<u>2022 JC2 H2 Math Paper 1 Suggested Solutions:</u>

1.

$$\begin{aligned}
\mathrm{f}'(x) &= \lim_{\delta x \to 0} \frac{\mathrm{f}\left(x+\delta x\right)-\mathrm{f}\left(x\right)}{\left(x+\delta x\right)-x} \\
&= \lim_{\delta x \to 0} \frac{\dfrac{1}{\left(x+\delta x\right)^2}-\dfrac{1}{x^2}}{\delta x} \\
&= \lim_{\delta x \to 0} \frac{\dfrac{x^2-\left(x+\delta x\right)^2}{\left(x+\delta x\right)^2 x^2}}{\delta x} \\
&= \lim_{\delta x \to 0} \frac{x^2-\left(x^2+2x\delta x+\left(\delta x\right)^2\right)}{\delta x\left(x+\delta x\right)^2 x^2} \\
&= \lim_{\delta x \to 0} \frac{-2x\delta x-\left(\delta x\right)^2}{\delta x\left(x+\delta x\right)^2 x^2} \\
&= \lim_{\delta x \to 0} \frac{-2x-\left(\delta x\right)}{\left(x+\delta x\right)^2 x^2} \\
&= \frac{-2x}{\left(x\right)^2 x^2} \\
&= -\frac{2}{x^3}
\end{aligned}$$

2.

Let the equation be $y = ax^2 + bx + c$, where a, b and c are constants.

Given that when $x = \frac{1}{4}$, $\frac{dy}{dx} = 2ax+b=0 \Rightarrow 2a(\frac{1}{4})+b=0 \Rightarrow \frac{1}{2}a+b=0$-----(1)

Also, since $\left(\frac{1}{4}, \frac{57}{8}\right)$ lies on the curve, $\frac{a}{16}+\frac{b}{4}+c=\frac{57}{8}$----(2)

Given when x = 1, $\Rightarrow 2a+b=$ -3-----(3)

Solving simultaneously using GC, a = -2, b = 1 and c = 7

Hence, equation of curve is $y=-2x^2+x+7$

Alternative Method

Let the equation of the curve be $y=-k\left(x-\frac{1}{4}\right)^2+\frac{57}{8}$.

Then $\frac{dy}{dx}=-2k\left(x-\frac{1}{4}\right)$

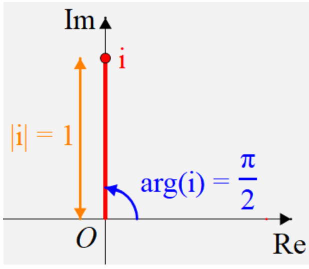
Using the given information, k = 2.

So the equation is $y=-2\left(x-\frac{1}{4}\right)^2+\frac{57}{8}$

3. (i)

$$r = \left|\frac{\mathrm{i}}{z}\right| = \frac{|\mathrm{i}|}{|z|} = \frac{1}{2}$$

$$\theta = \arg\left(\frac{\mathrm{i}}{z}\right) = \arg(\mathrm{i}) - \arg z$$



$$= \frac{\pi}{2} - \left(-\frac{5\pi}{6}\right)$$

$$= \frac{4\pi}{3} \equiv \frac{4\pi}{3} - 2\pi \pmod{2\pi}$$

$$= -\frac{2\pi}{3}$$

(ii)

$$\left|\left(\frac{\mathrm{i}}{z}\right)^*\right| = \left|\frac{\mathrm{i}}{z}\right| = \frac{1}{2}$$

$$\arg\left[\left(\frac{\mathrm{i}}{z}\right)^*\right] = -\arg\left(\frac{\mathrm{i}}{z}\right)$$

$$= \frac{2}{3}\pi$$

$$\therefore \left(\frac{\mathrm{i}}{z}\right)^* = \frac{1}{2}\left(\cos\frac{2\pi}{3} + \mathrm{i}\sin\frac{2\pi}{3}\right)$$

$$= \frac{1}{2}\left(-\frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2}\right)$$

$$= -\frac{1}{4} + \mathrm{i}\frac{\sqrt{3}}{4}$$

<u>Using Exponential Form</u>

$$\frac{\mathrm{i}}{z} = \frac{1}{2}\cdot e^{\mathrm{i}\left(-\frac{2\pi}{3}\right)}$$

$$\left(\frac{\mathrm{i}}{z}\right)^* = \left[\frac{1}{2}\cdot e^{\mathrm{i}\left(-\frac{2\pi}{3}\right)}\right]^*$$

$$= \frac{1}{2}\cdot e^{-\mathrm{i}\left(-\frac{2\pi}{3}\right)}$$

$$= \frac{1}{2}\cdot \underline{e^{\mathrm{i}\left(\frac{2\pi}{3}\right)}}$$

$$= \frac{1}{2}\underline{\left(\cos\frac{2\pi}{3} + \mathrm{i}\sin\frac{2\pi}{3}\right)}$$

$$= \frac{1}{2}\left(-\frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2}\right)$$

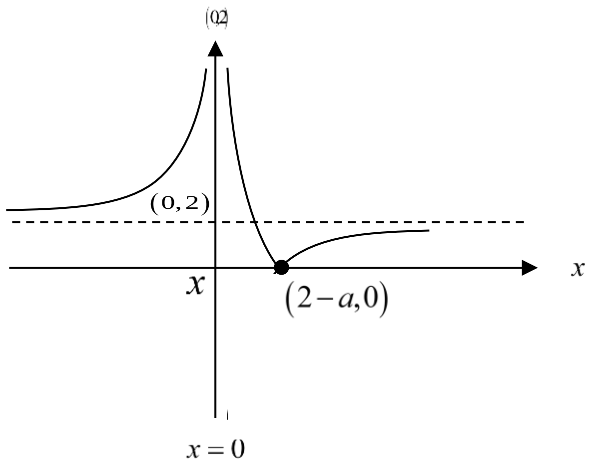
$$= -\frac{1}{4} + \mathrm{i}\frac{\sqrt{3}}{4}$$

4. (a) In any order:

Scale parallel to the y – axis by scale factor of $\frac{1}{2}$.

