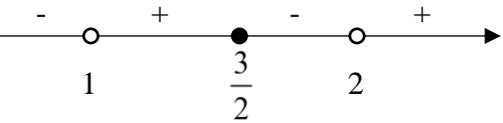


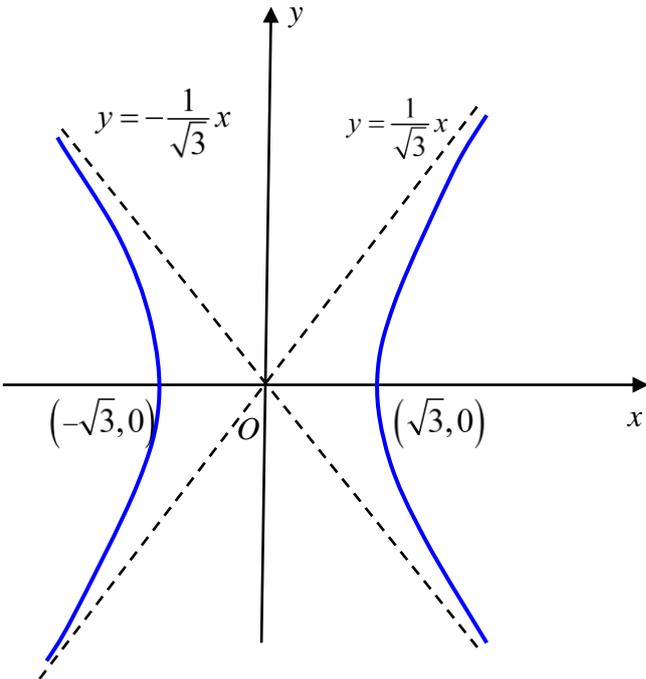
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Qn	Suggested Solutions
1(i)	$\frac{x-1}{x-2} \leq \frac{x-2}{x-1}$ $\frac{x-1}{x-2} - \frac{x-2}{x-1} \leq 0$ $\frac{(x-1)^2 - (x-2)^2}{(x-2)(x-1)} \leq 0$ $\frac{(x-1-x+2)(x-1+x-2)}{(x-2)(x-1)} \leq 0$ $\frac{(2x-3)}{(x-2)(x-1)} \leq 0$  <p>$\therefore x < 1$ or $\frac{3}{2} \leq x < 2$</p>
(ii)	$f(x) = \frac{x-1}{x-2}$ $f\left(x - \frac{1}{2}\right) = \frac{\left(x - \frac{1}{2}\right) - 1}{\left(x - \frac{1}{2}\right) - 2}$ $= \frac{2x-3}{2x-5}$ <p>OR</p> $\frac{x-1}{x-2} = 1 + \frac{1}{x-2}$ <p style="text-align: center;">↓ Replace x with $x - \frac{1}{2}$</p> $\frac{2x-3}{2x-5} = 1 + \frac{2}{2x-5}$ <p>$\therefore a = \frac{1}{2}$</p>
(iii)	<p>Replace x with $x - \frac{1}{2}$,</p> $x - \frac{1}{2} < 1 \quad \text{or} \quad \frac{3}{2} \leq x - \frac{1}{2} < 2$ <p>$\therefore x < \frac{3}{2}$ or $2 \leq x < \frac{5}{2}$</p>

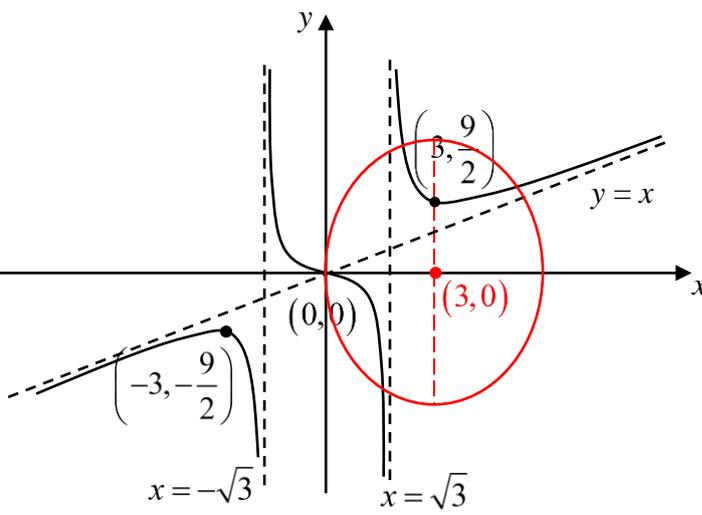
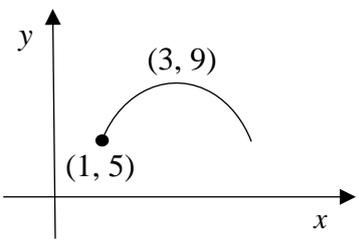
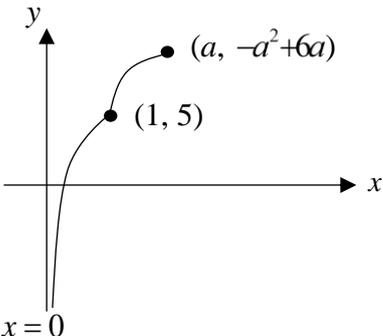
HCI C2 H2 Mathematics Preliminary Examinations 2022 Mark Scheme

