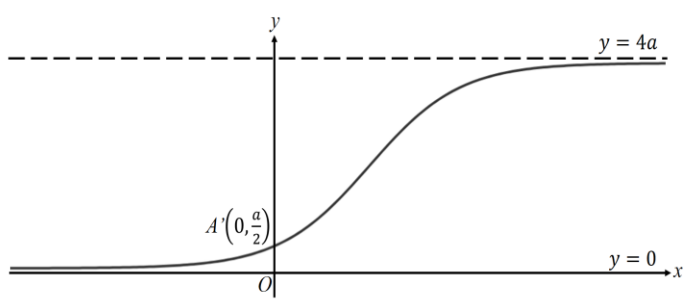
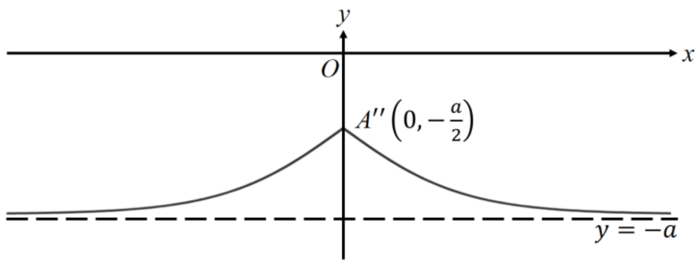
Promos Practice Paper 3 [YIJC 2022] Solutions

Qn	Solutions
1	Let V cm^3 represent the volume of water in the cup when the depth of water is h cm. $\frac{r}{h}=\frac{3}{9}$ $r=\frac{h}{3}$ --- (1) Substitute $r=\frac{h}{3}$ into $V=\frac{1}{3}\pi r^2 h$, $V=\frac{1}{3}\pi r^2 h$ $=\frac{1}{3}\pi\left(\frac{h}{3}\right)^2 h$ $=\frac{1}{27}\pi h^3$ **Alternatively,** Substitute $h=3r$ into $V=\frac{1}{3}\pi r^2 h$, $V=\pi r^3$ $\frac{dV}{dh}=\frac{1}{9}\pi h^2$ $\frac{dh}{dt}=\frac{dh}{dV}\times\frac{dV}{dt}$ $=\frac{9}{\pi h^2}\times 5$ $=\frac{45}{\pi h^2}$ **Alternatively,** Differentiate V wrt t, $\frac{dV}{dt}=\frac{3}{27}\pi h^2\frac{dh}{dt}$ $5=\frac{3}{27}\pi h^2\frac{dh}{dt}$ $\frac{dh}{dt}=\frac{45}{\pi h^2}$ **OR** Differentiate V wrt t, $\frac{dV}{dt}=3\pi r^2\frac{dr}{dt}$ $\frac{dh}{dt}=3\frac{dr}{dt}=\frac{5}{\pi r^2}$ When $r=2$, $\frac{dh}{dt}=\frac{5}{4\pi}\approx 0.398$ When $r=2$, $h=2\times 3=6$ When $h=6$, $\frac{dh}{dt}=\frac{45}{\pi(6)^2}=\frac{45}{36\pi}=\frac{5}{4\pi}\approx 0.398$ The rate of increase of the depth of the water is 0.398 (or $\frac{5}{4\pi}$) cm / s

2a	$y=\mathrm{f}(x)+a$ 
2b	$y=\mathrm{f}(-\lvert x\rvert)$


y
x
O
$A''\left(0, -\frac{a}{2}\right)$
$y = -a$

3(i) **[3]**	$\sin\left(\frac{\pi}{3}-\theta\right)=\sin\frac{\pi}{3}\cos\theta-\sin\theta\cos\frac{\pi}{3}$ $=\frac{\sqrt{3}}{2}\cos\theta-\frac{1}{2}\sin\theta$ $\approx\frac{\sqrt{3}}{2}\left(1-\frac{\theta^2}{2}\right)-\frac{1}{2}\theta$ $=\frac{\sqrt{3}}{2}-\frac{1}{2}\theta-\frac{\sqrt{3}}{4}\theta^2$ $\therefore a=\frac{\sqrt{3}}{2}, b=-\frac{1}{2}, c=-\frac{\sqrt{3}}{4}$
3(ii) **[3]**	$\sin\left(\frac{\pi}{3}-\frac{\pi}{10}\right)\approx\frac{\sqrt{3}}{2}-\frac{1}{2}\left(\frac{\pi}{10}\right)-\frac{\sqrt{3}}{4}\left(\frac{\pi}{10}\right)^2$ $\sin\left(\frac{7\pi}{30}\right)=0.66621\ (5\text{ d.p})$ using $\cos\theta=\sin\left(\frac{\pi}{2}-\theta\right)$, $\cos\left(\frac{4\pi}{15}\right)=\sin\left(\frac{\pi}{2}-\frac{4\pi}{15}\right)$ $=\sin\left(\frac{7}{30}\pi\right)$ $=0.66621\ (5\text{ d.p})$
3(iii)	Using calculator, $\cos\left(\frac{4\pi}{15}\right)=0.66913\ (5\text{ d.p.})$ The approximation is not good because $\frac{\pi}{10}$ is **not small enough**. ***<u>Alternative explanation:</u>*** The terms with θ^3 and higher powers **are not negligible** / **are significant** enough to affect the accuracy of the approximation.

