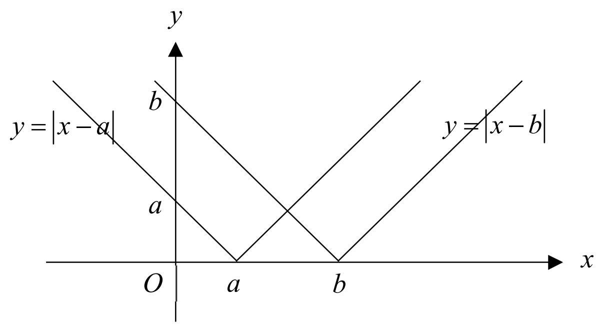
1



Point of intersection:

$x - a = -(x - b)$

$x = \dfrac{a+b}{2}$

$\therefore x < \dfrac{a+b}{2}$

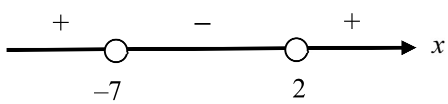
2(a) $\dfrac{17-5x}{x^2+5x-14}+1\geq 0$

$\dfrac{x^2+3}{x^2+5x-14}\geq 0$

$\dfrac{x^2+3}{(x+7)(x-2)}\geq 0$

As $x^2+3>0$ for all real values of x,

$(x+7)(x-2)>0$



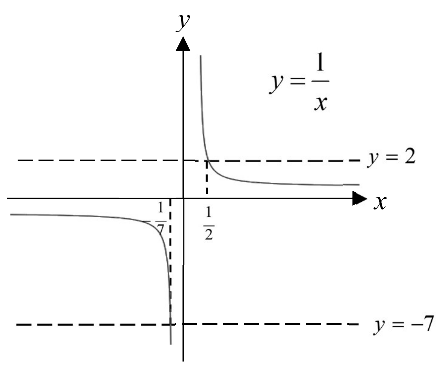
$x<-7$ OR $x>2$

2(b) Dividing numerator and denominator of fraction by x:

$$\frac{17-\frac{5}{x}}{\frac{1}{x^2}+\frac{5}{x}-14}\geq -1$$

Replacing x in (a) with $\dfrac{1}{x}$:

$\dfrac{1}{x}<-7$ OR $\dfrac{1}{x}>2$



$-\dfrac{1}{7}<x<0$ OR $0<x<\dfrac{1}{2}$

3(a) $\mathbf{c}\times 3\mathbf{b} = 5\mathbf{a}\times\mathbf{c}$

$\Rightarrow \mathbf{0} = (5\mathbf{a}\times\mathbf{c}) - (\mathbf{c}\times 3\mathbf{b}) = (5\mathbf{a}\times\mathbf{c}) + (3\mathbf{b}\times\mathbf{c})$

$\Rightarrow (5\mathbf{a}+3\mathbf{b})\times\mathbf{c} = \mathbf{0}$

Either $\mathbf{c}=\mathbf{0}$ or $5\mathbf{a}+3\mathbf{b}=\mathbf{0}$ or $\mathbf{c}$ is parallel to $5\mathbf{a}+3\mathbf{b}$

$\mathbf{c}\neq\mathbf{0}$, and since **a** and **b** are non-parallel, non-zero vectors, $5\mathbf{a}+3\mathbf{b}\neq\mathbf{0}$

Hence **c** is parallel to $5\mathbf{a}+3\mathbf{b}$.

(b) $\mathbf{d} = \dfrac{(1-\lambda)\mathbf{a}+\lambda\mathbf{b}}{\lambda+(1-\lambda)} = (1-\lambda)\mathbf{a}+\lambda\mathbf{b}$

(c) Method 1 (Comparing Coefficients)

Since D lies on OC, **d** is parallel to both **c** and $5\mathbf{a}+3\mathbf{b}$

Hence $\mathbf{d} = (1-\lambda)\mathbf{a}+\lambda\mathbf{b} = k(5\mathbf{a}+3\mathbf{b})$

Since **a** and **b** are non-parallel vectors, by comparing coefficients,

$1-\lambda = 5k$ and $\lambda = 3k$

Solving, $\lambda = \dfrac{3}{8}$ $\left(\text{and} \quad k = \dfrac{1}{8}\right)$

Method 2 (Cross Product)