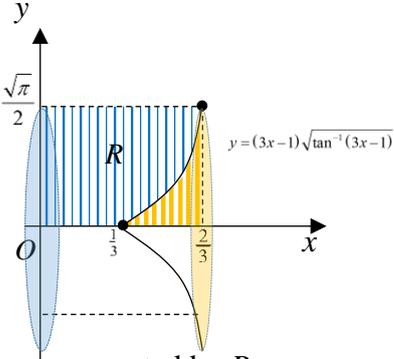
Qn	Suggested Solutions
2(i)	<p>Using cosine rule,</p> $BD^2 = AD^2 + AB^2 - 2(AD)(AB)\cos \angle BAD$ $= k^2 + 4k^2 - 4k^2 \cos \left(x + \frac{\pi}{3} \right)$ $= 5k^2 - 4k^2 \left(\cos x \cos \frac{\pi}{3} - \sin x \sin \frac{\pi}{3} \right)$ $= k^2 (5 - 2\cos x + 2\sqrt{3} \sin x) \quad (\text{Shown}).$
(ii)	<p>Since x is sufficiently small, then $\sin x \approx x$ and $\cos x \approx 1 - \frac{x^2}{2}$.</p> $BD^2 \approx k^2 \left[5 - 2 \left(1 - \frac{x^2}{2} \right) + 2\sqrt{3}(x) \right]$ $BD = k \left(3 + 2\sqrt{3}x + x^2 \right)^{\frac{1}{2}}$ $= \sqrt{3}k \left[1 + \left(\frac{2x}{\sqrt{3}} + \frac{x^2}{3} \right) \right]^{\frac{1}{2}}$ $= \sqrt{3}k \left[1 + \frac{1}{2} \left(\frac{2x}{\sqrt{3}} + \frac{x^2}{3} \right) + \frac{\left(\frac{1}{2} \right) \left(-\frac{1}{2} \right)}{2!} \left(\frac{2x}{\sqrt{3}} + \frac{x^2}{3} \right)^2 + \dots \right]$ $\approx \sqrt{3}k \left[1 + \frac{x}{\sqrt{3}} + \frac{x^2}{6} - \frac{1}{8} \left(\frac{4x^2}{3} \right) \right]$ $= \sqrt{3}k + kx$
3(i)	$RHS = u_r - 2u_{r+1} + u_{r+2}$ $= \frac{1}{r!} - \frac{2}{(r+1)!} + \frac{1}{(r+2)!}$ $= \frac{(r+1)(r+2) - 2(r+2) + 1}{(r+2)!}$ $= \frac{r^2 + 3r + 2 - 2r - 4 + 1}{(r+2)!}$ $= \frac{r^2 + r - 1}{(r+2)!} \quad (\text{Shown})$

Qn	Suggested Solutions
(ii)	$\sum_{r=1}^n \frac{r^2 + r - 1}{(r+2)!} = \sum_{r=1}^n (u_r - 2u_{r+1} + u_{r+2})$ $= u_1 - 2u_2 + u_3$ $+ u_2 - 2u_3 + u_4$ $+ u_3 - 2u_4 + u_5$ \vdots $+ u_{n-2} - 2u_{n-1} + u_n$ $+ u_{n-1} - 2u_n + u_{n+1}$ $+ u_n - 2u_{n+1} + u_{n+2}$ $= u_1 - 2u_2 + u_2 + u_{n+1} - 2u_{n+1} + u_{n+2}$ $= 1 - 1 + \frac{1}{2} + \frac{1}{(n+1)!} - \frac{2}{(n+1)!} + \frac{1}{(n+2)!}$ $= \frac{1}{2} - \frac{1}{(n+1)!} + \frac{1}{(n+2)!}$ <p>As $n \rightarrow \infty$, $-\frac{1}{(n+1)!} + \frac{1}{(n+2)!} \rightarrow 0$.</p> $\therefore \sum_{r=1}^{\infty} \frac{r^2 + r - 1}{(r+2)!} = \frac{1}{2}$
(iii)	<p>From MF26, see that</p> $e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{r!} + \dots$ $= 1 + \sum_{r=1}^{\infty} \frac{1}{r!}$ $= 1 + \sum_{r=1}^{\infty} u_r$ $\therefore \sum_{r=1}^{\infty} \left(u_r - \frac{r^2 + r - 1}{(r+2)!} \right)$ $= e - 1 - \frac{1}{2}$ $= e - \frac{3}{2}$

Qn	Suggested Solutions
4(i)	
(ii)	<p>$\operatorname{Re}(z) \rightarrow \infty \Rightarrow x \rightarrow \infty$</p> <p>As $x \rightarrow \infty$, $y \rightarrow \pm \frac{1}{\sqrt{3}}x$</p> <p>i.e. $\arg(z) = \tan^{-1}\left(\frac{y}{x}\right) \rightarrow \tan^{-1}\left(\pm \frac{1}{\sqrt{3}}\right) = \pm \frac{\pi}{6}$</p> <p>From the graph, z satisfies the right side of the hyperbola.</p> <p>$\operatorname{Re}(z) \rightarrow \infty$</p> <p>When $y > 0$,</p> <p>$\arg(z)$ increases from 0 to $\frac{\pi}{6}$</p> <p>When $y < 0$,</p> <p>$\arg(z)$ decreases from 0 to $-\frac{\pi}{6}$</p>
5(a)	$f(x) = \frac{x^3}{x^2 + k} = x - \frac{kx}{x^2 + k}$

