ANDERSON SERANGOON JUNIOR COLLEGE

MATHEMATICS **9758**

H2 Math Prelim Paper 1 (100 marks)

12 Sept 2022

3 hours

Additional Material(s): List of Formulae (MF26)

CANDIDATE NAME

CLASS   /

READ THESE INSTRUCTIONS FIRST

Write your name and class in the boxes above.
Please write clearly and use capital letters.
Write in dark blue or black pen. HB pencil may be used for graphs and diagrams only.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions and write your answers in this booklet.
Do not tear out any part of this booklet.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

All work must be handed in at the end of the examination. If you have used any additional paper, please insert them inside this booklet.
The number of marks is given in brackets [ ] at the end of each question or part question.

Question number	Marks
1	
2	
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Total	

This document consists of 22 printed pages and 2 blank pages.

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1 **(i)** For positive real constant c, state a sequence of three transformations in terms of c, that will transform the graph with equation of the form $y = \mathrm{f}(2x+3)+c$ onto the graph with equation $y = \mathrm{f}(x)$. [3]

(ii) The point with coordinates $(-2, 0)$ that lies on the curve with equation of the form $y = \mathrm{f}(2x+3)+c$ is mapped onto the point with coordinates $(-1, -1)$ that is on the curve with equation $y = \mathrm{f}(x)$. State the value of c. [1]

Solution

(i) A translation of c units in the negative y direction.
Scaling parallel to the x-axis by a scale factor of 2.
A translation of 3 units in the positive x-direction.

(ii) $c = 1$

2 The complex numbers z_1, z_2 and z_3 are given by $z_1 = \left(1-\sqrt{3}\mathrm{i}\right)^2$,

$z_2 = \left[\sqrt{2}\left(\cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}\right)\right]^6$ and $z_3 = -1+\sqrt{3}\mathrm{i}$.

(i) Using an algebraic method, find $\frac{z_2}{z_1}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and θ is an exact real constant such that $-\pi < \theta \le \pi$. [3]

(ii) Hence find $\frac{z_2}{z_1} + z_3$ in the form $p\mathrm{e}^{\mathrm{i}\alpha}$, where both r and θ are exact real constants such that $r > 0$ and $-\pi < \theta \le \pi$. [3]

Solution

(i) $$\frac{z_2}{z_1} = \frac{\left[\sqrt{2}\mathrm{e}^{\left(\frac{\pi}{4}\right)\mathrm{i}}\right]^6}{\left[2\mathrm{e}^{\left(-\frac{\pi}{3}\right)\mathrm{i}}\right]^2}$$

$$\frac{z_2}{z_1} = 2\mathrm{e}^{\left(\frac{13\pi}{6}\right)\mathrm{i}}$$

$$\therefore \frac{z_2}{z_1} = 2\mathrm{e}^{\left(\frac{\pi}{6}\right)\mathrm{i}}$$

(ii) $$\frac{z_2}{z_1} + z_3 = 2\mathrm{e}^{\left(\frac{\pi}{6}\right)\mathrm{i}} + 2\mathrm{e}^{\left(\frac{2\pi}{3}\right)\mathrm{i}}$$

$$= 2\mathrm{e}^{\left(\frac{\frac{\pi}{6}+\frac{2\pi}{3}}{2}\right)\mathrm{i}}\left[\mathrm{e}^{\left(\frac{\frac{\pi}{6}-\frac{2\pi}{3}}{2}\right)\mathrm{i}} + \mathrm{e}^{\left(-\frac{\frac{\pi}{6}-\frac{2\pi}{3}}{2}\right)\mathrm{i}}\right]$$

$$= 2\mathrm{e}^{\left(\frac{5\pi}{12}\right)\mathrm{i}}\left[2\cos\left(-\frac{\pi}{4}\right)\right]$$

$$= 2\sqrt{2}e^{\left(\frac{5\pi}{12}\right)i}$$

3 It is given that the curve C has equation $y = \dfrac{x^2 - x + 7}{x - 2}$, $x \in \mathbb{R}$, $x \neq 2$.

(i) Without using a calculator, find the set of values that y cannot take. [3]

(ii) Sketch C, stating clearly the equations of any asymptotes, the coordinates of the stationary points and the point(s) where the curve crosses the axes. [3]