Since D lies on OC, **d** is parallel to **c** and $5\mathbf{a}+3\mathbf{b}$

$\therefore \mathbf{d}\times(5\mathbf{a}+3\mathbf{b}) = \mathbf{0}$

$[(1-\lambda)\mathbf{a}+\lambda\mathbf{b}]\times(5\mathbf{a}+3\mathbf{b}) = \mathbf{0}$

$5(1-\lambda)\mathbf{a}\times\mathbf{a} + 5\lambda\mathbf{b}\times\mathbf{a} + 3(1-\lambda)\mathbf{a}\times\mathbf{b} + 3\lambda\mathbf{b}\times\mathbf{b} = \mathbf{0}$

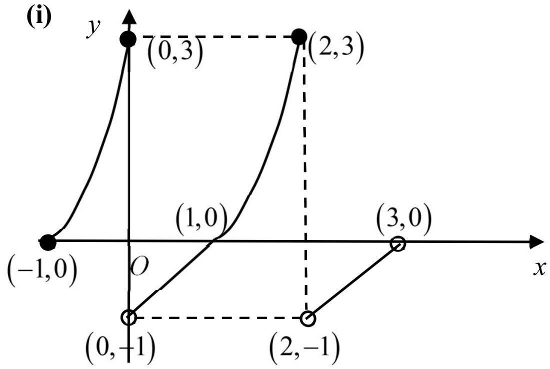
As $\mathbf{a}\times\mathbf{a}, \mathbf{b}\times\mathbf{b} = \mathbf{0}$, and $\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$:

$-5\lambda\mathbf{a}\times\mathbf{b} + 3(1-\lambda)\mathbf{a}\times\mathbf{b} = \mathbf{0}$

$(3-8\lambda)\mathbf{a}\times\mathbf{b} = \mathbf{0}$

a and **b** are non-parallel $\Rightarrow \mathbf{a}\times\mathbf{b}\neq\mathbf{0}$

$\therefore 3-8\lambda = 0, \quad \lambda = \dfrac{3}{8}$

4(a)	Let $y=(x+3)^2-1$ $y+1=(x+3)^2$ $x=-3+\sqrt{y+1}$ or $-3-\sqrt{y+1}$ (rej $\because x\geq -3$) So $\mathrm{f}^{-1}(x)=-3+\sqrt{x+1}$ $D_{\mathrm{f}^{-1}}=R_{\mathrm{f}}=[-1,\infty)$.
(b) **(i)**	 $\mathrm{R_g}=(-1,3]$
(c)(i)	Note that $R_{\mathrm{g}}=(-1,3]$ and $D_{\mathrm{f}^{-1}}=[-1,\infty)$. Since $R_{\mathrm{g}}\subseteq D_{\mathrm{f}^{-1}}$, $\mathrm{f}^{-1}\mathrm{g}$ exists.
(c)(ii)	For $0<x<1$, $\mathrm{f}^{-1}\mathrm{g}(x)=-3+\sqrt{(x-1)+1}=-3+\sqrt{x}$ For $-1\leq x<0$, note that $\mathrm{g}(x)=(x+2)^2-1$. Hence, $\mathrm{f}^{-1}\mathrm{g}(x)=-3+\sqrt{(x+2)^2-1+1}=-3+x+2=x-1$ $\mathrm{f}^{-1}\mathrm{g}(x)=\begin{cases}x-1 & \text{for } -1<x\leq 0\\ -3+\sqrt{x} & \text{for } 0<x<1\end{cases}$ Hence, $\mathrm{p}(x)=x-1$ and $\mathrm{q}(x)=-3+\sqrt{x}$

5(a) Horizontal Asymptote:

$y = 0$

Vertical Asymptote:

$x^2 + 2x - 3 = (x+3)(x-1)$

Vertical asymptotes are $x = -3,\ x = 1$.

(b) $yx^2 + 2yx - 3y = 2x - 6$

$yx^2 + (2y-2)x + (6-3y) = 0$

Since $x \in \mathbb{R}$,

Discriminant $(2y-2)^2 - 4y(6-3y) \geq 0$

$16y^2 - 32y + 4 \geq 0$

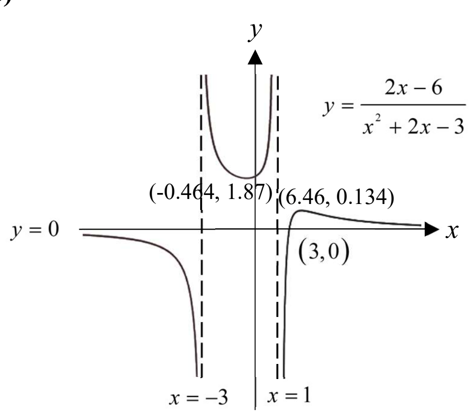
$4y^2 - 8y + 1 \geq 0$

As roots of $4y^2 - 8y + 1 = 0$ are:

$$y = \frac{8 \pm \sqrt{8^2 - 4(4)(1)}}{2(4)} = 1 \pm \frac{\sqrt{3}}{2},$$

$$y \leq 1 - \frac{\sqrt{3}}{2} \text{ OR } y \geq 1 + \frac{\sqrt{3}}{2}.$$

(c)