Translate 1 unit in the positive x – direction.

(b)(i)



(b)(ii) $x = -2,\ x = 2$

5.

(i) $y=\sqrt{1+\cos^2 x}$

Squaring both sides: $y^2=1+\cos^2 x$

Diff wrt x:

$$2y\frac{\mathrm{d}y}{\mathrm{d}x}=2\cos x(-\sin x)$$

$$2y\frac{\mathrm{d}y}{\mathrm{d}x}+\sin 2x=0 \quad \text{(shown)}$$

Diff wrt x:

$$2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\frac{\mathrm{d}y}{\mathrm{d}x}\left(2\frac{\mathrm{d}y}{\mathrm{d}x}\right)+2\cos 2x=0$$

$$\Rightarrow \quad 2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+2\cos 2x=0$$

When x = 0, $y=\sqrt{2}$, $\frac{\mathrm{d}y}{\mathrm{d}x}=0$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\frac{1}{\sqrt{2}}=-\frac{\sqrt{2}}{2}$

$$y=\sqrt{1+\cos^2 x}=\sqrt{2}-\frac{\sqrt{2}}{4}x^2+\ldots$$

(ii) $y=\sqrt{1+\cos^2 x}=\sqrt{1+\left(1-\frac{x^2}{2}\right)^2}$

$$=\sqrt{1+\left(1-x^2+\ldots\right)}$$

$$=\left(2-x^2\right)^{\frac{1}{2}}=\left(2\left(1-\frac{x^2}{2}\right)\right)^{\frac{1}{2}}$$

$$=\sqrt{2}\left(1-\frac{1}{2}\left(\frac{x^2}{2}\right)+\ldots\right)$$

$$=\sqrt{2}\left(1-\frac{1}{2}\left(\frac{x^2}{2}\right)+\ldots\right)=\sqrt{2}-\frac{\sqrt{2}}{4}x^2+\ldots \quad \text{(shown)}$$

Valid for $\left|\frac{x^2}{2}\right|<1\Rightarrow -\sqrt{2}<x<\sqrt{2}$

6. (i)(a)

$$\int \frac{x^3}{(1+x^4)^2}\,dx = \frac{1}{4}\int 4x^3\left(1+x^4\right)^{-2} dx$$
$$= \frac{1}{4}\frac{\left(1+x^4\right)^{-1}}{(-1)} + c$$
$$= -\frac{1}{4}\left(1+x^4\right)^{-1} + c$$

(i)(b)

<u>Method 1</u> <u>(use double angle formula for tangent and then trigo integration formula)</u>

$$\int \frac{2\tan x}{1-\tan^2 x}\,dx = \int \tan 2x\,dx$$
$$= \frac{1}{2}\ln\left|\sec 2x\right| + c$$

<u>Method 2</u> (use double angle formula for tangent and then f '(x) formulat)

$$\int \frac{2\tan x}{1-\tan^2 x}\,dx = \int \tan 2x\,dx$$
$$= \int \frac{\sin 2x}{\cos 2x}\,dx = -\frac{1}{2}\int \frac{2\sin 2x}{\cos 2x}\,dx$$
$$= -\frac{1}{2}\ln\left|\cos 2x\right| + c$$

(ii)

<u>Method 1</u> <u>(Simplify using Factor Formula)</u>

$$\int_0^{\pi} \cos\left[(a+1)x\right]\sin ax\,dx$$
$$= \frac{1}{2}\int_0^{\pi} \sin(2a+1)x - \sin x\,dx$$
$$= \frac{1}{2}\left[\frac{-\cos(2a+1)x}{2a+1} + \cos x\right]_0^{\pi}$$
$$= \frac{1}{2}\left[\left(\frac{-\cos(2a+1)\pi}{2a+1} + \cos\pi\right) - \left(\frac{-\cos(2a+1)0}{2a+1} + \cos 0\right)\right]$$
$$= \frac{1}{2}\left[\left(\frac{-1(-1)}{2a+1} + (-1)\right) - \left(\frac{-1}{2a+1} + 1\right)\right]$$
$$= \frac{1}{2}\left(\frac{2}{2a+1} - 2\right)$$
$$= \frac{1}{2a+1} - 1 \quad \text{or} \quad -\frac{2a}{2a+1}$$

<u>Method 2</u> <u>(By Parts)</u>

$u = \cos\left[(a+1)x\right]$	$dv = \sin ax$
$du = -(a+1)\sin\left[(a+1)x\right]$	