Solution

(i) $y = \dfrac{x^2 - x + 7}{x - 2}$

$x^2 - x + 7 = y(x - 2)$

$x^2 - (1 + y)x + 7 + 2y = 0$

For the equation to not have real solutions, discriminant < 0

$\left[-(1 + y)\right]^2 - 4(7 + 2y) < 0$

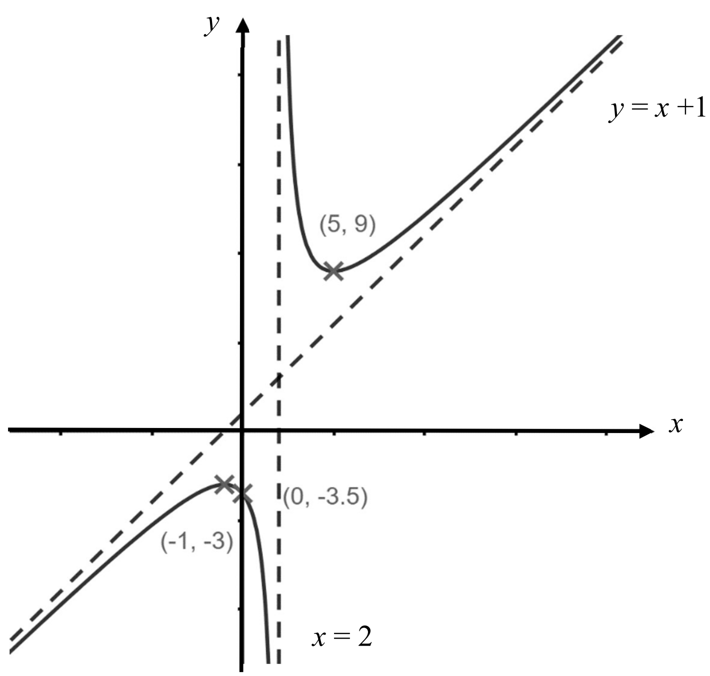
$y^2 - 6y - 27 < 0$

$(y - 9)(y + 3) < 0$

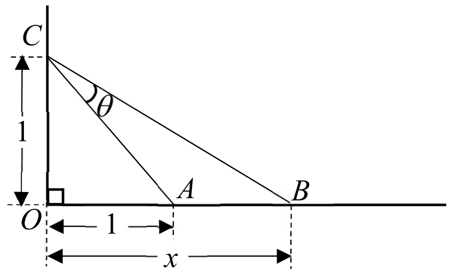
$-3 < y < 9$

$\therefore$ The set of values that C cannot take is $\{y \in \mathbb{R} : -3 < y < 9\}$.

(ii) $y = \dfrac{x^2 - x + 7}{x - 2} = x + 1 + \dfrac{9}{x - 2}$



4 **(i)** Show that the first two non-zero terms of the Maclaurin series for $\tan\theta$ is given by $\theta+\frac{1}{3}\theta^3$. You may use the standard results given in the List of Formulae (MF26). [2]



In the right-angle triangle OBC shown above, point A lies on OB such that $OA=1$, $OB=x$, where $x>1$ and $OC=1$. It is given that angle COB is $\frac{\pi}{2}$ radians and that angle ACB is θ radians (see diagram).

(ii) Show that $AB=\dfrac{2\tan\theta}{1-\tan\theta}$. [2]

(iii) Given that θ is a sufficently small angle, show that

$$AB\approx a\theta+b\theta^2+c\theta^3$$

for exact real constants a, b and c to be determined. [3]

Solutions

(i)

$$\begin{aligned}\tan\theta &= \frac{\sin\theta}{\cos\theta}\\ &= \sin\theta(\cos\theta)^{-1}\\ &\approx\left(\theta-\frac{\theta^3}{3!}\right)\left(1-\frac{\theta^2}{2!}\right)^{-1}\\ &\approx\left(\theta-\frac{\theta^3}{3!}\right)\left(1+\frac{\theta^2}{2!}\right)\\ &\approx\theta+\frac{\theta^3}{2!}-\frac{\theta^3}{3!}\\ &=\theta+\frac{1}{3}\theta^3\end{aligned}$$

(ii) $\tan\left(\dfrac{\pi}{4}+\theta\right)=\dfrac{\tan\frac{\pi}{4}+\tan\theta}{1-\tan\frac{\pi}{4}\tan\theta}$

$$x=\frac{1+\tan\theta}{1-\tan\theta}$$

$$AB=\frac{1+\tan\theta}{1-\tan\theta}-1$$

$$AB=\frac{2\tan\theta}{1-\tan\theta}$$

(iii) $AB = 2\tan\theta(1-\tan\theta)^{-1}$

$$\approx 2\left(\theta+\frac{\theta^3}{3}\right)\left(1-\left(\theta+\frac{\theta^3}{3}\right)\right)^{-1}$$

$$\approx 2\left(\theta+\frac{\theta^3}{3}\right)\left(1+\left(\theta+\frac{\theta^3}{3}\right)+\left(\theta+\frac{\theta^3}{3}\right)^2\right)$$

$$\approx \left(2\theta+\frac{2\theta^3}{3}\right)\left(1+\theta+\theta^2\right)$$

$$\approx 2\theta+2\theta^2+\frac{8\theta^3}{3}$$

$a=2,\ b=2,\ c=\frac{8}{3}$

5 **(i)** By considering $u_n - u_{n+1}$, where $u_n = \frac{1}{n(n+1)(n+2)}$,

find $\sum_{n=1}^{N}\frac{1}{n(n+1)(n+2)(n+3)}$ in terms of N. [3]