(d) Translation of *C* by 1 unit in the positive *x*-direction.

6(a) $x=\sqrt{23}\cos\left(t+\frac{\pi}{6}\right)$, $y=2\sin t$

At $\left(0,\sqrt{3}\right)$, $t=\frac{\pi}{3}$

$\frac{\mathrm{d}x}{\mathrm{d}t}=-\sqrt{23}\sin\left(t+\frac{\pi}{6}\right)$, $\frac{\mathrm{d}y}{\mathrm{d}t}=2\cos t$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2\cos t}{-\sqrt{23}\sin\left(t+\frac{\pi}{6}\right)}$$

When $t=\frac{\pi}{3}$, $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{1}{\sqrt{23}}$

Hence, gradient of normal $=\sqrt{23}$

Equation of normal: $y=\sqrt{23}x+\sqrt{3}$

(b) $x=\sqrt{23}\cos\left(t+\frac{\pi}{6}\right)$

To find min x-coordinate,

Method 1 (range of values of cosine)

Note that $-1\le\cos\left(t+\frac{\pi}{6}\right)\le 1$. Hence, minimum x-coordinate occurs when $x=-\sqrt{23}$. This occurs when $t=\frac{5\pi}{6}$.

Method 2 (differentiation)

$\frac{\mathrm{d}x}{\mathrm{d}t}=-\sqrt{23}\sin\left(t+\frac{\pi}{6}\right)=0$

$\Rightarrow t=\frac{5\pi}{6}$ or $-\frac{\pi}{6}$(rejected $\because 0\le t\le\pi$)

When $t=\frac{5\pi}{6}$, $\frac{\mathrm{d}y}{\mathrm{d}x}$ is undefined. Hence the tangent of curve at $t=\frac{5\pi}{6}$ is **parallel to the *y*-axis**. Therefore, equation of tangent is $x=-\sqrt{23}$

When tangent intersects l,

$x=-\sqrt{23}\Rightarrow y=\sqrt{23}\left(-\sqrt{23}\right)+\sqrt{3}=\sqrt{3}-23$

$\therefore R\left(-\sqrt{23},\sqrt{3}-23\right)$

(c) $\dfrac{dy}{dx} = \dfrac{2\cos t}{-\sqrt{23}\sin\left(t+\dfrac{\pi}{6}\right)}$

For stationary points, $\dfrac{dy}{dx} = 0$:

$$\frac{2\cos t}{-\sqrt{23}\sin\left(t+\frac{\pi}{6}\right)} = 0 \Rightarrow \cos t = 0$$

Since $0 \le t \le \pi$, then $t = \dfrac{\pi}{2}$

Since there is **only one** value of t such that $\dfrac{dy}{dx} = 0$, then C has only one stationary point. (Shown)

When $t = \dfrac{\pi}{2}$, $x = \sqrt{23}\cos\left(\dfrac{\pi}{2}+\dfrac{\pi}{6}\right) = \dfrac{-\sqrt{23}}{2}$

x	$\left(\dfrac{-\sqrt{23}}{2}\right)^-$	$\dfrac{-\sqrt{23}}{2}$	$\left(\dfrac{-\sqrt{23}}{2}\right)^+$
t	$\left(\dfrac{\pi}{2}\right)^+$ (for e.g. 1.58)	$\dfrac{\pi}{2}$	$\left(\dfrac{\pi}{2}\right)^-$ (for e.g. 1.56)
Value of $\dfrac{dy}{dx}$	0.00445577	0	-0.0051669
Explain	$\cos t < 0$ and $\sin\left(t+\dfrac{\pi}{6}\right) > 0$	0	$\cos t > 0$ and $\sin\left(t+\dfrac{\pi}{6}\right) > 0$
Sign of $\dfrac{dy}{dx}$	+	0	-
Nature of stationary value	Maximum value		

Hence, maximum value occurs when $t = \dfrac{\pi}{2}$.