$u' = \sin\left[(a+1)x\right]$	$dv' = \cos ax$
$du' = (a+1)\cos\left[(a+1)x\right]$	$v' = \frac{1}{a}\sin ax$

$$\int_0^{\pi} \cos\left[(a+1)x\right]\sin ax\,dx$$
$$= \left[-\frac{1}{a}\cos(a+1)x\cos ax\right]_0^{\pi} - \int_0^{\pi}\frac{a+1}{a}\sin\left[(a+1)x\right]\cos ax\,dx$$

$$=\left[-\frac{1}{a}\cos(a+1)x\cos ax\right]_0^\pi-\left(\frac{a+1}{a}\right)\left\{\left[\frac{1}{a}\sin(a+1)x\sin ax\right]_0^\pi-\int_0^\pi\left(\frac{a+1}{a}\right)\cos\left[(a+1)x\right]\sin ax\ \mathrm{d}x\right\}$$

$$=\left[-\frac{1}{a}\cos(a+1)x\cos ax\right]_0^\pi-\left(\frac{a+1}{a}\right)\left[\frac{1}{a}\sin(a+1)x\sin ax\right]_0^\pi+\left(\frac{a+1}{a}\right)^2\int_0^\pi\cos\left[(a+1)x\right]\sin ax\ \mathrm{d}x$$

$$\therefore\ \left[1-\left(\frac{a+1}{a}\right)^2\right]\int_0^\pi\cos\left[(a+1)x\right]\sin ax\ \mathrm{d}x=\left[-\frac{1}{a}\cos(a+1)x\cos ax-\left(\frac{a+1}{a^2}\right)\sin(a+1)x\sin ax\right]_0^\pi$$

$$\left[1-\frac{a^2+2a+1}{a^2}\right]\int_0^\pi\cos\left[(a+1)x\right]\sin ax\ \mathrm{d}x=-\frac{1}{a}\left[\cos(a+1)x\cos ax+\left(\frac{a+1}{a}\right)\sin(a+1)x\sin ax\right]_0^\pi$$

$$\left[-\frac{2a+1}{a^2}\right]\int_0^\pi\cos\left[(a+1)x\right]\sin ax\ \mathrm{d}x=\begin{cases}-\frac{1}{a}\left[(1)(-1)+0-(1)-0\right] & \text{when } a \text{ is odd integer}\\ -\frac{1}{a}\left[(-1)(1)+0-(1)-0\right] & \text{when } a \text{ is even integer}\end{cases}$$

$$\int_0^\pi\cos\left[(a+1)x\right]\sin ax\ \mathrm{d}x=\frac{2}{a}\left(-\frac{a^2}{2a+1}\right)=-\frac{2a}{2a+1}$$

<u>**Method 3** (By Parts)</u>

$u=\sin ax$	$\mathrm{d}v=\cos(a+1)x$
$\mathrm{d}u=a\cos ax$	$v=\dfrac{\sin(a+1)x}{a+1}$

$u'=\cos ax$	$\mathrm{d}v'=\sin(a+1)x$
$\mathrm{d}u'=-a\sin ax$	$v'=-\dfrac{1}{a+1}\cos(a+1)x$

$$\int_0^\pi\cos\left[(a+1)x\right]\sin ax\ \mathrm{d}x$$

$$=\left[\frac{1}{a+1}\sin ax\sin(a+1)x\right]_0^\pi-\int_0^\pi\frac{a}{a+1}\cos ax\sin(a+1)x\ \mathrm{d}x$$

$$=\left[\frac{1}{a+1}\sin ax\sin(a+1)x\right]_0^\pi-\left(\frac{a}{a+1}\right)\left\{\left[-\frac{1}{a+1}\cos ax\cos(a+1)x\right]_0^\pi-\int_0^\pi\left(\frac{a}{a+1}\right)\sin ax\cos\left[(a+1)x\right]\ \mathrm{d}x\right\}$$

$$=\left[\frac{1}{a+1}\sin ax\sin(a+1)x\right]_0^\pi+\frac{a}{(a+1)^2}\left[\cos ax\cos(a+1)x\right]_0^\pi+\frac{a^2}{(a+1)^2}\int_0^\pi\sin ax\cos\left[(a+1)x\right]\ \mathrm{d}x$$

$$\therefore\ \left[1-\left(\frac{a}{a+1}\right)^2\right]\int_0^\pi\cos\left[(a+1)x\right]\sin ax\ \mathrm{d}x=\left[\frac{1}{a+1}\sin ax\sin(a+1)x+\frac{a}{(a+1)^2}\cos ax\cos(a+1)x\right]_0^\pi$$

$$\left[\frac{(a+1)^2-a^2}{(a+1)^2}\right]\int_0^\pi\cos\left[(a+1)x\right]\sin ax\ \mathrm{d}x=\left[\frac{1}{a+1}\sin ax\sin(a+1)x+\frac{a}{(a+1)^2}\cos ax\cos(a+1)x\right]_0^\pi$$

$$\left[\frac{2a+1}{(a+1)^2}\right]\int_0^\pi\cos\left[(a+1)x\right]\sin ax\ \mathrm{d}x=\begin{cases}\left[0+\frac{a}{(a+1)^2}(-1)(1)-0-\frac{a}{(a+1)^2}(1)(1)\right] & \text{when } a \text{ is odd integer}\\ \left[0+\frac{a}{(a+1)^2}(1)(-1)-0-\frac{a}{(a+1)^2}(1)(1)\right] & \text{when } a \text{ is even integer}\end{cases}$$

$$\int_0^\pi\cos\left[(a+1)x\right]\sin ax\ \mathrm{d}x=-\frac{2a}{(a+1)^2}\left(\frac{(a+1)^2}{2a+1}\right)=-\frac{2a}{2a+1}$$

7. (i) Given that 1980 = (area of trapezium)(y) = $11x^2y$

$$\Rightarrow y = \frac{1980}{11x^2} = \frac{180}{x^2}$$

$HD^2 = x^2 + x^2$ (Pythagoras Theorem)

$HD = x\sqrt{2}$

So A = twice the area of each trapezium + twice the area of each rectangle + area of base

$$= 22x^2 + 2xy\sqrt{2} + 10xy$$

$$= 22x^2 + 2xy\left(5+\sqrt{2}\right)$$

$$= 22x^2 + \frac{360}{x}\left(5+\sqrt{2}\right)$$

(i) At stationary point, $\frac{\mathrm{d}A}{\mathrm{d}x} = 44x - \frac{360(5+\sqrt{2})}{x^2} = 0$

Solving for x gives 3.7440 cm.