(ii) Hence or otherwise, find $\sum_{n=5}^{N+3}\frac{1}{n(n-1)(n-2)(n-3)}$. [3]

(iii) Deduce that

$$\frac{1}{6^2}+\frac{1}{12^2}+\frac{1}{20^2}+\frac{1}{30^2}+\frac{1}{42^2}+\ldots$$

is less than $\frac{1}{18}$. Show your workings clearly. [3]

Solution:

(i)

$$u_n - u_{n+1} = \frac{1}{n(n+1)(n+2)} - \frac{1}{(n+1)(n+2)(n+3)}$$

$$= \frac{(n+3)-n}{n(n+1)(n+2)(n+3)}$$

$$\Rightarrow u_n - u_{n+1} = \frac{3}{n(n+1)(n+2)(n+3)}$$

$$\sum_{n=1}^{N}\frac{1}{n(n+1)(n+2)(n+3)} = \frac{1}{3}\sum_{n=1}^{N}\left[u_n - u_{n+1}\right]$$

$$
\begin{aligned}
&= \frac{1}{3}\Big[\ u_1 - u_2 \\
&\quad + u_2 - u_3 \\
&\quad + u_3 - u_4 \\
&\quad + \\
&\quad + u_{N-1} - u_N \\
&\quad + u_N - u_{N+1}\Big] \\
&= \frac{1}{3}\left[u_1 - u_{N+1}\right] \\
&= \frac{1}{3}\left[\frac{1}{(1)(2)(3)} - \frac{1}{(N+1)(N+2)(N+3)}\right] \\
&= \frac{1}{18} - \frac{1}{3(N+1)(N+2)(N+3)}
\end{aligned}
$$

(ii) By replacing n with $(n+3)$,

$$
\begin{aligned}
\sum_{n=5}^{N+3} \frac{1}{n(n-1)(n-2)(n-3)} &= \sum_{n+3=5}^{n+3=N+3} \frac{1}{(n+3)(n+3-1)(n+3-2)(n+3-3)} \\
&= \sum_{n=2}^{N} \frac{1}{(n)(n+1)(n+2)(n+3)} \\
&= \left[\frac{1}{18} - \frac{1}{3(N+1)(N+2)(N+3)}\right] - \left[\frac{1}{(1)(2)(3)(4)}\right] \\
&= \frac{1}{72} - \frac{1}{3(N+1)(N+2)(N+3)}
\end{aligned}
$$

(iii) For positive integers n

$$n^2 + 3n < n^2 + 3n + 2$$

$$n(n+3) < (n+1)(n+2)$$

$$n(n+1)(n+2)(n+3) < (n+1)^2(n+2)^2$$

$$\therefore \frac{1}{n(n+1)(n+2)(n+3)} > \frac{1}{(n+1)^2(n+2)^2} \qquad \forall n > 0$$

So $\displaystyle\sum_{n=1}^{N} \frac{1}{(n+1)^2(n+2)^2} < \sum_{n=1}^{N} \frac{1}{n(n+1)(n+2)(n+3)}$

As $\displaystyle\frac{1}{6^2} + \frac{1}{12^2} + \frac{1}{20^2} + \ldots = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{(n+1)^2(n+2)^2}$

$$
\begin{aligned}
&< \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n(n+1)(n+2)(n+3)} \\
&= \lim_{N\to\infty}\left[\frac{1}{18} - \frac{1}{3(N+1)(N+2)(N+3)}\right] \\
&= \frac{1}{18}
\end{aligned}
$$

Thus $\displaystyle\frac{1}{6^2} + \frac{1}{12^2} + \frac{1}{20^2} + \ldots < \frac{1}{18}$ (deduced)

6 **(a)** Find $\int \frac{\sin^{-1}(2x-1)}{\sqrt{1-x}}\,\mathrm{d}x$ for $0<x<1$. [3]

(b) **(i)** Sketch the graphs of $y=\left|x^2-7\right|$ and $y=x+5$ on the same diagram. Indicate clearly the x-intercepts and the values of x where the two curves intersect. Hence solve the inequality $\left|x^2-7\right| \geq x+5$. [4]

(ii) Hence, for $a>5$, find $\int_3^a \left| \left|x^2-7\right|-x-5\right| \mathrm{d}x$ in terms of a. Leave your answer in exact form. [3]

Solution

(a)