7(a) Two vectors parallel to p are $\begin{pmatrix}1\\-1\\2\end{pmatrix}$ and $\begin{pmatrix}2\\0\\-1\end{pmatrix}-\begin{pmatrix}11\\-3\\-1\end{pmatrix}=3\begin{pmatrix}-3\\1\\0\end{pmatrix}$.

$$\begin{pmatrix}1\\-1\\2\end{pmatrix}\times\begin{pmatrix}-3\\1\\0\end{pmatrix}=\begin{pmatrix}-2\\-6\\-2\end{pmatrix}=-2\begin{pmatrix}1\\3\\1\end{pmatrix}.$$

So a vector normal to p is $\begin{pmatrix}1\\3\\1\end{pmatrix}$.

Since $A(2,0,-1)$ lies on p, equation of p is $\mathbf{r}\cdot\begin{pmatrix}1\\3\\1\end{pmatrix}=\begin{pmatrix}2\\0\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\3\\1\end{pmatrix}=1$.

So a cartesian equation of p is $x+3y+z=1$.

(b) Let G be the foot of perpendicular from A to l_1.

Since G lies on l_1, $\overrightarrow{OG}=\begin{pmatrix}11\\-3\\-1\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\2\end{pmatrix}$ for some λ.

$$\overrightarrow{AG}\cdot\begin{pmatrix}1\\-1\\2\end{pmatrix}=0 \Rightarrow \left[\begin{pmatrix}11\\-3\\-1\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\2\end{pmatrix}-\begin{pmatrix}2\\0\\-1\end{pmatrix}\right]\cdot\begin{pmatrix}1\\-1\\2\end{pmatrix}=0$$
$$\Rightarrow 12+6\lambda-0=0$$
$$\Rightarrow \lambda=-2$$

Hence, $\overrightarrow{OG}=\begin{pmatrix}9\\-1\\-5\end{pmatrix}$.

Using Ratio Theorem, $\overrightarrow{OG}=\dfrac{\overrightarrow{OA}+\overrightarrow{OA'}}{2}$.

$$\therefore\quad \overrightarrow{OA'}=2\overrightarrow{OG}-\overrightarrow{OA}=2\begin{pmatrix}9\\-1\\-5\end{pmatrix}-\begin{pmatrix}2\\0\\-1\end{pmatrix}=\begin{pmatrix}16\\-2\\-9\end{pmatrix}.$$

<u>Alternative solution (using projection vector to find $\overrightarrow{OG}$)</u>

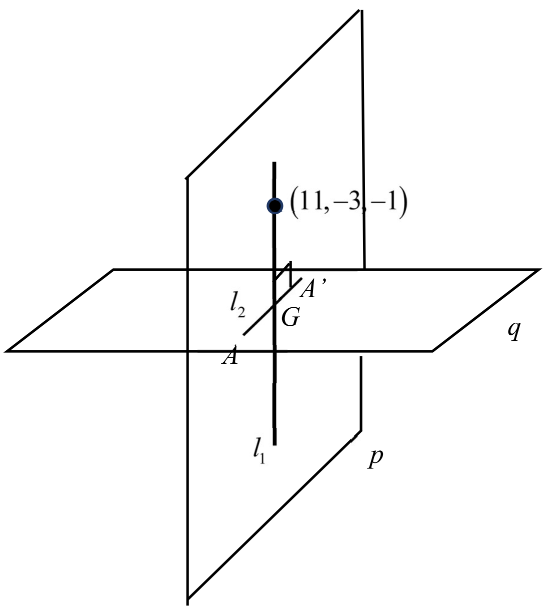
$$\overrightarrow{BG} = \left[\overrightarrow{BA} \cdot \frac{1}{\sqrt{6}}\begin{pmatrix}1\\-1\\2\end{pmatrix}\right]\frac{1}{\sqrt{6}}\begin{pmatrix}1\\-1\\2\end{pmatrix}$$

$$= \frac{1}{6}\left[\begin{pmatrix}-9\\3\\0\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\2\end{pmatrix}\right]\begin{pmatrix}1\\-1\\2\end{pmatrix}$$

$$= -2\begin{pmatrix}1\\-1\\2\end{pmatrix}$$

$$\therefore \quad \overrightarrow{OG} = -2\begin{pmatrix}1\\-1\\2\end{pmatrix} + \begin{pmatrix}11\\-3\\-1\end{pmatrix} = \begin{pmatrix}9\\-1\\-5\end{pmatrix}.$$

(c)



A direction vector of l_2 is $\overrightarrow{AG} = \begin{pmatrix}7\\-1\\-4\end{pmatrix}$.

