

**ANGLO-CHINESE JUNIOR COLLEGE
JC2 PRELIMINARY EXAMINATION**

Higher 2

/100

CANDIDATE
NAME

TUTORIAL/
FORM CLASS

INDEX
NUMBER

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MATHEMATICS

9758/01

Paper 1

23 August 2022

3 hours

Candidates answer on the Question Paper.

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your index number, class and name on all the work you hand in.
Write in dark blue or black pen.

You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.

Write your answers in the spaces provided in the question paper.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

The use of an approved graphing calculator is expected, where appropriate.

Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.

Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [] at the end of each question or part question.

The total number of marks for this paper is 100.

Question	Marks
1	/3
2	/4
3	/7
4	/6
5	/7
6	/7
7	/10
8	/10
9	/10
10	/10
11	/12
12	/14

This document consists of **30** printed pages and **4** blank pages.

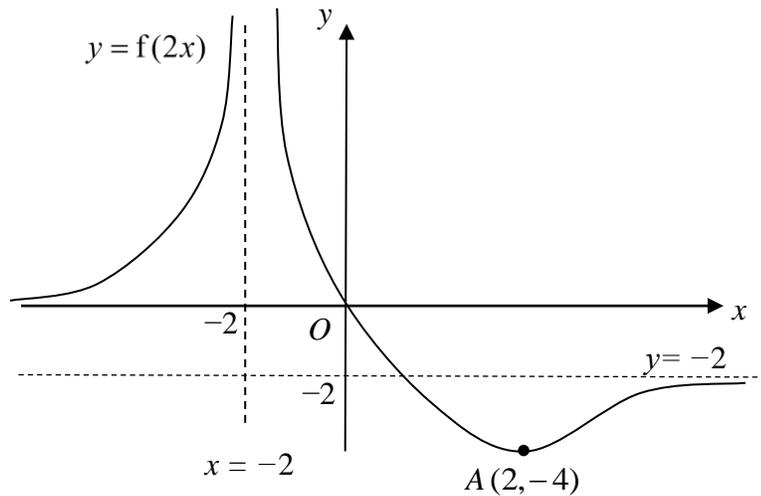


Anglo-Chinese Junior College

[Turn over

- 1 The diagram shows the graph of $y = f(2x)$. The lines $x = -2$ and $y = -2$ are asymptotes to the curve. The minimum point A has coordinates $(2, -4)$ and the curve passes through the origin $(0, 0)$.

Sketch the graph of $y = \frac{1}{f(x)}$, indicating clearly the equations of any asymptotes, axial intercepts and turning points. [3]



- 2 Find $\int x \tan^{-1}(3x) dx$. [4]

- 3 With reference to the origin O , the points A , B and C have position vectors \mathbf{a} , \mathbf{b} and \mathbf{c} respectively, where points O , A , B and C are not collinear.

(i) It is given that $|\mathbf{a}| = \sqrt{10}$, $|\mathbf{b}| = 20$, $\mathbf{a} \cdot \mathbf{b} = 40$ and the point N has position vector $\frac{5\mathbf{a} + 3\mathbf{b}}{8}$. Find the exact area of triangle ONB . [5]

(ii) Find the angle between the vector $\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}$ and the plane π containing points A , B and C . [2]

- 4 (a) The continuous function $f(x)$, where $f(x) > 0$, is strictly decreasing for $x \geq 1$. Sketch the curve $y = f(x)$ for $k < x < k+1$, where k is an integer and $k \geq 1$.