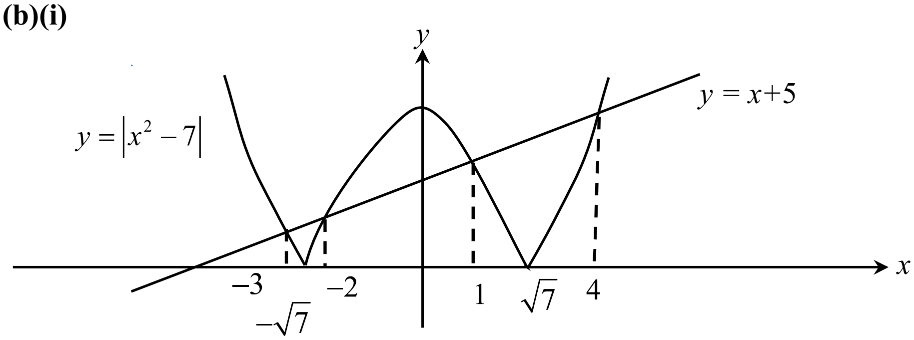
$$\int \frac{\sin^{-1}(2x-1)}{\sqrt{1-x}}\,\mathrm{d}x = -2\sqrt{1-x}\cdot\sin^{-1}(2x-1)+\int 2\sqrt{1-x}\ \frac{2}{\sqrt{1^2-(2x-1)^2}}\,\mathrm{d}x$$

$$= -2\sqrt{1-x}\cdot\sin^{-1}(2x-1)+4\int \frac{\sqrt{1-x}}{\sqrt{1-x}\sqrt{4x}}\,\mathrm{d}x$$

$$= -2\sqrt{1-x}\cdot\sin^{-1}(2x-1)+2\int (x)^{-\frac{1}{2}}\ \mathrm{d}x$$

$$= -2\sqrt{1-x}\sin^{-1}(2x-1)+4\sqrt{x}+C$$

(b)(i)



From the sketch, $x \leq -3$ or $-2 \leq x \leq 1$ or $x \geq 4$

(ii) $\int_3^a \left| \left|x^2-7\right|-x-5\right| \mathrm{d}x = \int_3^4 x+5-\left|x^2-7\right| \mathrm{d}x + \int_4^a \left|x^2-7\right|-x-5\ \mathrm{d}x$

$$=\frac{19}{6}+\int_4^a x^2-7\ \mathrm{d}x-\int_4^a x+5\ \mathrm{d}x$$

$$=\frac{19}{6}+\left[\frac{x^3}{3}-7x\right]_4^a-\left[\frac{x^2}{2}+5x\right]_4^a$$

$$=\frac{19}{6}+\left(\frac{a^3}{3}-7a\right)+\frac{20}{3}-\left(\frac{a^2}{2}+5a\right)+28$$

$$=\frac{a^3}{3}-\frac{a^2}{2}-12a+\frac{227}{6}$$

7 A curve C has parametric equations

$$x=\sin^3 t\ ,\quad y=\cos^2 t,\ -\frac{\pi}{2}<t<0\,.$$

The tangent at the point $P(\sin^3 p, \cos^2 p)$, $-\frac{\pi}{2}<p<0$, meets the x-axis and y-axis at Q and R respectively.

(i) By finding the equation of the tangent at the point P, show that the area of the triangle OQR is $-\frac{1}{12}\sin p\ (2+\cos^2 p)^2$. [6]

(ii) Find a cartesian equation of the locus of the mid-point of QR as p varies. You need not indicate its domain. [5]

Solution

(i)

$x=\sin^3 t \qquad\qquad y=\cos^2 t$

$\dfrac{dx}{dt}=3\sin^2 t\cos t \qquad \dfrac{dy}{dt}=-2\sin t\cos t$

$\dfrac{dy}{dx}=\dfrac{-2\sin t\cos t}{3\sin^2 t\cos t}=-\dfrac{2}{3\sin t}$

At the point P, $x=\sin^3 p$

$$y=\cos^2 p$$

$$\frac{dy}{dx}=-\frac{2}{3\sin p}$$

Equation of the tangent at the point P:

$$y-\cos^2 p=-\frac{2}{3\sin p}(x-\sin^3 p)$$

When $y = 0$, $-\cos^2 p=-\dfrac{2}{3\sin p}(x-\sin^3 p)$

$$x=\sin^3 p+\frac{3}{2}\sin p\cos^2 p$$

$$x=\frac{1}{2}\sin p(2\sin^2 p+3\cos^2 p)$$

$$x=\frac{1}{2}\sin p(2+\cos^2 p)$$

$Q\left(\frac{1}{2}\sin p(2+\cos^2 p),0\right)$

When $x = 0$, $y-\cos^2 p=-\dfrac{2}{3\sin p}(0-\sin^3 p)$

$$y=\frac{2}{3}\sin^2 p+\cos^2 p$$

$$y=\frac{1}{3}(2\sin^2 p+3\cos^2 p)=\frac{1}{3}(2+\cos^2 p)$$

$R\left(0,\frac{1}{3}(2+\cos^2 p)\right)$

Area of the triange OQR

$$=\frac{1}{2}\times\left[0-\frac{1}{2}\sin p(2+\cos^2 p)\right]\times\frac{1}{3}(2+\cos^2 p)$$