A vector equation of l_2 is $\mathbf{r} = \begin{pmatrix}2\\0\\-1\end{pmatrix} + \mu\begin{pmatrix}7\\-1\\-4\end{pmatrix}, \quad \mu \in \mathbb{R}$.

<u>Alternative solution (consider l_2 as intersection of p and q)</u>

$x + 3y + z = 1$ --- (1)

$$\mathbf{r}\cdot\begin{pmatrix}1\\-1\\2\end{pmatrix}=\begin{pmatrix}2\\0\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\2\end{pmatrix}=0\Rightarrow x-y+2z=0 \quad \text{--- (2)}$$

Using GC to solve (1) and (2),

A equation of l_2 is $\mathbf{r}=\begin{pmatrix}\frac{1}{4}\\\frac{1}{4}\\0\end{pmatrix}+\mu\begin{pmatrix}-\frac{7}{4}\\\frac{1}{4}\\1\end{pmatrix}$, $\mu\in\mathbb{R}$.

(d) Note that the points $G(9,-1,-5)$ on q and $B(11,-3,-1)$ on $\prod$, lie on the same line l_1, and l_1 is perpendicular to both q and $\prod$.

Perpendicular distance between $\prod$ and q $=\left|\overrightarrow{GB}\right|=\left|\begin{pmatrix}11\\-3\\-1\end{pmatrix}-\begin{pmatrix}9\\-1\\-5\end{pmatrix}\right|=\left|\begin{pmatrix}2\\-2\\4\end{pmatrix}\right|=2\sqrt{6}$ units

Alternative solution (using formula)

Plane $\prod$: $\mathbf{r}\cdot\begin{pmatrix}1\\-1\\2\end{pmatrix}=\begin{pmatrix}11\\-3\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\2\end{pmatrix}=12$

Plane q: $\mathbf{r}\cdot\begin{pmatrix}1\\-1\\2\end{pmatrix}=0$

Perpendicular distance $=\dfrac{|d_1-d_2|}{|\mathbf{n}|}=\dfrac{|12-0|}{\left|\begin{pmatrix}1\\-1\\2\end{pmatrix}\right|}=\dfrac{12}{\sqrt{6}}$ unit

Alternative solution (using projection of $\overrightarrow{AB}$ on normal)

Perp dist $=\dfrac{\left|\overrightarrow{AB}\cdot\begin{pmatrix}1\\-1\\2\end{pmatrix}\right|}{\left|\begin{pmatrix}1\\-1\\2\end{pmatrix}\right|}=\dfrac{\left|\begin{pmatrix}9\\-3\\0\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\2\end{pmatrix}\right|}{\left|\begin{pmatrix}1\\-1\\2\end{pmatrix}\right|}=\dfrac{12}{\sqrt{6}}$

units

Alternative solution (using cross product)

Perpendicular distance $=\dfrac{\left|\overrightarrow{AB}\times\overrightarrow{AG}\right|}{\left|\overrightarrow{AG}\right|}$

$=\dfrac{\left|\begin{pmatrix}9\\-3\\0\end{pmatrix}\times\begin{pmatrix}7\\-1\\-4\end{pmatrix}\right|}{\left|\begin{pmatrix}7\\-1\\-4\end{pmatrix}\right|}=\dfrac{\left|\begin{pmatrix}12\\36\\12\end{pmatrix}\right|}{\left|\begin{pmatrix}7\\-1\\-4\end{pmatrix}\right|}=\dfrac{\sqrt{1584}}{\sqrt{66}}=\sqrt{24}$ units