Qn	Suggested Solutions
	$f'(x) = 1 - \frac{k(x^2 + k) - 2x(kx)}{(x^2 + k)^2}$ $= \frac{(x^2 + k)^2 - (k^2 - kx^2)}{(x^2 + k)^2}$ $= \frac{x^4 + 2kx^2 + k^2 - k^2 + kx^2}{(x^2 + k)^2}$ $= \frac{x^4 + 3kx^2}{(x^2 + k)^2}$ $= \frac{x^2(x^2 + 3k)}{(x^2 + k)^2}$ <p>Since $k > 0$, $x^2(x^2 + 3k) \geq 0$ and $(x^2 + k)^2 > 0$, $f'(x) \geq 0$. Therefore f is an increasing function. (Shown)</p>
(b)(i)	<p>When $k = -3$,</p> $f(x) = \frac{x^3}{x^2 - 3} = x - \frac{3x}{x^2 - 3}$ <p>Vertical asymptotes: $x = \pm\sqrt{3}$ Oblique asymptote: $y = x$</p> <p>When $f'(x) = 0$,</p> $x^4 - 9x^2 = 0$ $x^2(x^2 - 9) = 0$ $x = 0 \text{ or } x = \pm 3$ <p>Turning points: $(0, 0)$, $(3, \frac{9}{2})$, $(-3, -\frac{9}{2})$</p> <p>Axial intercept: $(0, 0)$</p>

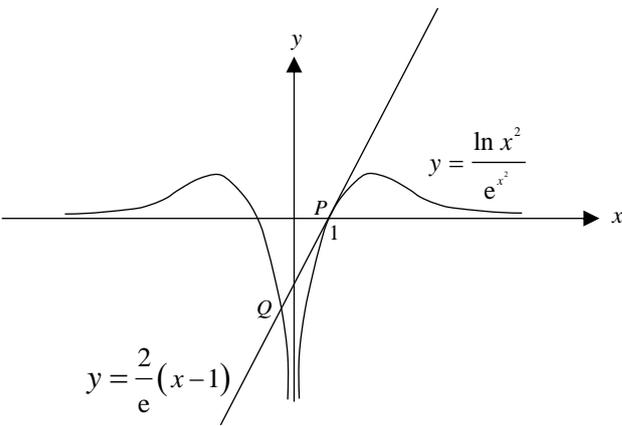
Qn	Suggested Solutions
(iii)	 <p>For equation to have more than 3 real roots, $a > \frac{81}{4}$.</p>
6(i)	<p>Complete the square $-x^2 + 6x = -[x^2 - 6x] = -(x - 3)^2 + 9$</p>  <p>For f to be 1-1 on $[1, a]$, $a \leq 3$ \therefore the range of values of a is $1 \leq a \leq 3$</p>
(ii)	

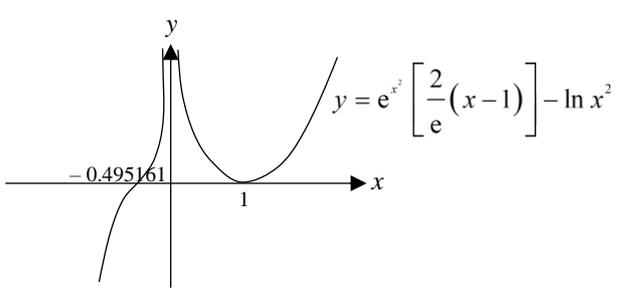
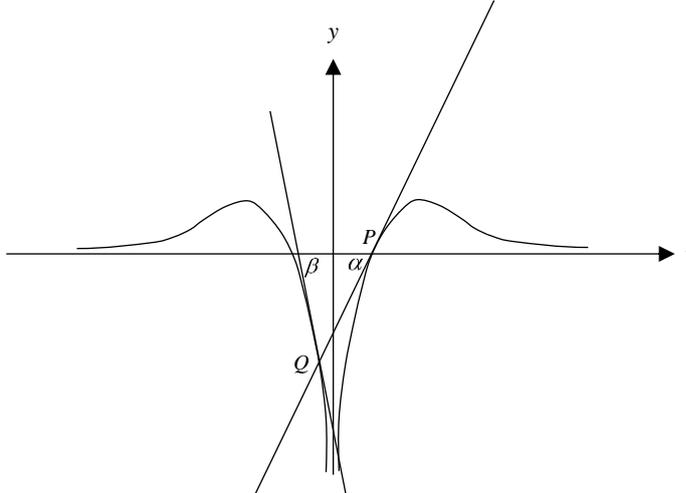
Qn	Suggested Solutions
(iii)	<p>Let $y = f(x) \Leftrightarrow x = f^{-1}(y)$</p> <p>For $0 < x < 1$</p> $5 + \ln x = y$ $\ln x = y - 5$ $x = e^{y-5}, \quad y < 5$ <p>For $0 \leq x \leq a \leq 3$</p> $y = -(x-3)^2 + 9$ $(x-3)^2 = 9 - y$ $x = 3 \pm \sqrt{9-y}$ <p>Since $x \leq 3$</p> $x = 3 - \sqrt{9-y}, \quad 5 \leq y \leq -a^2 + 6a$ $\therefore f^{-1} : x \mapsto \begin{cases} e^{x-5}, & x < 5 \\ 3 - \sqrt{9-x}, & 5 \leq x \leq -a^2 + 6a \end{cases}$
7(i)	$\int w^2 \tan^{-1} w \, dw$ $= \frac{w^3}{3} \tan^{-1} w - \frac{1}{3} \int \frac{w^3}{1+w^2} \, dw$ $= \frac{w^3}{3} \tan^{-1} w - \frac{1}{3} \int w - \frac{w}{w^2+1} \, dw$ $= \frac{w^3}{3} \tan^{-1} w - \frac{w^2}{6} + \frac{1}{6} \ln(w^2+1) + C$
(ii)	 <p>Volume generated by R</p> $= (\pi) \left(\frac{\sqrt{\pi}}{2} \right)^2 \left(\frac{2}{3} \right) - \pi \int_{\frac{1}{3}}^{\frac{2}{3}} (3x-1)^2 \tan^{-1}(3x-1) \, dx$