$$=-\frac{1}{12}\sin p\left(2+\cos^2 p\right)^2$$

(ii)

$$\text{Mid point of } QR=\left(\frac{\frac{1}{2}\sin p(2+\cos^2 p)+0}{2},\frac{0+\frac{1}{3}(2+\cos^2 p)}{2}\right)$$

$$=\left(\frac{1}{4}\sin p(2+\cos^2 p),\frac{1}{6}(2+\cos^2 p)\right)$$

$$x=\frac{1}{4}\sin p(2+\cos^2 p)------------(1)$$

$$y=\frac{1}{6}(2+\cos^2 p)\qquad ------------(2)$$

$\frac{(1)}{(2)}$ gives

$$\frac{x}{y}=\frac{\frac{1}{4}\sin p(2+\cos^2 p)}{\frac{1}{6}(2+\cos^2 p)}$$

$$\frac{x}{y}=\frac{3}{2}\sin p$$

$$\sin p=\frac{2x}{3y}$$

$$y=\frac{1}{6}(2+\cos^2 p)$$

$$y=\frac{1}{6}\left(2+(1-\sin^2 p)\right)$$

$$y=\frac{1}{6}\left(3-\frac{4x^2}{9y^2}\right)$$

$$y=\frac{1}{54y^2}\left(27y^2-4x^2\right)$$

$$54y^3=27y^2-4x^2$$

Cartesian equation of the locus of the mid-point of QR is $54y^3 = 27y^2 - 4x^2$

8 **(a)** Functions f and g are defined by

$$\mathrm{f}: x\mapsto x^2,\quad x<0,$$

$$\mathrm{g}: x\mapsto \frac{1}{x},\quad x>0.$$

(i) Explain why the composite function gf exists. [1]

(ii) Find the exact value of $\mathrm{f}^{-1}\mathrm{g}^{-1}(3)$. Show your workings clearly. [3]

(b) For real values a, the function h is defined by

$$\mathrm{h}: x \mapsto ax - \frac{1}{x}, \quad x < 0.$$

(i) If a is negative, explain clearly with a well-labelled diagram, why h^{-1} does not exist. [4]

(ii) If $a = 1$, find h^{-1} in similar form. [3]

Solution

(ai) $\mathrm{R_f} = (0, \infty)$ and $\mathrm{D_g} = (0, \infty)$

Since $\mathrm{R_f} \subseteq \mathrm{D_g}$, the composite function gf exists.

(ii) Let $\mathrm{f}^{-1}\mathrm{g}^{-1}(3) = k$

$\mathrm{g}^{-1}(3) = \mathrm{f}(k) = k^2$

$\mathrm{g}(k^2) = 3$

$\frac{1}{k^2} = 3$

$k = -\frac{\sqrt{3}}{3} \left(\because \mathrm{D_f} = (-\infty, 0)\right)$

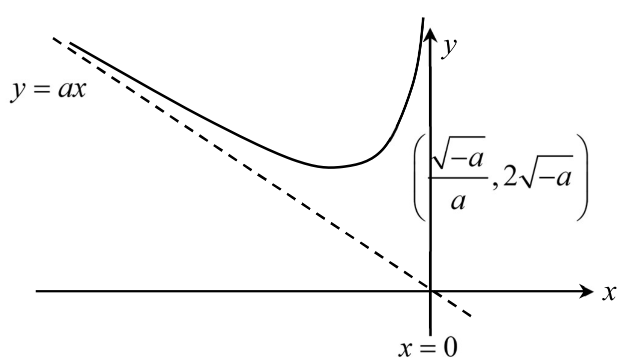
(bi)

$\mathrm{h}(x) = ax - \frac{1}{x}$

$\mathrm{h}'(x) = a + \frac{1}{x^2}$

For $a + \frac{1}{x^2} = 0 \Rightarrow x = \frac{-1}{\sqrt{-a}} = \frac{\sqrt{-a}}{a} (\because x < 0)$

$\therefore \mathrm{h}\left(\frac{\sqrt{-a}}{a}\right) = \sqrt{-a} + \sqrt{-a} = 2\sqrt{-a}$



Since the horizontal line $y = 2\sqrt{-a} + 1$ cuts the curve twice, the function is not a 1-1 function and so h^{-1} does not exist.

(ii) Let $y = \mathrm{h}(x) = x - \frac{1}{x}$

$y = x - \frac{1}{x}$

$yx = x^2 - 1$

$x^2 - xy - 1 = 0$

$x = \frac{y \pm \sqrt{y^2+4}}{2}$

$x = \frac{y - \sqrt{y^2+4}}{2} \quad (\because x < 0)$

$\text{h}^{-1} : x \mapsto \frac{x - \sqrt{x^2+4}}{2}, \quad x \in \mathbb{R}$

9 **(a)** An arithmetic progression has first term a and common difference d, where $a > 0$ and $d \neq 0$. The eighth, third and second term of the progression are the first three terms of an infinite geometric progression.