8(a)	When $n=1$ $u_2 = 3(2)+A(1)+B=5.5 \Rightarrow A+B=-0.5$ ---(1) When $n=2$, $u_3=3(5.5)+A(2)+B=17.5 \Rightarrow 2A+B=1$ ---(2) Solving, $A=1.5, B=-2$ So $u_{n+1}=3u_n+1.5n-2$ When $n=3$, $u_4 = 3u_3+1.5(3)-2$ $=3(17.5)+4.5-2$ $=55$
(b)(i)	$\mathrm{f}(r)-\mathrm{f}(r-1)=\left(2r^3+3r^2+4r+5\right)$ $-\left(2(r-1)^3+3(r-1)^2+4(r-1)+5\right)$ $=\left(2r^3+3r^2+4r+5\right)$ $-\left(2r^3-6r^2+6r-2+3r^2-6r+3+4r-4+5\right)$ $=6r^2+3$ $\therefore \mathrm{f}(r)-\mathrm{f}(r-1)=6r^2+3$ $\Rightarrow \sum_{r=1}^{n}\left(6r^2+3\right)=\sum_{r=1}^{n}\left[\mathrm{f}(r)-\mathrm{f}(r-1)\right]$ $\Rightarrow 6\left(\sum_{r=1}^{n}r^2\right)+3n=\sum_{r=1}^{n}\left[\mathrm{f}(r)-\mathrm{f}(r-1)\right]$ $\Rightarrow \sum_{r=1}^{n}r^2=\frac{1}{6}\left\{\sum_{r=1}^{n}\left[\mathrm{f}(r)-\mathrm{f}(r-1)\right]-3n\right\}$ $\sum_{r=1}^{n}\left[\mathrm{f}(r)-\mathrm{f}(r-1)\right]=\begin{bmatrix} \mathrm{f}(\cancel{1})-\mathrm{f}(0) \\ +\mathrm{f}(\cancel{2})-\mathrm{f}(\cancel{1}) \\ +\mathrm{f}(\cancel{3})-\mathrm{f}(\cancel{2}) \\ \vdots \\ +\mathrm{f}(\cancel{n-2})-\mathrm{f}(\cancel{n-3}) \\ +\mathrm{f}(\cancel{n-1})-\mathrm{f}(\cancel{n-2}) \\ +\mathrm{f}(n)-\mathrm{f}(\cancel{n-1}) \end{bmatrix}$ $=\mathrm{f}(n)-\mathrm{f}(0)$ Then $\sum_{r=1}^{n}r^2=\frac{1}{6}\left[\mathrm{f}(n)-\mathrm{f}(0)-3n\right]$

$$\Rightarrow \quad \sum_{r=1}^{n} r^2 = \frac{2n^3+3n^2+4n+5-5-3n}{6}$$
$$= \frac{n\left(2n^2+3n+1\right)}{6}$$
$$= \frac{n(n+1)(2n+1)}{6} \text{ (shown)}$$

(b)(ii) $\sum_{r=1}^{n} \mathrm{f}(r) = \sum_{r=1}^{n}\left(2r^3+3r^2+4r+5\right)$

$$= 2\sum_{r=1}^{n}\left(r^3\right) + 3\sum_{r=1}^{n}\left(r^2\right) + 4\sum_{r=1}^{n}(r) + \sum_{r=1}^{n}(5)$$
$$= 2\left[\frac{n^2(n+1)^2}{4}\right] + 3\left[\frac{n(n+1)(2n+1)}{6}\right] + 4\left[\frac{n(n+1)}{2}\right] + 5n$$
$$= \frac{\left[n(n+1)\right]\left[n(n+1)\right]}{2} + \frac{n(n+1)(2n+1)}{2} + \frac{4n(n+1)}{2} + 5n$$
$$= \frac{n(n+1)\left[n(n+1)+2n+1+4\right]}{2} + 5n$$
$$= \frac{n(n+1)\left(n^2+3n+5\right)}{2} + 5n \text{ (shown)}$$

9(a) Note that the respective coordinates are $\mathrm{C}(0,2)$, $\mathrm{D}(2,0)$ and $\mathrm{E}(3.5,8.25)$.

Method 1 (area w.r.t. y-axis)

$$\text{Area of cross-section} = 2\left[\int_0^{8.25}\sqrt{y+4}\ \mathrm{d}y - \int_0^2\sqrt{\frac{8}{\pi}\cos^{-1}\frac{y}{2}}\ \mathrm{d}y\right]$$
$$= 40.3\ \text{cm}^2$$

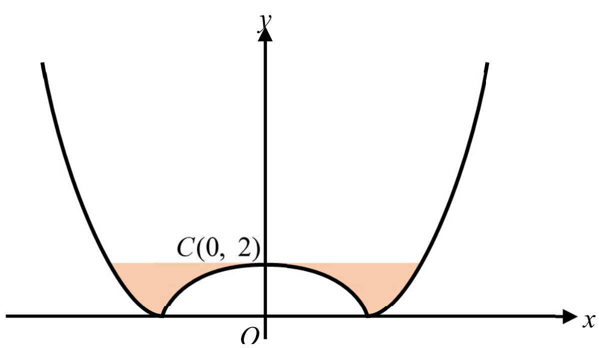
Method 2 (mixed area between x- and y-axis)

$$\text{Area of cross-section} = 2\left[\int_0^{8.25}\sqrt{y+4}\ \mathrm{d}y - \int_0^2 2\cos\left(\frac{\pi x^2}{8}\right)\mathrm{d}x\right]$$
$$= 40.3\ \text{cm}^2$$