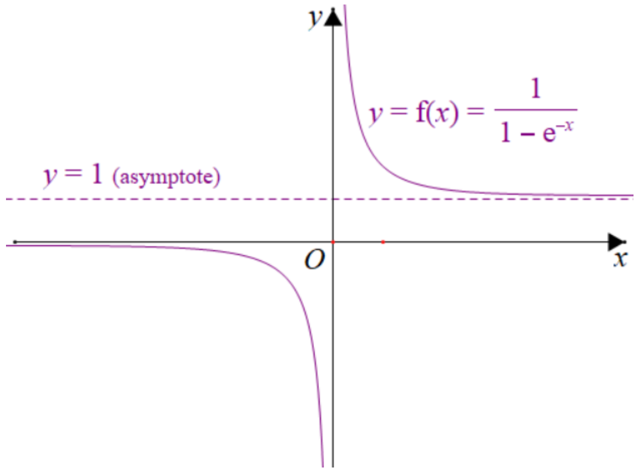
$\frac{\mathrm{d}^2A}{\mathrm{d}x^2} = 44 + \frac{720(5+\sqrt{2})}{x^3} > 0$ for all $x > 0$.

So A is a minimum.

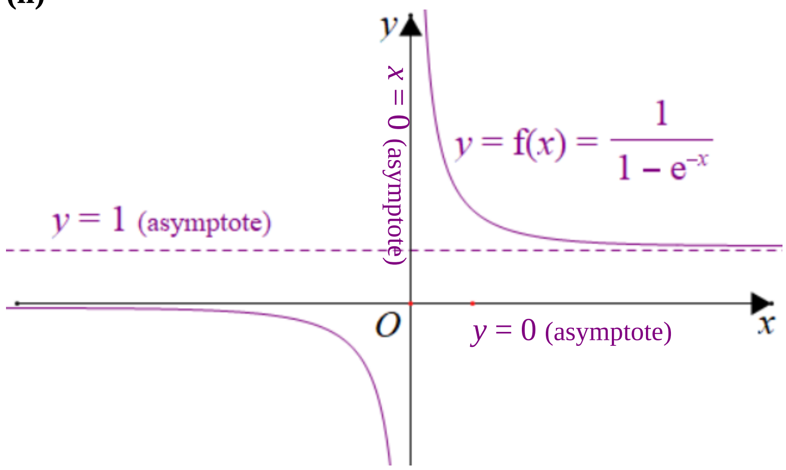
Alternatively, accept first derivative test.

Minimum A is given by $A = 925.1371 \approx 925\text{cm}^{\mathbf{2}}$ cm^3

8. (i)



(ii)



Since (use one of the following clauses or equivalent)

- Every horizontal line $y=k$ cuts the graph of $y=\mathrm{f}(x)$ at most once, OR
- Every horizontal line $y=k$ that cuts the graph of $y=\mathrm{f}(x)$ does so just once, OR
- No horizontal line $y=k$ cuts the graph of $y=\mathrm{f}(x)$ more than once,

$\therefore$ f is a one-to-one function, and its inverse function f^{-1} exists.

$$\mathrm{D}_{\mathrm{f}^{-1}}=\mathrm{R}_\mathrm{f}=(-\infty,\ 0)\cup(1,\ \infty),\quad \text{OR}$$
$$=\mathbb{R}\setminus[0,\ 1]$$

(iii)

$$\begin{aligned}\mathrm{fg}(x)&=\mathrm{f}\left(\mathrm{g}(x)\right)\\&=\mathrm{f}\left(2\ln|x-4|\right)\\&=\frac{1}{1-\mathrm{e}^{-2\ln|x-4|}}\\&=\frac{1}{1-\left(\mathrm{e}^{\ln|x-4|}\right)^{-2}}\\&=\frac{1}{1-|x-4|^{-2}}\end{aligned}$$