(i) Find the common ratio of the geometric progression. [3]

(ii) Find the exact sum of the odd-numbered terms of the geometric progression in terms of a. [3]

(b) A programmer coded a program involving a rabbit-fox chase along a straight path to model the actual hunt for a rabbit by a fox.
The rabbit first hop is 1.75 m. In each subsequent hop, the distance covered is 1% less than its previous hop. The fox first leaps 3 m. In each subsequent leap, the distance covered is 0.02 m less than its previous leap. Initially the rabbit is 60 m ahead of the fox and assume that the rabbit and the fox start and end each hop and leap at the same time.

(i) By finding the total distance travelled by the fox and the rabbit after n leaps and hops respectively, find the minimum number of hops and leaps for the fox to catch up with the rabbit. [4]

(ii) Find the number of leaps the fox takes before it comes to a stop. Hence, find the minimum starting distance, in metre, between the fox and the rabbit such that the fox will never catch up with the rabbit. Leave your answer to the nearest integer. [2]

Solution

(ai) Let b and r be the first term and common ratio of the G.P.

$b = a + 7d$ ----------------------- (1)

$br = a + 2d$ ----------------------- (2)

$br^2 = a + d$ ----------------------- (3)

From (1) and (2) gives

$b - br = 5d$ ----------------------- (4)

From (2) and (3) gives

$br - br^2 = d$ ----------------------- (5)

(4) divides (5) gives

$\frac{1-r}{r-r^2} = 5$

$5r^2 - 6r + 1 = 0$

$(5r-1)(r-1) = 0$

$r=\frac{1}{5}$ or $r=1$(rejected since $d\neq 0$)

(ii) From (5), $\frac{4}{25}b=d$.

And from (3), $b=-\frac{25}{3}a$

$$(S_n)_{\text{odd}}=\frac{b}{1-r^2}$$

$$=\frac{-\frac{25}{3}a}{1-\frac{1}{25}}$$

$$=-\frac{625a}{72}$$

(bi) $(S_n)_{\text{fox}}=\frac{n}{2}\left[2(3)+(n-1)(-0.02)\right]=n(3.01-0.01n)$

$$(S_n)_{\text{rabbit}}=\frac{1.75\left[1-0.99^n\right]}{1-0.99}=175\left(1-0.99^n\right)$$

For the fox to catch the rabbit,

$n(3.01-0.01n)\ -\ 175\left(1-0.99^n\right)\geq 60$

Let $Y=n(3.01-0.01n)-175\left(1-0.99^n\right)$

From GC,

n	Y
53	$59.171\ <60$
54	$60.084>60$
55	$60.987>60$

Least n = 54

(ii) Let k be the starting distance between fox and rabbit.

For the fox to never catch up with the rabbit,

$k>\text{Max}\left[n(3.01-0.01n)-175\left(1-0.99^n\right)\right]$

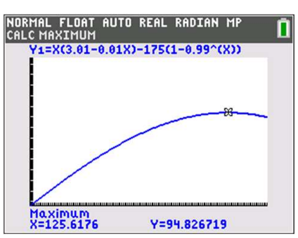
To find n for which the fox stop moving, $T_n=0$

$3+(n-1)(-0.02)=0$

$n=151$

The fox takes 150 leaps before it stops moving.

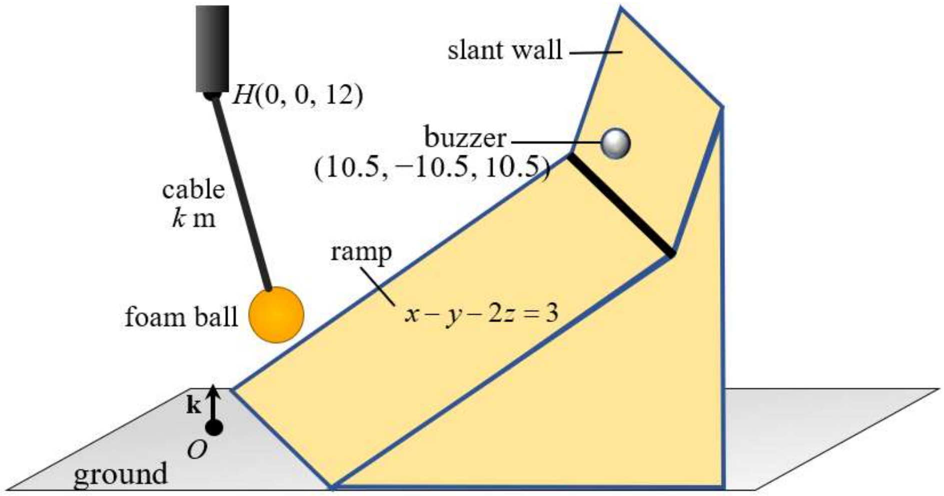
From GC, for $0\leq n\leq 150$,



$k>\text{Max}\left[n(3.01-0.01n)-175\left(1-0.99^n\right)\right]=94.827$

Hence minimum k is 95 m (to the nearest integer)

10 The production team of a popular variety show, *Sprinting Man*, is preparing a site for a segment of the show. In this segment, each participant is to sprint from the starting point, go up a ramp and press a buzzer to complete the challenge.