Method 3 (area w.r.t. x-axis)

$$\text{Area of cross-section} = 2\left[3.5(8.25) - \int_0^2 2\cos\left(\frac{\pi x^2}{8}\right)\mathrm{d}x - \int_2^{3.5}(x^2-4)\ \mathrm{d}x\right]$$
$$= 40.3\ \text{cm}^2$$

(b)



When volume is k cm^3, espresso level just touches $C(0,\ 2)$.

Method 1: Integration by parts

$$k = \pi\int_0^2 (y+4)\ \mathrm{d}y - \pi\int_0^2 \frac{8}{\pi}\cos^{-1}\frac{y}{2}\ \mathrm{d}y$$
$$= \pi\left[\frac{y^2}{2}+4y\right]_0^2 - 8\int_0^2\cos^{-1}\frac{y}{2}\ \mathrm{d}y$$
$$= 10\pi - 8\left\{\left[y\cos^{-1}\frac{y}{2}\right]_0^2 + \frac{1}{2}\int_0^2\frac{y}{\sqrt{1-(\frac{y}{2})^2}}\ \mathrm{d}y\right\}$$
$$= 10\pi + 16\left[\sqrt{1-(\tfrac{y}{2})^2}\right]_0^2$$
$$= 10\pi - 16$$

Method 2: Integration using substitution

$$k=\pi\int_0^2 (y+4)\ \mathrm{d}y-\pi\int_0^2 \frac{8}{\pi}\cos^{-1}\frac{y}{2}\ \mathrm{d}y$$

$$=\pi\left[\frac{y^2}{2}+4y\right]_0^2-8\int_0^2\cos^{-1}\frac{y}{2}\ \mathrm{d}y$$

$$=10\pi-8\int_0^2\cos^{-1}\frac{y}{2}\ \mathrm{d}y$$

Let $u=\cos^{-1}\frac{y}{2}\Rightarrow y=2\cos u$

$$\frac{\mathrm{d}y}{\mathrm{d}u}=-2\sin u$$

When $y=0,\ u=\frac{\pi}{2}$

$y=2,\ u=0$

$$\int_0^2\cos^{-1}\frac{y}{2}\ \mathrm{d}y=\int_{\frac{\pi}{2}}^0 u\ \left(-2\sin u\ \mathrm{d}u\right)$$

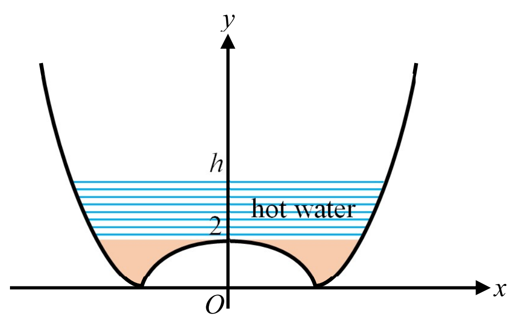
$$=\int_0^{\frac{\pi}{2}}2u\sin u\ \mathrm{d}u$$

$$=\left[-2u\cos u\right]_0^{\frac{\pi}{2}}+\int_0^{\frac{\pi}{2}}2\cos u\ \mathrm{d}u$$

$$=2$$

$\therefore k=10\pi-8\left(2\right)=10\pi-16$

(c)



Let the height be h cm after hot water is added.

Volume of hot water $=\pi\int_2^h (y+4)\ \mathrm{d}y$

$$=\pi\left[\frac{y^2}{2}+4y\right]_2^h=14\pi$$

$$\Rightarrow\frac{h^2}{2}+4h-10=14$$

$$\Rightarrow h^2+8h-48=0$$

Using GC, $h=4$ or -12 (rej.).

$\therefore$ Radius of the top surface $=\sqrt{h+4}=2\sqrt{2}$ cm.