Qn	Suggested Solutions
	<p>From (i),</p> $\int w^2 \tan^{-1} w \, dw = \frac{w^3}{3} \tan^{-1} w - \frac{w^2}{6} + \frac{1}{6} \ln(w^2 + 1) + C$ <p>Let</p> $w = 3x - 1 \Rightarrow \frac{dw}{dx} = 3$ $\int_{\frac{1}{3}}^{\frac{2}{3}} (3x - 1)^2 \tan^{-1}(3x - 1) \, dx$ $= \frac{1}{3} \int_0^1 (w)^2 \tan^{-1}(w) \, dw$ $= \frac{1}{3} \left[\frac{w^3}{3} \tan^{-1} w - \frac{w^2}{6} + \frac{1}{6} \ln(w^2 + 1) \right]_0^1$ $= \frac{1}{3} \left[\frac{\pi}{12} - \frac{1}{6} + \frac{1}{6} \ln 2 \right]$ <p>Volume generated by R</p> $= (\pi) \left(\frac{\sqrt{\pi}}{2} \right)^2 \left(\frac{2}{3} \right) - \pi \int_{\frac{1}{3}}^{\frac{2}{3}} (3x - 1)^2 \tan^{-1}(3x - 1) \, dx$ $= \frac{\pi^2}{6} - \frac{\pi}{3} \left[\frac{\pi}{12} - \frac{1}{6} + \frac{1}{6} \ln 2 \right]$ $= \frac{5\pi^2}{36} + \frac{\pi}{18} - \frac{\pi}{18} \ln 2 \text{ unit}^3$
8(i)	<p>Since the coefficients of the polynomial are all real and $a + i$ is a root, then $a - i$ is also a root.</p> $z^2 - 2\sqrt{2}z + b = [z - (a + i)][z - (a - i)]$ $= z^2 - 2az + (a^2 + 1)$ <p>By comparing z term, $-2\sqrt{2} = -2a$ $a = \sqrt{2}$.</p> <p>By comparing constant term, $b = a^2 + 1 = (\sqrt{2})^2 + 1 = 3$.</p>
	<p>Alternative Method</p> $(a + i)^2 - 2\sqrt{2}(a + i) + b = 0$ $(a^2 - 1 - 2\sqrt{2}a + b) + (2a - 2\sqrt{2})i = 0$ <p>By comparing real and imaginary parts,</p>

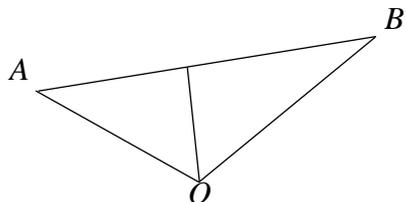
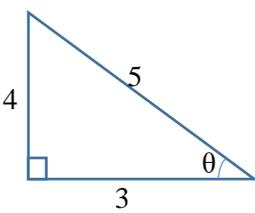
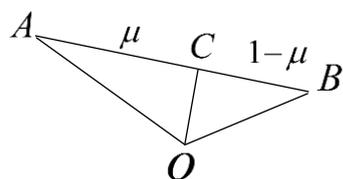
HCI C2 H2 Mathematics Preliminary Examinations 2022 Mark Scheme

Qn	Suggested Solutions
	$(2a - 2\sqrt{2}) = 0$ $a = \sqrt{2}$ $a^2 - 1 - 2\sqrt{2}a + b = 0$ $2 - 1 - 4 + b = 0$ $b = 3$
(ii)	$f(-z) = (-z)^6 - 3(-z)^4 + 11(-z)^2 - 9$ $= z^6 - 3z^4 + 11z^2 - 9$ $= f(z)$
(iii)	<p>Since the coefficients of $f(z)$ are all real and $\sqrt{2} + i$ is a root, then $\sqrt{2} - i$ is also a root. Thus $z^2 - 2\sqrt{2}z + 3$ is a quadratic factor of $f(z)$.</p> <p>Since $f(z) = f(-z)$, then $-\sqrt{2} + i$ and $-\sqrt{2} - i$ are also roots of $f(z) = 0$. $\left[z - (-\sqrt{2} - i) \right] \left[z - (-\sqrt{2} + i) \right] = (z + \sqrt{2})^2 - i^2$ $= z^2 + 2\sqrt{2}z + 3$</p> <p>Thus $z^2 + 2\sqrt{2}z + 3$ is a quadratic factor of $f(z)$.</p> $f(z) = z^6 - 3z^4 + 11z^2 - 9$ $= (z^2 - 2\sqrt{2}z + 3)(z^2 + 2\sqrt{2}z + 3)(z^2 + Cz + D)$ <p>By comparing constant term, $-9 = 9D \Rightarrow D = -1$</p> $f(z) = (z^2 - 2\sqrt{2}z + 3)(z^2 + 2\sqrt{2}z + 3)(z^2 + Cz - 1)$ $= \left[(z^2 + 3)^2 - (2\sqrt{2}z)^2 \right] (z^2 + Cz - 1)$ $= (z^4 + 14z^2 + 9)(z^2 + Cz - 1)$ <p>By comparing coefficient of z term, then $0 = 9Cz \Rightarrow C = 0$.</p> <p>Therefore,</p> $f(z) = (z^2 - 2\sqrt{2}z + 3)(z^2 + 2\sqrt{2}z + 3)(z^2 - 1).$