Referring the starting point as the origin O and the horizontal ground as the x-y plane, the top surface of the ramp has equation $x - y - 2z = 3$ (see diagram that is not drawn to scale). Distances are measured in metres.

(i) Find the angle of inclination of the ramp. [2]

A spherical polyurethane foam ball of radius 1 m is suspended from a point H with coordinates $(0, 0, 12)$ by a cable of length k m, that is taut all the time. The ball will be swung in various directions during the challenge to increase the level of difficulty.

(ii) If the production team wants to ensure that the foam ball will never come in contact with the ramp, find the range of values that k can take. [3]

The buzzer that the participants are to press is located at the point with coordinates $(10.5, -10.5, 10.5)$. This point lies on a flat slant wall which intersects the ramp along the line l with cartesian equation $x = y + 20,\ z = 8.5$.

(iii) Find a cartesian equation of the slant wall. [3]

A camera is to be placed along a line L with equation $\mathbf{r} = 12\mathbf{k} + t(\mathbf{i} + 3\mathbf{j}),\ t \in \mathbb{R}$, with its position denoted by C.

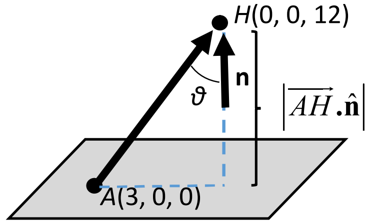
(iv) If the camera is at a distance of $\sqrt{254}$ m from a point P with coordinates $(10, -10, 10)$, determine the possible coordinates of C exactly, showing your workings. Hence deduce the point on L that is nearest to P. [4]

Solution

(i) Angle of inclination of the ramp $= \cos^{-1} \dfrac{\left| \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} . \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right|}{\sqrt{1^2 + 1^2 + 2^2}}$

$$= \cos^{-1} \frac{2}{\sqrt{6}} \approx 35.264° = 35.3° \text{ (1 d.p.)}$$

(ii) A point on the ramp is $A(3,\,0,\,0)$. Let the normal to the ramp be $\mathbf{n}=\begin{pmatrix}1\\-1\\-2\end{pmatrix}$.



Shortest distance from H to ramp

$$=\left|\overrightarrow{AH}.\hat{\mathbf{n}}\right|$$

$$=\frac{\left|\begin{pmatrix}-3\\0\\12\end{pmatrix}.\begin{pmatrix}1\\-1\\-2\end{pmatrix}\right|}{\sqrt{1+1+4}}=\frac{27}{\sqrt{6}}\quad(\approx 11.0227)$$

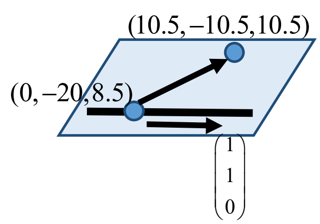
Since the diameter of the ball is 2 m, $0<k<\frac{27}{\sqrt{6}}-2$ [or $0<k<9.02$ (3 s.f.)].

(iii) $l:\mathbf{r}=\begin{pmatrix}0\\-20\\8.5\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\0\end{pmatrix},\ \lambda\in\mathbb{R}$.

A vector parallel to the slant wall is $\begin{pmatrix}10.5\\-10.5\\10.5\end{pmatrix}-\begin{pmatrix}0\\-20\\8.5\end{pmatrix}=\begin{pmatrix}10.5\\9.5\\2\end{pmatrix}$.

Therefore, a normal to the slant wall is

$$\begin{pmatrix}1\\1\\0\end{pmatrix}\times\begin{pmatrix}10.5\\9.5\\2\end{pmatrix}=\begin{pmatrix}2\\-2\\-1\end{pmatrix}.$$

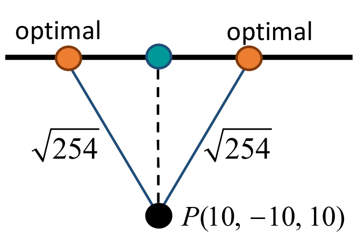


$$\mathbf{r}.\begin{pmatrix}2\\-2\\-1\end{pmatrix}=\begin{pmatrix}10.5\\-10.5\\10.5\end{pmatrix}.\begin{pmatrix}2\\-2\\-1\end{pmatrix}=31.5$$

A cartesian equation of the slant wall: $4x-4y-2z=63$.

(iv) Camera lies along the line with equation $\mathbf{r}=\begin{pmatrix}0\\0\\12\end{pmatrix}+t\begin{pmatrix}1\\3\\0\end{pmatrix}, t\in\mathbb{R}$.