By comparing the areas of appropriate rectangles and the area under the curve $y = f(x)$, show that for any integer $k \geq 1$,

$$f(k+1) < \int_k^{k+1} f(x) dx < f(k). \quad [2]$$

- (b) The region under the curve $y = \frac{1}{x}$ between $x = 1$ and $x = 10$, is split into 9 vertical strips of equal width. Use the result in part (a) to prove

$$(i) \int_1^{10} \frac{1}{x} dx < \sum_{k=1}^9 \frac{1}{k}, \quad [1]$$

$$(ii) \sum_{k=1}^9 \frac{1}{k} < 1 + \int_1^9 \frac{1}{x} dx. \quad [2]$$

$$\text{Hence show that } \ln 10 < 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{9} < 1 + \ln 9. \quad [1]$$

5 **Do not use a calculator in answering this question.**

Two complex numbers are $z_1 = 2 \left(\cos \frac{\pi}{18} - i \sin \frac{\pi}{18} \right)$ and $z_2 = 2i$.

$$(i) \text{ Show that } \frac{z_1^2}{z_1^*} + z_2 \text{ is } \sqrt{3} + i. \quad [3]$$

(ii) A third complex number, z_3 , is such that $\left(\frac{z_1^2}{z_1^*} + z_2 \right) z_3$ is real and

$$\left| \left(\frac{z_1^2}{z_1^*} + z_2 \right) z_3 \right| = \frac{2}{3}. \text{ Find the possible values of } z_3 \text{ in the form of } r(\cos \theta + i \sin \theta),$$

where $r > 0$ and $-\pi < \theta \leq \pi$. [4]

6 The function h is defined by $h : x \mapsto |\ln(2x)| + 1, x \in \mathbb{R}, 0 < x \leq \lambda$.

- (i) Find the maximum value of λ for which the inverse function h exist. [1]
- (ii) Using $\lambda = \frac{1}{2}$,
- (a) find h^{-1} and state its domain, [4]
- (b) sketch the graph of $y = hh^{-1}(x)$, [1]
- (c) find the solution set for $h(x) = h^{-1}(x)$. [1]

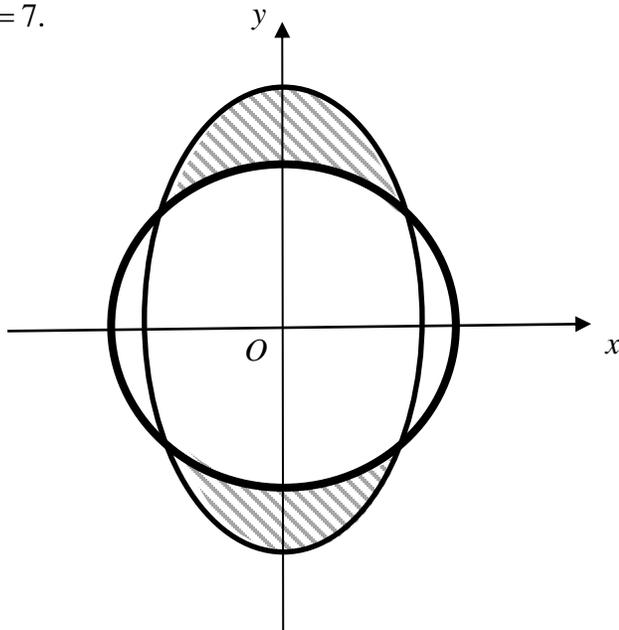
7 It is given that $y = \ln(2 + \sin 2x)$.

- (i) Show that $e^y \frac{d^2y}{dx^2} + e^y \left(\frac{dy}{dx} \right)^2 = -4 \sin 2x$. [2]
- (ii) By further differentiation of the above results, find the Maclaurin series for y , up to and including the term in x^3 . [3]
- (iii) Verify that the series for $\ln(2 + \sin 2x)$ is the same as the result obtained in part (ii), if the standard series from the List of Formulae (MF26) are used. [3]
- (iv) Hence deduce the series expansion for $\frac{\ln(2 + \sin 2x)}{\sqrt{1-x}}$, up to and including the term in x^2 . [2]

- 8 (i) Use the substitution $x = k \sin \theta$, to show that

$$\int \sqrt{k^2 - x^2} \, dx = \frac{x}{2} \sqrt{k^2 - x^2} + \frac{k^2}{2} \sin^{-1} \left(\frac{x}{k} \right) + c. \quad [4]$$

- (ii) The diagram shows the shaded area R , enclosed between two curves, $\frac{x^2}{4} + \frac{y^2}{16} = 1$ and $x^2 + y^2 = 7$.