Qn	Suggested Solutions
9(i)	$e^{x^2} y = \ln x^2$ <p>Differentiating wrt x both sides,</p> $2xe^{x^2} y + e^{x^2} \frac{dy}{dx} = \frac{1}{x^2}(2x)$ $2x \ln x^2 + e^{x^2} \frac{dy}{dx} = \frac{2}{x} \quad (\text{Shown})$ $(\because e^{x^2} y = \ln x^2)$
9(ii)	<p>When $x = 1$,</p> $y = \frac{\ln 1^2}{e^{1^2}} = 0$ $\frac{dy}{dx} = \frac{\frac{2}{x} - 2(1)\ln 1^2}{e^{1^2}} = \frac{2}{e} \quad (\text{from (i)})$ <p>Hence equation of tangent at P: $y = \frac{2}{e}(x-1)$</p> <p>Method 1</p> $\frac{2}{e}(x-1) = \frac{\ln x^2}{e^{x^2}}$ <p>Using GC, another point of intersection between the tangent $y = \frac{2}{e}(x-1)$ and $C: y = \frac{\ln x^2}{e^{x^2}}$ is</p> <p>$Q (-0.49516, -1.1001)$ (to 5 sf).</p>  <p>The graph shows a Cartesian coordinate system with x and y axes. A curve, labeled $y = \frac{\ln x^2}{e^{x^2}}$, passes through the origin and has a local maximum in the first quadrant and a local minimum in the third quadrant. A straight line, labeled $y = \frac{2}{e}(x-1)$, is tangent to the curve at point $P(1, 0)$. Another point Q is marked on the curve in the third quadrant, representing another point of intersection between the curve and the tangent line.</p> <p>Method 2</p> <p>Substitute $y = \frac{2}{e}(x-1)$ into the equation for C:</p>

Qn	Suggested Solutions
	$e^{x^2} \left[\frac{2}{e}(x-1) \right] = \ln x^2 \quad \dots (1)$ $e^{x^2} \left[\frac{2}{e}(x-1) \right] - \ln x^2 = 0$  <p>At Q, $x = -0.495161$</p> $y = \frac{2}{e}(-0.495161 - 1) = -1.100078256$ <p>Hence, $Q (-0.49516, -1.1001)$ (to 5 sf).</p>
(iii)	<p>At Q,</p> $\left. \frac{d}{dx} \left[\frac{\ln(x^2)}{e^{x^2}} \right] \right _{x=-0.495161} = -4.2502646 \text{ (from GC)}$ <p>(Alternatively, the above value can be obtained by substituting $x = -0.495161$ into equation in (i))</p>  $\alpha = \tan^{-1} \left(\frac{2}{e} \right) = 36.3441^\circ$ $\beta = \tan^{-1} (4.2502646) = 76.7603^\circ$ <p>Required acute angle between the 2 tangents</p>

HCI C2 H2 Mathematics Preliminary Examinations 2022 Mark Scheme

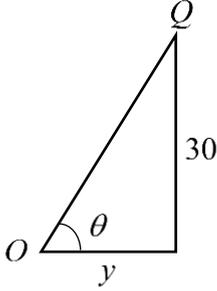
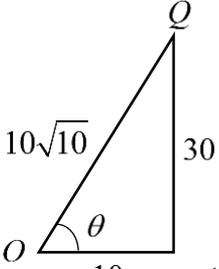
Qn	Suggested Solutions
	$= 180^\circ - 36.3441^\circ - 76.7603^\circ$ $= 66.9^\circ \text{ (1 d.p.)}$
<p>10(i)</p>	<div style="display: flex; align-items: flex-start;"> <div style="flex: 1;"> $\frac{1}{2} \underline{a} \times \underline{b} = 16$ $\Rightarrow \underline{a} \times \underline{b} = 32$ <p>Since $\hat{b} = \frac{\underline{b}}{ \underline{b} }$</p> $\therefore \underline{b} = \underline{b} \hat{b} = 5\underline{d}$ $\Rightarrow \underline{a} \times 5\underline{d} = 32$ $\underline{a} \times \underline{d} = \frac{32}{5}$ <p>$\underline{a} \times \underline{d}$ is the shortest distance from A to OB OR the perpendicular height of the triangle OAB with OB as the base.</p> </div> <div style="flex: 1; text-align: center;">  </div> </div>
<p>(ii)</p>	<div style="display: flex; align-items: flex-start;"> <div style="flex: 1;"> <p>Let θ be the angle between \underline{a} and \underline{b}</p> $\underline{a} \times \underline{b} = 32$ $\underline{a} \underline{b} \sin \theta = 32$ $\sin \theta = \frac{4}{5}$ <p>Since θ is obtuse, $\cos \theta = -\frac{3}{5}$</p> $\underline{a} \cdot \underline{b} = \underline{a} \underline{b} \cos \theta$ $\underline{a} \cdot \underline{b} = (8 \times 5) \left(-\frac{3}{5} \right) = -24$ </div> <div style="flex: 1; text-align: center;">  </div> </div>
<p>(iii)</p>	<div style="display: flex; align-items: flex-start;"> <div style="flex: 1;">  <p>By ratio theorem,</p> $\overline{OC} = \mu \underline{b} + (1 - \mu) \underline{a}$ </div> </div>