As the camera needs to be $\sqrt{254}$ m from P,



$$\sqrt{(t-10)^2+(3t+10)^2+(12-10)^2}=\sqrt{254}$$

$$\left(t^2-20t+100\right)+\left(9t^2+60t+100\right)+4=254$$

$$10t^2+40t-50=0$$

$$(t+5)(t-1)=0$$

$$\therefore t=1 \text{ or } t=-5.$$

The corresponding optimal positions are $(1,\,3,\,12)$ and $(-5,\,-15,\,12)$.

By symmetry, the point along the line closest to P is the **midpoint** of the two optimal positions. Therefore, this point has coordinates $\left(\frac{1-5}{2}, \frac{3-15}{2}, 12\right) = (-2, -6, 12)$.

11 The cylindrical tank in a research laboratory has a cross-sectional area of 4 m^2. To cool the tank, water is pumped in and out of the tank simultaneously. The volume and height of the water in the tank at any time t minutes is given by V (m^3) and h (metres) respectively. Clean water is pumped into the tank at a rate that is proportional to h^2 and the water is pumped out from the tank at a rate that is proportional to h.

(i) Assume that the water does not overflow and that there is no change to the height of the water when h is 10, show that $\frac{dh}{dt} = \frac{kh(h-10)}{4}$ where k is a real constant. [4]

The tank was initially filled with clean water to a height of 2 metres. When the height of the water is 5 metres, the volume of water is increasing at a rate of 5.5 m^3 per minute.

(ii) Find the exact value of k. Hence find h in terms of t. [5]

(iii) Sketch a graph of h against t. Hence write down the minimum height of the cylindrical tank that will not result in the overflow of the water. [3]

Solution

(i)

$$\frac{dV}{dt} = \frac{dV_{in}}{dt_{in}} - \frac{dV_{out}}{dt_{out}}$$

$$\frac{dV}{dt} = Ah^2 - Bh, \quad A, B \in \mathbb{R}$$

When $h = 10$, $\frac{dV}{dt} = 0$.

$B = 10A$

Since $V = \pi r^2 h$ (and given that base area is 4 m^2)

$\therefore V = 4\ h$

$$\frac{dV}{dt} = 4\frac{dh}{dt}$$

$$\Rightarrow 4\frac{dh}{dt} = Ah^2 - 10Ah$$

$$\Rightarrow \frac{dh}{dt} = \frac{kh(h-10)}{4}, \text{ where } A = k$$

(ii)

$$\frac{dV}{dt} = 5.5$$

$$5.5 = \frac{5k(5-10)}{4} \times 4$$

$$k = -\frac{11}{50}$$

$$\frac{dh}{dt} = -\frac{11h(h-10)}{200}$$

$$\int \frac{1}{h^2-10h}\ dh = -\int \frac{11}{200}\ dt$$

$$\int \frac{1}{(h-5)^2-5^2}\ dh = -\frac{11}{200}t + c$$

$$\frac{1}{10}\ln\left|\frac{(h-5)-5}{h}\right| = -\frac{11}{200}t + c$$

$$\ln\left|\frac{h-10}{h}\right| = -\frac{11}{20}t + 10c$$

$$\left|1-\frac{10}{h}\right| = e^{-\frac{11}{20}t+10c}$$

$$1-\frac{10}{h} = \pm e^{-\frac{11}{20}t+10c}$$

$$\frac{10}{h} = 1 + Ae^{-\frac{11}{20}t} \qquad A = \pm e^{10c}$$

$$h = \frac{10}{1+Ae^{-\frac{11}{20}t}}$$

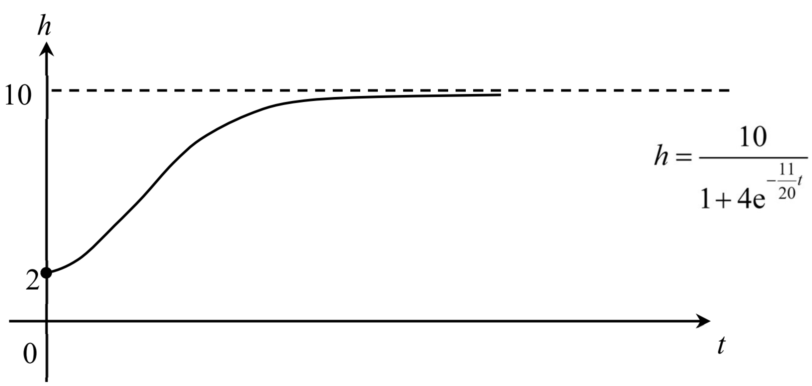
When $t = 0,\ h = 2$

$$2 = \frac{10}{1+A}$$

$A = 4$

$$h = \frac{10}{1+4e^{-\frac{11}{20}t}}$$

(iii)



Minimum height of the cylindrical tank = 10 metres