10(a)	$x^2+(y-5)^2=25$
(b)(i)	$\dfrac{\mathrm{d}V}{\mathrm{d}t}=k$
(ii)	As the volume of water in the container is decreasing with time, $k<0$.
(iii)	$\int \dfrac{\mathrm{d}V}{\mathrm{d}t}\,\mathrm{d}t=\int k\ \mathrm{d}t$ $\therefore V=kt+C$, where C is an arbitrary constant When $t=0$, $\dfrac{4}{3}\pi(5)^3=k(0)+C \Rightarrow C=\dfrac{500\pi}{3}$
(c)(i)	Area of hole, $\alpha=\pi\left(\dfrac{1}{100}\right)^2=\dfrac{\pi}{10000}\text{m}^2$ So $\dfrac{\mathrm{d}V}{\mathrm{d}t}=-\dfrac{\pi}{10000}\sqrt{20h}=-\dfrac{\sqrt{5}\pi}{5000}\sqrt{h}$ By Chain Rule, $\dfrac{\mathrm{d}V}{\mathrm{d}h}\times\dfrac{\mathrm{d}h}{\mathrm{d}t}=\dfrac{\mathrm{d}V}{\mathrm{d}t}$ (so we need to find $\dfrac{\mathrm{d}V}{\mathrm{d}h}$) <u>Method 1 (use direct expression for *V*)</u> $V=\pi\int_0^h 25-(y-5)^2\ \mathrm{d}y$ $=\pi\int_0^h 10y-y^2\ \mathrm{d}y$ $=\pi\left[5y^2-\dfrac{y^3}{3}\right]_0^h$, $=\pi\left[5h^2-\dfrac{h^3}{3}\right]$ $\dfrac{\mathrm{d}V}{\mathrm{d}h}=\pi\left(10h-h^2\right)$ Then we have $\pi\left(25-(h-5)^2\right)\times\dfrac{\mathrm{d}h}{\mathrm{d}t}=-\dfrac{\sqrt{5}\pi}{5000}\sqrt{h}$ $\left(10h-h^2\right)\dfrac{\mathrm{d}h}{\mathrm{d}t}=-\dfrac{\sqrt{5}}{5000}\sqrt{h}$ $\left(10h^{\frac{1}{2}}-h^{\frac{3}{2}}\right)\dfrac{\mathrm{d}h}{\mathrm{d}t}=-\dfrac{\sqrt{5}}{5000}$ (shown) <u>Method 2 (different expression of *V*)</u>

		$V=\dfrac{500\pi}{3}-\pi\int_h^{10} 25-(y-5)^2\ \mathrm{d}y$ $=\dfrac{500\pi}{3}-\pi\int_h^{10} 10y-y^2\ \mathrm{d}y$ $=\dfrac{500\pi}{3}-\pi\left[5y^2-\dfrac{y^3}{3}\right]_h^{10}$ $=\dfrac{500\pi}{3}-\pi\left[\left(5(10)^2-\dfrac{(10)^3}{3}\right)-\left(5h^2-\dfrac{h^3}{3}\right)\right]$ $=\pi\left[5h^2-\dfrac{h^3}{3}\right]$ $\dfrac{\mathrm{d}V}{\mathrm{d}h}=\pi\left(10h-h^2\right)$ Then we have $\pi\left(10h-h^2\right)\dfrac{\mathrm{d}h}{\mathrm{d}t}=-\dfrac{\sqrt{5}\pi}{5000}\sqrt{h}$ $\left(10h^{\frac{1}{2}}-h^{\frac{3}{2}}\right)\dfrac{\mathrm{d}h}{\mathrm{d}t}=-\dfrac{\sqrt{5}}{5000}$ (shown)
	(ii)	From the DE, $\left(10h^{\frac{1}{2}}-h^{\frac{3}{2}}\right)\dfrac{\mathrm{d}h}{\mathrm{d}t}=-\dfrac{\sqrt{5}}{5000}$ $\int 10h^{\frac{1}{2}}-h^{\frac{3}{2}}\ \mathrm{d}h=\int -\dfrac{\sqrt{5}}{5000}\ \mathrm{d}t$ $10\left(\dfrac{2}{3}h^{\frac{3}{2}}\right)-\dfrac{2}{5}h^{\frac{5}{2}}=-\dfrac{\sqrt{5}}{5000}t+B$ When $t=0$, $h=10$: $B=\dfrac{20}{3}(10)^{\frac{3}{2}}-\dfrac{2}{5}(10)^{\frac{5}{2}}$ $=84.327$ (to 5sf) When $h=0$: $\dfrac{\sqrt{5}}{5000}t=84.327$ $t=188561$ $=189000$ (to 3sf)