HCI C2 H2 Mathematics Preliminary Examinations 2022 Mark Scheme

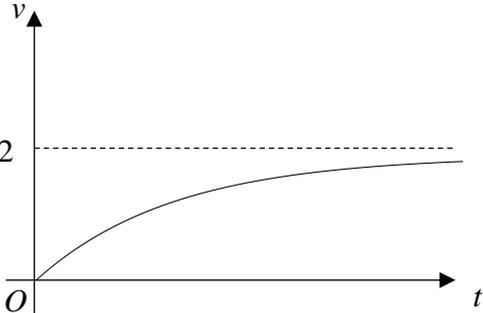
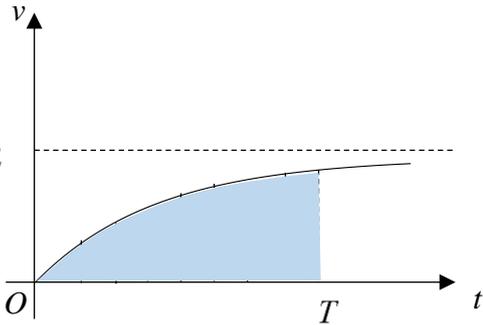
Qn	Suggested Solutions
	$ \overrightarrow{OC} = \mu \underline{b} + (1 - \mu) \underline{a} $ $ \overrightarrow{OC} ^2 = \mu \underline{b} + (1 - \mu) \underline{a} ^2$ $\frac{401}{49} = [\mu \underline{b} + (1 - \mu) \underline{a}] \cdot [\mu \underline{b} + (1 - \mu) \underline{a}]$ $\frac{401}{49} = \mu^2 \underline{b} ^2 + 2\mu(1 - \mu) \underline{a} \cdot \underline{b} + (1 - \mu)^2 \underline{a} ^2$ $\frac{401}{49} = 25\mu^2 - 48\mu(1 - \mu) + 64(1 - \mu)^2$ $401 = 1225\mu^2 - 2352\mu(1 - \mu) + 3136(1 - 2\mu + \mu^2)$ $6713\mu^2 - 8624\mu + 2735 = 0$ $\mu = \frac{5}{7}, \frac{547}{959}$ $\overrightarrow{OC} = \frac{2}{7} \underline{a} + \frac{5}{7} \underline{b}, \quad \overrightarrow{OC} = \frac{412}{959} \underline{a} + \frac{547}{959} \underline{b}$

Qn	Suggested Solutions
<p>11(a) (i)</p>	<p>Let the point where the diagonals meet be M.</p> $AM = \sqrt{a^2 - x^2}$ $MC = \sqrt{b^2 - x^2}$ <p>Method 1: Area of kite $ABCD$ = Area of $\triangle ABD$ + Area of $\triangle BDC$ = $\frac{1}{2}(2x)\sqrt{a^2 - x^2} + \frac{1}{2}(2x)\sqrt{b^2 - x^2}$ = $(x)\sqrt{a^2 - x^2} + (x)\sqrt{b^2 - x^2}$ $\therefore K = x\sqrt{a^2 - x^2} + x\sqrt{b^2 - x^2}$</p> <p>Method 2: $AC = \sqrt{a^2 - x^2} + \sqrt{b^2 - x^2}$ Area of kite $ABCD$ = $\frac{1}{2}(AC)(BD)$ = $\frac{1}{2}(\sqrt{a^2 - x^2} + \sqrt{b^2 - x^2})(2x)$ $\therefore K = x\sqrt{a^2 - x^2} + x\sqrt{b^2 - x^2}$</p>
<p>(a) (ii)</p>	$\frac{dK}{dx} = \frac{x}{2}(-2x)(a^2 - x^2)^{-\frac{1}{2}} + (a^2 - x^2)^{\frac{1}{2}} + \frac{x}{2}(-2x)(b^2 - x^2)^{-\frac{1}{2}} + (b^2 - x^2)^{\frac{1}{2}}$ $\frac{dK}{dx} = (a^2 - x^2)^{\frac{1}{2}} + (b^2 - x^2)^{\frac{1}{2}} - x^2 \left[\frac{1}{\sqrt{a^2 - x^2}} + \frac{1}{\sqrt{b^2 - x^2}} \right]$ $= \sqrt{a^2 - x^2} + \sqrt{b^2 - x^2} - x^2 \left[\frac{\sqrt{a^2 - x^2} + \sqrt{b^2 - x^2}}{\sqrt{a^2 - x^2}\sqrt{b^2 - x^2}} \right]$ $= (\sqrt{a^2 - x^2} + \sqrt{b^2 - x^2}) \left[1 - \frac{x^2}{\sqrt{a^2 - x^2}\sqrt{b^2 - x^2}} \right]$ <p>For stationary value of K, $\frac{dK}{dx} = 0$</p> $\frac{dK}{dx} = \frac{(\sqrt{a^2 - x^2} + \sqrt{b^2 - x^2})[\sqrt{a^2 - x^2}\sqrt{b^2 - x^2} - x^2]}{\sqrt{a^2 - x^2}\sqrt{b^2 - x^2}} \text{ ----} (*)$ $\sqrt{a^2 - x^2} > 0, \quad \sqrt{b^2 - x^2} > 0$ $\sqrt{a^2 - x^2} + \sqrt{b^2 - x^2} > 0$