Alternatively

$$=\frac{1}{1-\mathrm{e}^{\ln\left(|x-4|^{-2}\right)}}$$

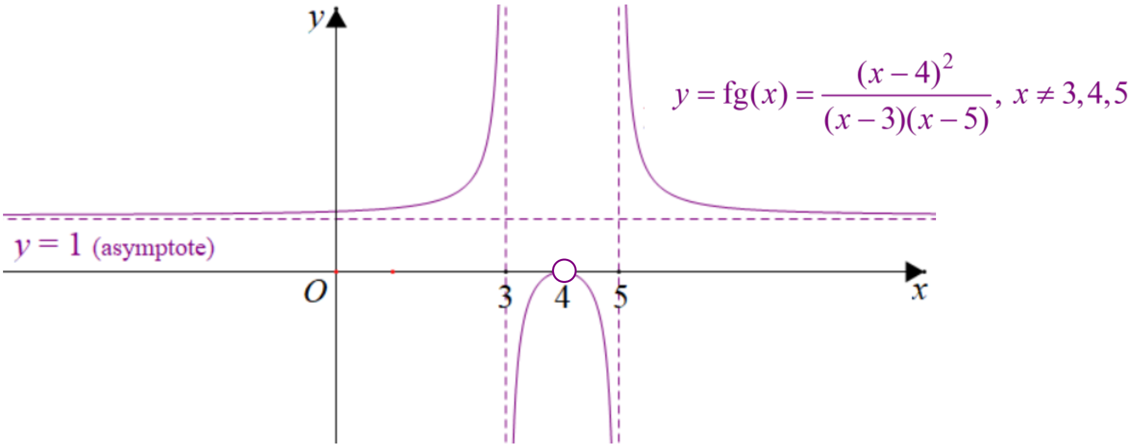
$\because\ \forall A\in\mathbb{R}^+,\ \mathrm{e}^{\ln A}=A.\ \left(\text{i.e. } \mathrm{e}^{\ln\square}=\square\right)$

$$= \frac{1}{1-\frac{1}{(x-4)^2}}$$

$$= \frac{(x-4)^2}{(x-4)^2-1}$$

$$= \frac{(x-4)^2}{(x-4+1)(x-4-1)} = \frac{(x-4)^2}{(x-3)(x-5)}. \quad \text{(shown)}$$

- Method 1: Using graph of y = fg(x)



$\therefore \; \mathrm{R_{fg}} = (-\infty, \; 0) \cup (1, \; \infty)$, OR

$= \mathbb{R} \setminus [0, \; 1]$, OR

$= \{ y \in \mathbb{R} : \; y < 0 \;\text{ or }\; y > 1 \}$.

- Method 2: Using both the graphs of y = f(x) and y = g(x), along with the “mapping” method.

$\mathrm{R_{fg}} = \mathrm{R}_{\text{f restricted to } \mathrm{R_g}}$

$= \mathrm{R}_{\text{f restricted to } \mathbb{R}\setminus\{0\}}$

$= \mathrm{R_f} \; = (-\infty, \; 0) \cup (1, \; \infty)$, OR

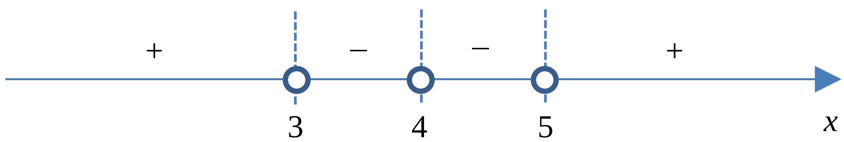
$= \mathbb{R} \setminus [0, \; 1]$.

(iv) $\mathrm{fg}(x) < 0$

$$\frac{(x-4)^2}{(x-3)(x-5)} < 0$$

Method 1 : Test-point approach

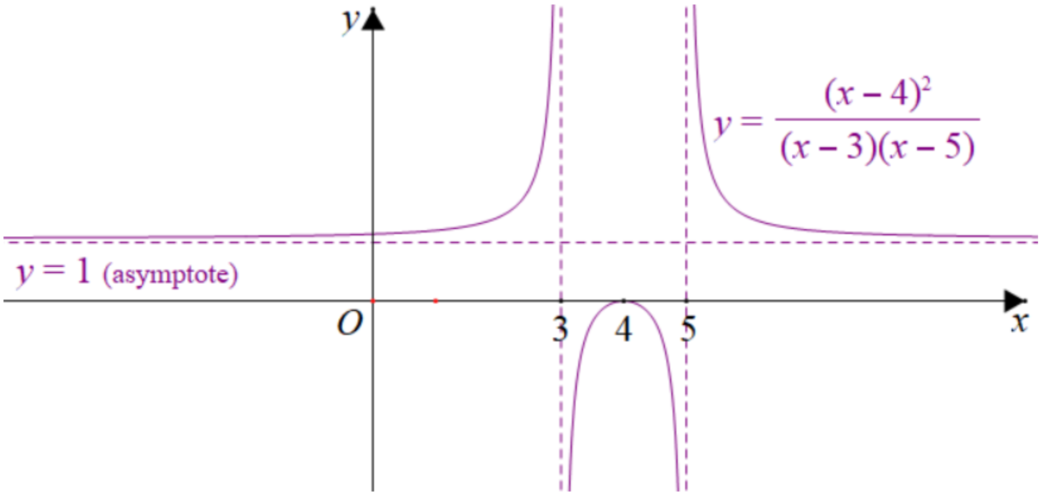
Critical values : $x = 3, \; 4, \; 5$



$\therefore \; 3 < x < 4 \;\text{ or }\; 4 < x < 5$

Accept : $(3 < x < 5 \;\text{ and }\; x \neq 4)$

<u>Method 2</u> : Graphical approach



$\therefore\ 3 < x < 5$ and $x \neq 4$

Accept : $(3 < x < 4$ or $4 < x < 5)$

9.

(a)(i) $h=\left(1-\frac{1}{2}At\right)^2 \Rightarrow \frac{dh}{dt}=2\left(1-\frac{1}{2}At\right)\left(-\frac{1}{2}A\right)=-A\sqrt{h}$

(a)(ii) Subst h = 0 and t = 20 into the general solution, we have $0=\left(1-\frac{1}{2}A(20)\right)^2 \Rightarrow A=\frac{1}{10}$

When h = 0.5, $\frac{1}{2}=\left(1-\frac{1}{20}t\right)^2 \Rightarrow t=20\left(\frac{\sqrt{2}-1}{\sqrt{2}}\right)$ or 5.86s since t cannot exceed 20s.

