mathrm{P}(M \le 19) = 0.19429 < 0.3$ $n \ge 29$ There must be at least 29 students in the class.	

Qn	Solution	Notes
9(i)	$V \sim \mathrm{N}\left(56,8^2\right)$ and $W \sim \mathrm{N}\left(60,12^2\right)$ Required probability = P(V <55) . P(W >55) = 0.29787 $\approx$ 0.298	
(ii)	$V_1+V_2+V_3-3W \sim \mathrm{N}(-12,1488)$ $\mathrm{P}(\lvert V_1+V_2+V_3-3W\rvert<15)$ = 0.28901 $\approx$0.289	
(iii)	Let $M = \dfrac{V_1+....V_5+W_1+....W_6}{11}$ $\mathrm{E}(M) = \dfrac{640}{11}$ and $\mathrm{Var}(M) = \dfrac{1184}{121}$ $M \sim \mathrm{N}\left(\dfrac{640}{11},\dfrac{1184}{121}\right)$ P(M < 60) = 0.71946 $\approx$0.719	
(iv)	$T = 2V+W$ $T \sim \mathrm{N}(172,400)$ P($T > t$) = 0.35 t = 179.71 $\approx$180	

Qn	Solution	Notes
10 (a) (i)	$\bar{x} = 40 + \frac{1}{150}\left[\Sigma(x-40)\right] = 43.9$	
	$s^2 = \frac{1}{150-1}\left\{\Sigma(x-40)^2 - \frac{\left[\Sigma(x-40)\right]^2}{150}\right\}$ $= 58.5$	
(ii)	X = speed of a car on the expressway μ = average speed of a car on the expressway $H_0: \mu = 45$ $H_1: \mu < 45$ Under H_0, since n = 150 is large, by Central Limit Theorem, $\bar{X} \sim N\left(45, \frac{58.5}{150}\right)$ approximately. Test statistic, $z = \frac{\bar{x}-\mu_0}{\sqrt{\frac{s^2}{n}}} = \frac{43.9-45}{\sqrt{\frac{58.5}{150}}} = -1.76141$ From GC, p-value = 0.039084 $\approx$ 0.0391 Level of significance, α = 0.05 Since p-value < α, we reject H_0. There is sufficient evidence, at the 5% level, to indicate that the average speed of a car is less than 45km/h. Hence, the ERP rate needs to be increased for this particular expressway.	
(iii)	The p-value is 0.0391 and it means that there is a probability of 0.0391 of observing a test statistic, $z < -1.76$, given that the population average speed of a car is 45 km/h.	

(b)	$H_0: \mu = 45$ $H_1: \mu > 45$ Under H_0, since n is large, by Central Limit Theorem, $\overline{X} \sim N\left(45, \dfrac{7.79^2}{n}\right)$ approximately. Test statistic, $z = \dfrac{\bar{x}-\mu_0}{\frac{\sigma}{\sqrt{n}}} = \dfrac{45.9-45}{\frac{7.79}{\sqrt{n}}} = \dfrac{0.9\sqrt{n}}{7.79}$ Level of significance, $\alpha = 0.05$ Critical region is $z > 1.64485$ Reject H_0 if $z > 1.64485$	
	$\dfrac{0.9\sqrt{n}}{7.79} > 1.64485$ $n > 202.6$ least $n = 203$	