Qn	Suggested Solutions
	$\sqrt{a^2 - x^2} \sqrt{b^2 - x^2} - x^2 = 0$ $x^2 = \sqrt{a^2 - x^2} \sqrt{b^2 - x^2}$ $x^4 = (a^2 - x^2)(b^2 - x^2)$ $x^4 = a^2b^2 - x^2(a^2 + b^2) + x^4$ $x^2(a^2 + b^2) = a^2b^2$ $x^2 = \frac{a^2b^2}{a^2 + b^2}$ $x = \frac{ab}{\sqrt{a^2 + b^2}}, \quad x > 0$
	<p>Alternative Method:</p> $\frac{dK}{dx} = \frac{a^2 - 2x^2}{\sqrt{a^2 - x^2}} + \frac{b^2 - 2x^2}{\sqrt{b^2 - x^2}} = 0$ $\frac{a^2 - 2x^2}{\sqrt{a^2 - x^2}} = \frac{2x^2 - b^2}{\sqrt{b^2 - x^2}}$ $(a^2 - 2x^2)\sqrt{b^2 - x^2} = (2x^2 - b^2)\sqrt{a^2 - x^2}$ $(a^2 - 2x^2)^2(b^2 - x^2) = (2x^2 - b^2)^2(a^2 - x^2)$ $(a^4 - 4a^2x^2 + 4x^4)(b^2 - x^2) = (4x^4 - 4b^2x^2 + b^4)(a^2 - x^2)$ $x^2(a^4 - b^4) = a^4b^2 - a^2b^4$ $x^2 = \frac{a^2b^2(a^2 - b^2)}{(a^2 - b^2)(a^2 + b^2)}$ $x^2 = \frac{a^2b^2}{(a^2 + b^2)}$
<p>(a) (iii)</p>	$AC = \sqrt{a^2 - \frac{a^2b^2}{a^2 + b^2}} + \sqrt{b^2 - \frac{a^2b^2}{a^2 + b^2}}$ $= \sqrt{\frac{a^4 + a^2b^2 - a^2b^2}{a^2 + b^2}} + \sqrt{\frac{b^4 + a^2b^2 - a^2b^2}{a^2 + b^2}}$ $= \sqrt{\frac{a^4}{a^2 + b^2}} + \sqrt{\frac{b^4}{a^2 + b^2}}$ $= \frac{a^2 + b^2}{\sqrt{a^2 + b^2}}$ $\therefore AC = \sqrt{a^2 + b^2}$
<p>(a) (iv)</p>	<p>From (iii)</p> <p>Since $AC^2 = AB^2 + BC^2$</p> $\therefore \angle ABC = 90^\circ \text{ or } \frac{\pi}{2} \text{ radians}$

Qn	Suggested Solutions
(b)	<p>Let the horizontal distance travelled by the kite be y m.</p> $\tan \theta = \frac{30}{y}$ $\theta = \tan^{-1}\left(\frac{30}{y}\right)$ $\frac{d\theta}{dy} = \frac{1}{1 + \left(\frac{30}{y}\right)^2} \left(-\frac{30}{y^2}\right)$ $= -\frac{30}{y^2 + 30^2}$  <p>When $y = \sqrt{(10\sqrt{10})^2 - 30^2} = 10$ and $\frac{dy}{dt} = 2.4$,</p> $\frac{d\theta}{dt} = \frac{d\theta}{dy} \times \frac{dy}{dt}$ $= -\frac{30}{10^2 + 30^2} (2.4)$ $= -\frac{9}{125} \text{ rad/s}$ <p>The angle is decreasing at $\frac{9}{125}$ or 0.072 rad/s</p> <p>Alternatively,</p> $\tan \theta = \frac{30}{y}$ <p>diff wrt t,</p> $\sec^2 \theta \frac{d\theta}{dt} = -\frac{30}{y^2} \frac{dy}{dt}$  <p>When $OQ = 10\sqrt{10}$, $y = 10$ and $\sec^2 \theta = \left(\frac{10\sqrt{10}}{10}\right)^2 = 10$</p> $\therefore 10 \frac{d\theta}{dt} = -\frac{30}{100} \frac{dy}{dt}$ $\frac{d\theta}{dt} = -\frac{3}{100} (2.4)$ $= -\frac{9}{125} \text{ rad/s}$ <p>The angle is decreasing at $\frac{9}{125}$ or 0.072 rad/s.</p>
12(i)	$v = \frac{dx}{dt}$