(b) $\frac{d^2x}{dt^2}=\frac{1}{\sqrt{3-t^2}} \Rightarrow \frac{dx}{dt}=\int\frac{1}{\sqrt{3-t^2}}\,dt$

$$= \sin^{-1}\left(\frac{t}{\sqrt{3}}\right)+C$$

Integrate w.r.t. on both sides again gives $x=\int\sin^{-1}\left(\frac{t}{\sqrt{3}}\right)+Cdt$

By parts, we have $\int\sin^{-1}\left(\frac{t}{\sqrt{3}}\right)dt=t\sin^{-1}\left(\frac{t}{\sqrt{3}}\right)-\int t\left(3-t^2\right)^{-\frac{1}{2}}dt$

We have $\int t\left(3-t^2\right)^{-\frac{1}{2}}dt=-\frac{1}{2}\int -2t\left(3-t^2\right)^{-\frac{1}{2}}dt=-\frac{1}{2}\times 2\left(3-t^2\right)^{\frac{1}{2}}=-\left(3-t^2\right)^{\frac{1}{2}}$

Thus, $x= t\sin^{-1}\left(\frac{t}{\sqrt{3}}\right)+\left(3-t^2\right)^{\frac{1}{2}}+Ct+D$, where C and D are arbitrary constants.

10.(i) Normal to plane OABC

$$= \vec{OA} \times \vec{OC}$$

$$= \begin{pmatrix} 0 \\ -7 \\ 1 \end{pmatrix} \times \begin{pmatrix} 4 \\ -5 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} -21+5 \\ -(0-4) \\ 0+28 \end{pmatrix}$$

$$= \begin{pmatrix} -16 \\ 4 \\ 28 \end{pmatrix} = 4\begin{pmatrix} -4 \\ 1 \\ 7 \end{pmatrix}$$

Equation of rooftop plane is $\mathbf{r} \bullet \begin{pmatrix} -4 \\ 1 \\ 7 \end{pmatrix} = 0$

Hence cartesian equation is $-4x + y + 7z = 0$

(ii)

Angle between rooftop and ground

$$= \cos^{-1} \frac{\left| \begin{pmatrix} -4 \\ 1 \\ 7 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right|}{\sqrt{16+1+49}\sqrt{1}}$$

$$= \cos^{-1} \frac{|7|}{\sqrt{66}}$$

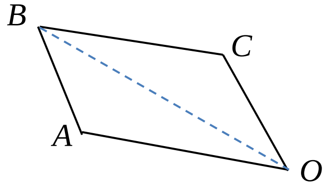
$$= 30.5^0$$

(iii)

Note that rooftop OABC is <u>**not**</u> a parallelogram.

Surface area of rooftop (where units are in tens of metres)
= sum of area of triangle OAB and triangle OBC

$$= \frac{1}{2}\left|\vec{OA} \times \vec{OB}\right| + \frac{1}{2}\left|\vec{OC} \times \vec{OB}\right|$$



$$= \frac{1}{2}\left| \begin{pmatrix} 0 \\ -7 \\ 1 \end{pmatrix} \times \begin{pmatrix} 3 \\ -9 \\ 3 \end{pmatrix} \right| + \frac{1}{2}\left| \begin{pmatrix} 4 \\ -5 \\ 3 \end{pmatrix} \times \begin{pmatrix} 3 \\ -9 \\ 3 \end{pmatrix} \right|$$

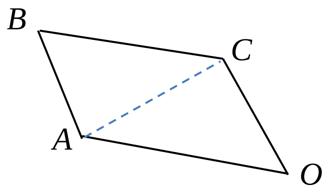
$$= \frac{3}{2}\left| \begin{pmatrix} 0 \\ -7 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} \right| + \frac{3}{2}\left| \begin{pmatrix} 4 \\ -5 \\ 3 \end{pmatrix} \times \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} \right|$$

$$= \frac{3}{2}\left| \begin{pmatrix} -4 \\ 1 \\ 7 \end{pmatrix} \right| + \frac{3}{2}\left| \begin{pmatrix} 4 \\ -1 \\ -7 \end{pmatrix} \right|$$

$$= 3\sqrt{66} \text{ units}^2$$

Alternative Method

Surface area of rooftop (where units are in tens of metres)
= sum of area of triangle OAC and triangle ABC



$$=\frac{1}{2}\left|\vec{OA}\times\vec{OC}\right|+\frac{1}{2}\left|\vec{AB}\times\vec{AC}\right|$$

$$=\frac{1}{2}\left|\begin{pmatrix}0\\-7\\1\end{pmatrix}\times\begin{pmatrix}4\\-5\\3\end{pmatrix}\right|+\frac{1}{2}\left|\begin{pmatrix}3-0\\-9+7\\3-1\end{pmatrix}\times\begin{pmatrix}4-0\\-5+7\\3-1\end{pmatrix}\right|$$

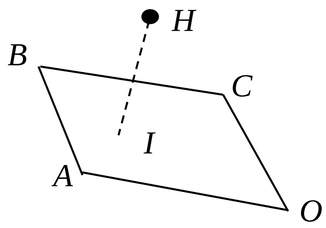
$$=\frac{1}{2}\left|\begin{pmatrix}0\\-7\\1\end{pmatrix}\times\begin{pmatrix}4\\-5\\3\end{pmatrix}\right|+\frac{1}{2}\left|\begin{pmatrix}3\\-2\\2\end{pmatrix}\times\begin{pmatrix}4\\2\\2\end{pmatrix}\right|$$

$$=\frac{1}{2}\left|\begin{pmatrix}-16\\4\\28\end{pmatrix}\right|+\frac{1}{2}\left|\begin{pmatrix}-8\\2\\14\end{pmatrix}\right|$$

$$=\frac{1}{2}\sqrt{1056}+\frac{1}{2}\sqrt{264}=3\sqrt{66}\ \text{units}^2$$

Note:
Students may notice that the cross product is the same as that for the normal of plane *OABC* and so can use the result from (i) instead of solving the cross product from scratch.
Students may notice that plane *OABC* is a kite and so the area can be taken as twice area of triangle OAB.