- (a) Using the result shown in part (i), find the exact area of region R in the form $A\pi + B \sin^{-1} C$ where A , B and C are exact constants to be determined. [4]
- (b) Find the volume generated when region R is rotated through π radians about the y -axis, giving your answer to two decimal places. [2]

- 9 (a) Mr Chan's monthly pay for the first quarter of the year (i.e. first three months) is \$3000. For the second quarter monthly pay, he gets an increment of 60% of his first quarter monthly pay. Subsequently, from the third quarter onwards, he gets an increment of 60% of his **previous increment** every quarter.

Q_n denotes his monthly pay for the n^{th} quarter, where $n \geq 1$.

- (i) Prove that $Q_n = 7500(1 - 0.6^n)$. [3]
- (ii) Find to the nearest dollar, his pay for his 17th month of work. [1]
- (iii) Find to the nearest dollar, his total salary for the first two years of work. [2]
- (iv) Mr Chan decides to quit his job when his increment is less than \$80. How many months will he stay in this job? [1]
- (b) Mr Chan's company is planning to implement a loyalty incentive scheme to retain workers with relevant skills and experience. 10 chosen workers will receive a loyalty incentive of \$90 in their pay every month. This monthly loyalty incentive will increase by \$40 every year. The company sets aside a budget of \$200,000 for this scheme.

If the 10 workers continue to work with the company,

- (i) what is the maximum number of years for which this scheme can be implemented with this budget? [2]
- (ii) what is the highest loyalty incentive amount Mr Chan can hope to receive if he is selected for this scheme? [1]

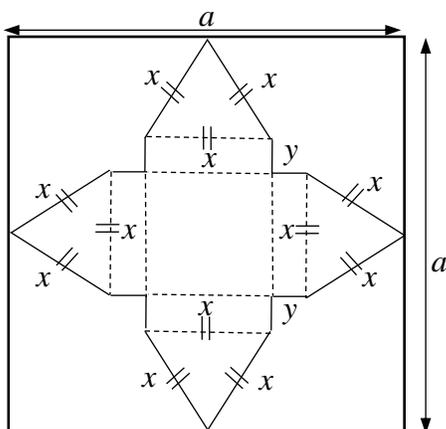


Figure 1

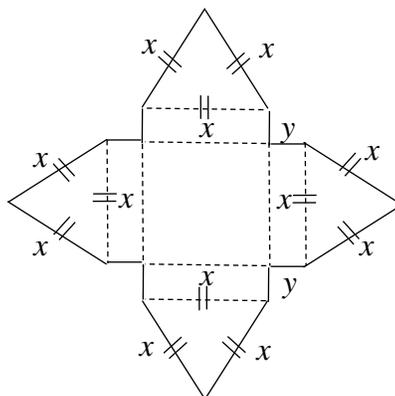


Figure 2

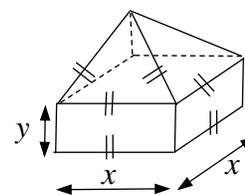


Figure 3

Figure 1 shows a piece of plastic sheet in the shape of a square with sides a metres. Parts of the plastic sheet is cut from each corner to give the shape shown in **Figure 2** which consists of a square, four identical rectangles and four identical equilateral triangles. The sides of the square are x metres each. Each rectangle has a length of x metres and breadth of y metres. Each equilateral triangle has sides of x metres. The remaining plastic sheet shown in **Figure 2** is then folded along the dotted lines to form a container, made up of a cuboid and a square base pyramid, as shown in **Figure 3**. The volume of the container is denoted by V .

(i) Show that $y = \frac{1}{2}(a - (1 + \sqrt{3})x)$. [2]

(ii) Use differentiation to find, in terms of a , the value of x that gives a maximum possible value of V , proving that it is a maximum. [5]

[The volume of a square-based pyramid is $\frac{1}{3} \times \text{base area} \times \text{height}$.]

(iii) The container is inverted and is held with its axis vertical and vertex downwards as shown in **Figure 4**. Water is poured into the container at a rate of 0.1 cubic metres per minute. At time t minutes after the start, the depth of water in the container is h metres as shown in the front view diagram of the inverted container in **Figure 5**.