Qn	Suggested Solutions
(ii)	<p>Speed $v = \frac{dx}{dt} \Rightarrow \frac{dv}{dt} = \frac{d^2x}{dt^2}$</p> <p>Given $\frac{d^2x}{dt^2} + \alpha \left(\frac{dx}{dt}\right)^2 = 10$</p> <p>$\Rightarrow \frac{dv}{dt} + \alpha(v)^2 = 10$</p> <p>$\Rightarrow \frac{dv}{dt} = 10 - \alpha v^2$</p>
(iii)	<p>$\frac{d^2x}{dt^2} + \alpha \left(\frac{dx}{dt}\right)^2 = 10.$</p> <p>Given that $\frac{d^2x}{dt^2} = 4.375$ when $\frac{dx}{dt} = 1.5.$</p> <p>$4.375 + \alpha(1.5)^2 = 10$</p> <p>$\Rightarrow \alpha = \frac{5}{2}$</p> <p>$\Rightarrow \frac{dv}{dt} = 10 - \frac{5}{2}v^2 = \frac{5}{2}(4 - v^2)$</p> <p>$\Rightarrow \frac{dt}{dv} = \frac{2}{5(4 - v^2)}$</p> <p>$\Rightarrow \int \frac{5}{2} dt = \int \frac{1}{4 - v^2} dv$</p> <p>$\Rightarrow \frac{5}{2}t + C = \frac{1}{4} \ln \left \frac{2+v}{2-v} \right$</p> <p>$\Rightarrow 10t + C' = \ln \left \frac{2+v}{2-v} \right$</p> <p>$\Rightarrow e^{10t+C'} = \left \frac{2+v}{2-v} \right$</p> <p>$\Rightarrow Ae^{10t} = \frac{2+v}{2-v},$ where $A = \pm e^{C'}$</p> <p>When $t = 0, x = 0, v = 0$</p> <p>$\Rightarrow Ae^0 = \frac{2+(0)}{2-(0)}$</p> <p>$\Rightarrow A = 1$</p> <p>$\Rightarrow e^{10t} = \frac{2+v}{2-v}$</p> <p>$\Rightarrow e^{10t}(2-v) = 2+v$</p> <p>$\Rightarrow 2e^{10t} - ve^{10t} = 2+v$</p> <p>$\Rightarrow 2e^{10t} - 2 = v + ve^{10t}$</p> <p>$\Rightarrow v = \frac{2e^{10t} - 2}{e^{10t} + 1}$</p>

Qn	Suggested Solutions
	$\Rightarrow v = \frac{2 - 2e^{-10t}}{1 + e^{-10t}} \text{ where } k = 2, m = 1$
(iv)	 <p>As $t \rightarrow \infty, e^{-10t} \rightarrow 0, v \rightarrow 2$ and $\frac{dv}{dt} \rightarrow 0$ according to graph.</p> <p>Thus $\frac{dx}{dt}$ will increase and approach 2 m/s.</p> <p>$\therefore \frac{d^2x}{dt^2} = \frac{dv}{dt}$ will decrease and approach 0 m/s².</p>
(v)	
	<p>Area under the graph</p> $\int_0^T \frac{2 - 2e^{-10t}}{1 + e^{-10t}} dt$ $= 2 \int_0^T \frac{e^{5t} - e^{-5t}}{e^{5t} + e^{-5t}} dt$ $= \frac{2}{5} \int_0^T \frac{5e^{5t} - 5e^{-5t}}{e^{5t} + e^{-5t}} dt$ $= \frac{2}{5} \left[\ln(e^{5t} + e^{-5t}) \right]_0^T$ $= \frac{2}{5} \left[\ln(e^{5T} + e^{-5T}) - \ln(e^0 + e^0) \right]$ $= \frac{2}{5} \ln \left(\frac{e^{5T} + e^{-5T}}{2} \right) \text{ where } \beta = 5$
	Alternative Method:

Qn	Suggested Solutions
	$\int \frac{2e^{10t} - 2}{e^{10t} + 1} dt$ $= \int_0^T \int \frac{2e^{10t}}{e^{10t} + 1} - \frac{-2e^{-10t}}{1 + e^{-10t}} dt$ $= \frac{1}{5} \int_0^T \frac{10e^{10t}}{e^{10t} + 1} dt + \int_0^T \frac{-10e^{-10t}}{1 + e^{-10t}} dt$ $= \left[\frac{1}{5} \ln(e^{10t} + 1) + \frac{1}{5} \ln(e^{-10t} + 1) \right]_0^T$ $\frac{1}{5} \ln \left[\frac{(e^{10T} + 1)(e^{-10T} + 1)}{4} \right]$ $= \frac{1}{5} \ln \left[\frac{(e^{5T} + e^{-5T})^2}{4} \right]$
	<p>Alternative Method</p> $\int_0^T 2 - \frac{4e^{-10t}}{e^{-10t} + 1} dt$ $= 2T + \frac{4}{10} \int_0^T \frac{-10e^{-10t}}{e^{-10t} + 1} dt$ $= 2T + \frac{2}{5} \ln(1 + e^{-10T}) - \frac{2}{5} \ln 2$ $= \frac{2}{5} e^{5T} + \frac{2}{5} \ln \left(\frac{1 + e^{-10t}}{2} \right)$ $= \frac{2}{5} \ln \left(\frac{e^{5T} + e^{-5T}}{2} \right)$
(v)	<p>$\frac{2}{5} \ln \left(\frac{e^{5T} + e^{-5T}}{2} \right)$ represents the distance the First Aid Kit dropped from the cargo drone in T seconds</p>