(iv)
For the pole to be shortest, the pole must be perpendicular to the rooftop from point *H*.
Let *I* be the foot of perpendicular from *H* to plane *OABC*.



$$L_{HI}:\ \mathbf{r}=\begin{pmatrix}-2\\-5\\9\end{pmatrix}+\lambda\begin{pmatrix}-4\\1\\7\end{pmatrix}$$

Since *I* lies on plane *OABC*,

$$\begin{pmatrix}-2-4\lambda\\-5+\lambda\\9+7\lambda\end{pmatrix}\bullet\begin{pmatrix}-4\\1\\7\end{pmatrix}=0$$

$$8+16\lambda-5+\lambda+63+49\lambda=0$$

$$\lambda=-1$$

So, $\vec{OI}=\begin{pmatrix}-2+4\\-5-1\\9-7\end{pmatrix}=\begin{pmatrix}2\\-6\\2\end{pmatrix}$

So point is $(2,-6,2)$

Alternative method 1:

$$\pi_{OABC}:\ \mathbf{r}=s\begin{pmatrix}0\\-7\\1\end{pmatrix}+t\begin{pmatrix}1\\-3\\1\end{pmatrix}=\begin{pmatrix}0+\\-49/5\\7/5\end{pmatrix}$$

$$\overrightarrow{IH}=\begin{pmatrix}-2-t\\-5+7s+3t\\9-s-t\end{pmatrix}$$

Since $\overrightarrow{IH}\perp\pi_{OABC}$,

$$\begin{pmatrix}-2-t\\-5+7s+3t\\9-s-t\end{pmatrix}\bullet\begin{pmatrix}0\\-7\\1\end{pmatrix}=0$$

$$35-49s-21t+9-s-t=0 \quad \text{-----}(1)$$

$$\begin{pmatrix}-2-t\\-5+7s+3t\\9-s-t\end{pmatrix}\bullet\begin{pmatrix}1\\-3\\1\end{pmatrix}=0$$

$$-2-t+15-21s-9t+9-s-t=0 \quad \text{-----}(2)$$

Solving, $s=0,\ t=2$

Coordinates: $(2,-6,2)$

Alternative method 2:

$$\overrightarrow{BH}=\begin{pmatrix}-5\\4\\6\end{pmatrix}$$

$$\overrightarrow{IH}=\left(\overrightarrow{BH}\bullet\mathbf{n}\right)\mathbf{n}$$

$$=\left(\frac{\begin{pmatrix}-5\\4\\6\end{pmatrix}\bullet\begin{pmatrix}-4\\1\\7\end{pmatrix}}{\sqrt{4^2+1^2+7^2}}\right)\frac{\begin{pmatrix}-4\\1\\7\end{pmatrix}}{\sqrt{4^2+1^2+7^2}}$$

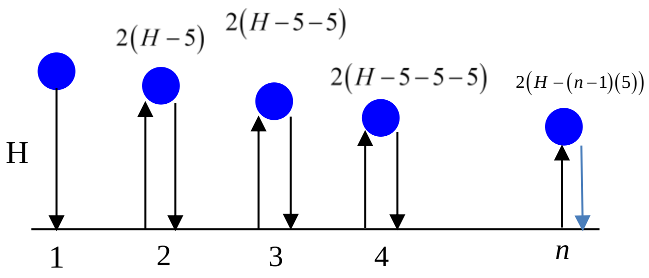
$$=\begin{pmatrix}-4\\1\\7\end{pmatrix}$$

$$\overrightarrow{OI}=\overrightarrow{OH}-\overrightarrow{IH}$$

$$=\begin{pmatrix}-2\\-5\\9\end{pmatrix}-\begin{pmatrix}-4\\1\\7\end{pmatrix}$$

Coordinates: $(2,-6,2)$

11. (i)



Height reached after 1st bounce = $H-5$
Height reached after 2nd bounce = $H-(2)5$
Height reached after 3rd bounce = $H-(3)5$
…
Height reached after the $(n-1)$th bounce = $H-(n-1)(5)$

(ii)
Total distance travelled by the ball just before it rebounds off the floor for the n^{th} time

$$=H+2(H-5)+2\big(H-2(5)\big)+2\big(H-3(5)\big)+\ldots+2\big(H-(n-1)(5)\big)$$

$$=H+\underbrace{2H+2H+\ldots+2H}_{(n-1)terms}-2\big[5+(5+5)+(5+5+5)+\ldots+(n-1)(5)\big]$$

$$=H+2(n-1)H-2\left[\frac{n-1}{2}\big(5+(n-1)5\big)\right]$$

$$=H+2(n-1)H-5n(n-1)$$

OR

Total distance travelled by the ball just before it rebounds off the floor for the n^{th} time

$$=H+2(H-5)+2\big(H-2(5)\big)+2\big(H-3(5)\big)+\ldots+2\big(H-(n-1)(5)\big)$$

$$=H+\underbrace{2H+2H+\ldots+2H}_{(n-1)terms}-2\big[5+(5+5)+(5+5+5)+\ldots+(n-1)(5)\big]$$

$$=H+2(n-1)H-2\left[\frac{n-1}{2}\big(2[5]+([n-1]-1)5\big)\right]$$

$$=H+2(n-1)H-(n-1)(10+5n-10)$$

$$=H+2(n-1)H-(n-1)(5n)$$

$$=H+2(n-1)H-5n(n-1)$$

(iii)
State an assumption made in the calculation above.
- The ball only travels vertically
- Upon hitting the floor, the tennis ball rebounds instantly