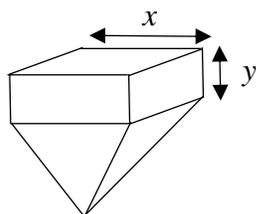


Figure 4

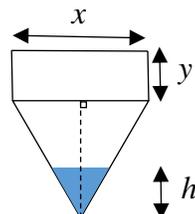


Figure 5

Given that the dimensions of the container, x and y are constants and the water level is still within the pyramid. Find in terms of a , the rate of increase of the water level when $h = 0.05a$ metres. [3]

- 10** With the proliferation of online social network, sociologists recognised a phenomenon called social diffusion which is the spreading of a piece of information through the population. The members of the population can be classified into two categories namely those who have the information and those who do not.

It is given that N denotes the number of people, in thousands, who have the information in a fixed population size P , in thousands. The rate of diffusion of a piece of information on social media, $\frac{dN}{dt}$, where t represents the time taken in hours, can be modelled as being proportional, with a constant of proportionality k , to the product of the number of people who have the information and the number of people who have yet to receive it.

- (i) Write down the differential equation involving N , t , k and P . [1]

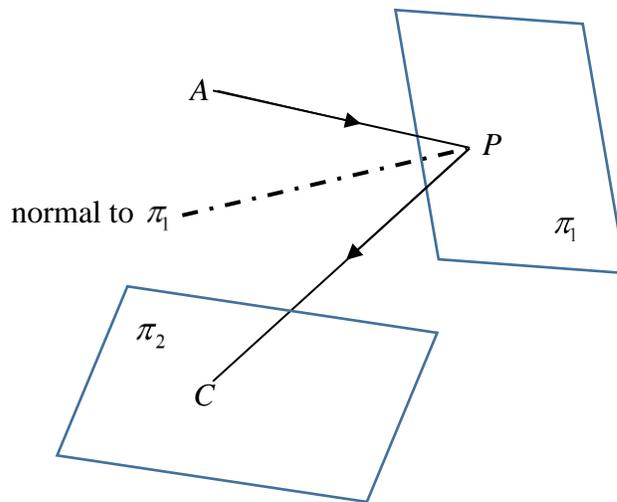
It is given that $N = 1$ and $\frac{dN}{dt} = \frac{P-1}{10}$ when $t = 0$.

- (ii) Show that the general solution of the differential equation in part (i) is

$$N = \frac{AP}{A + e^{-0.1Pt}}, \text{ where } A \text{ is an arbitrary constant.} \quad [6]$$

- (iii) Given that the population size is 1 million, find the particular solution and hence sketch the graph of N against t . [3]
- (iv) With reference to the graph of N against t , explain in context, the long term implication of the model used. [1]
- (v) Find the time taken, to the nearest minute, for a piece of information to reach 99% of the population of one million. [1]

- 11 (In this question you may assume that a laser beam travels in a straight line.)



A laser pointer is used to fire a beam in the direction $\begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$ from point A with coordinates

$(4, -5, 10)$. The beam is reflected at point P off the surface of the mirror π_1 which then, strikes a target plane π_2 at point C as shown in the diagram. It is given that the equation

of the plane π_1 is $\mathbf{r} \cdot \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} = 5$.

- (i) Show that the coordinates of the point P is $(12, -13, 6)$. [3]

It is given that the angle between AP and the normal to π_1 at P is equal to the angle between PC and the same normal.

- (ii) Find the vector \overrightarrow{AF} , where F is the foot of perpendicular from A to the normal to π_1 at P . [3]

- (iii) Find the vector equation of the line PC . [3]

Express the vector equation of the line PC in the form $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0}$, where \mathbf{a} and \mathbf{b} are constant vectors. [1]

The equation of the target plane π_2 is $\mathbf{r} \cdot \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = 1$.

- (iv) Show that the laser beam AP is parallel to π_2 . [1]
- (v) Find the shortest distance between the laser beam AP and π_2 . [3]