(iv)

Time taken between 1st and 2nd bounce = 1.2

Time taken between 2nd and 3rd bounce = $1.2(0.75)$

Time taken between 3rd and 4th bounce = $1.2(0.75)^2$

…

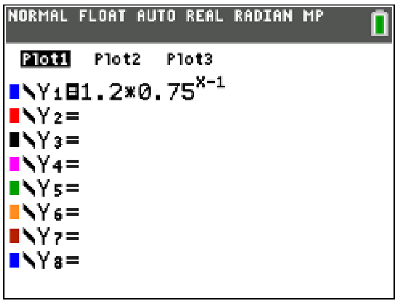
Time taken between mth and $(m+1)$th bounce = $1.2(0.75)^{m-1}$

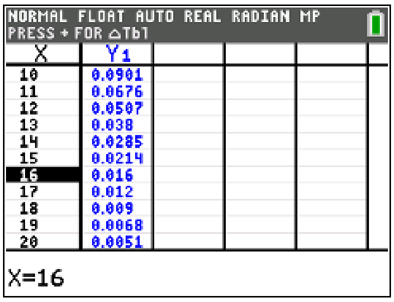
$$1.2(0.75)^{m-1} < 0.02$$
$$(0.75)^{m-1} < \frac{0.02}{1.2}$$
$$m-1 > \frac{\ln\left(\frac{0.02}{1.2}\right)}{\ln(0.75)}$$
$$m-1 > 14.232$$
$$m > 15.232$$

Least m = 16

Alternative Method (using GC table)

$$1.2(0.75)^{m-1} < 0.02$$





Least m = 16

(v)

total time taken before the tennis balls comes to a stop $= \dfrac{1.2}{1-0.75} + 2$

$= 6.8\text{s}$

12. (i)

$x = 2r\cos\theta$

$0 = 2r\cos\theta$

$0 = \cos\theta \Leftrightarrow \theta = \dfrac{\pi}{2}$

$y = 2r\sin\theta$

$0 = 2r\sin\theta$

$0 = \sin\theta \Leftrightarrow \theta = 0, \pi$

$y = 2r\sin\dfrac{\pi}{2} = 2r$

$\therefore (0, 2r)$

$x = 2r\cos 0 = 2r$ or $x = 2r\cos\pi = -2r$

$\therefore (2r, 0)$ and $(-2r, 0)$

(ii)

The exact area of R

$$A = \int_{\pi}^{0} 2r\sin\theta \frac{\mathrm{d}x}{\mathrm{d}\theta}\mathrm{d}\theta$$

$$= \int_{\pi}^{0} 2r\sin\theta(-2r\sin\theta)\mathrm{d}\theta$$

$$= \int_{0}^{\pi} 4r^2\sin^2\theta \,\mathrm{d}\theta$$

$$= \int_{0}^{\pi} 4r^2 \frac{1-\cos 2\theta}{2}\,\mathrm{d}\theta$$

$$= 2r^2\int_{0}^{\pi} 1-\cos 2\theta \,\mathrm{d}\theta$$

$$= 2r^2\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi}$$

$$= 2\pi r^2$$

(iii)

$$S = \int_{\alpha}^{\beta} 2\pi y\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\mathrm{d}\theta$$

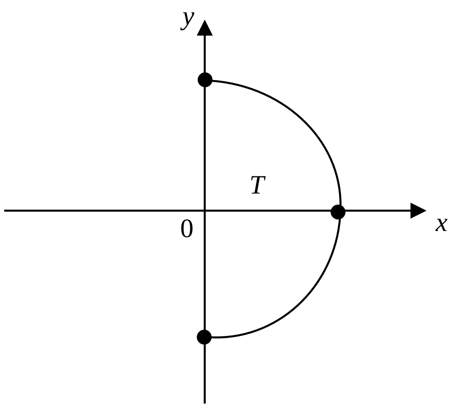
$$= \int_{0}^{\pi} 2\pi(2r\sin\theta)\sqrt{(-2r\sin\theta)^2 + (2r\cos\theta)^2}\mathrm{d}\theta$$

$$= \int_{0}^{\pi} 2\pi(2r\sin\theta)\sqrt{4r^2}\mathrm{d}\theta$$

$$= 8\pi r^2\int_{0}^{\pi} \sin\theta \,\mathrm{d}\theta = 8\pi r^2\left[-\cos\theta\right]_0^{\pi}$$

$$= 16\,\pi r^2$$

(iv)



When the region Γ is rotated about the y-axis completely, a sphere with radius $2r$ is obtained.
Using the **symmetry** property from the first curve C, the surface area obtained is the same as the first curve with radius $2r$.
Since $2r = 8$, therefore $r = 4$.
Hence, its surface area is $16\pi r^2 = 16\pi 4^2 = 256\pi$ units^2.