4(i)	Since all coefficients of the equation $z^3+2z^2-19z+30=0$ are real and $z=2+\mathrm{i}$ is a root, $z=2-\mathrm{i}$ is another root. A quadratic factor is

	$\left[z-(2+\mathrm{i})\right]\left[z-(2-\mathrm{i})\right]$ $=(z-2-\mathrm{i})(z-2+\mathrm{i})$ $=(z-2)^2-\mathrm{i}^2$ $=z^2-2(z)(2)+2^2-(-1)$ $=z^2-4z+4+1$ $=z^2-4z+5$ $\therefore z^3+2z^2-19z+30=(z^2-4z+5)(z+a)$ **To get the 3rd root:** **<u>Method 1</u>** Comparing constant terms $30=5a$ $\Rightarrow a=6$ $z+6=0 \Rightarrow z=-6$ **<u>Method 2</u>** By long division $$\begin{array}{r} z+6 \\ z^2-4z+5 \overline{\smash{\big)}\, z^3+2z^2-19z+30} \\ \underline{-(z^3-4z^2+5z)} \\ 6z^2-24z+30 \\ \underline{-(6z^2-24z+30)} \\ 0 \end{array}$$ $z+6=0 \Rightarrow z=-6$ $\therefore$ Besides $z=2+\mathrm{i}$, the other roots are $z=2-\mathrm{i}$ and $z=-6$.

5(i)	$xy^2-3x^2y+144=0$ Differentiating w.r.t x: $2xy\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)+y^2-6xy-3x^2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)=0$ $\frac{\mathrm{d}y}{\mathrm{d}x}(2xy-3x^2)=6xy-y^2$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{6xy-y^2}{2xy-3x^2}$ $=\frac{y(6x-y)}{x(2y-3x)}$

	At the turning point, $\frac{dy}{dx}=0$ $y(6x-y)=0$ $y=0$ or $x=\frac{y}{6}$ For $y=0$, $x(0)^2-3x^2(0)+144=144\neq 0$ Therefore, we reject $y=0$. Substitute $x=\frac{y}{6}$ into the equation for curve C $\left(\frac{y}{6}\right)y^2-3\left(\frac{y}{6}\right)^2y+144=0$ $\frac{y^3}{12}=-144$ $y^3=-1728$ $y=-12$ $x=-2$ The coordinates of the turning point are $(-2,-12)$ **Alternatively,** Substitute $y=6x$ into the eqn, $x(6x)^2-3x^2(6x)+144=0$ $18x^3=-144$ $x=-2$ $y=-12$
(ii)	At $x=4$, $4k^2-3(4)^2k+144=0$ $4k^2-48k+144=0$ $k^2-12k+36=0$ $k=6$ At $x=4$, $\frac{dy}{dx}=\frac{6(24-6)}{4(12-12)}$ which is undefined. Hence the tangent is parallel to the y-axis. $\therefore$ the equation of the tangent is $x=4$.

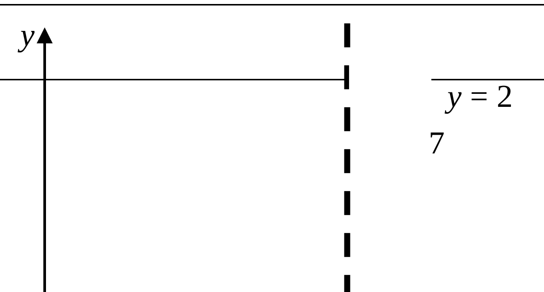
6	$\frac{3+6d}{3+36d}=\frac{3}{3+6d}$ $\frac{1+2d}{1+12d}=\frac{1}{1+2d}$ $(1+2d)^2=1+12d$ $1+4d+4d^2=1+12d$ $4d^2-8d=0$ Since $d\neq 0$, $d-2=0$ $d=2$

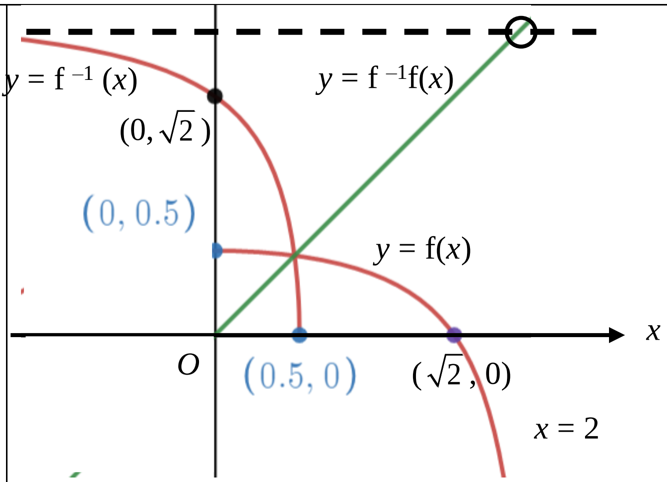
	$r=\dfrac{3}{3+6d}=\dfrac{3}{3+6(2)}=\dfrac{1}{5}$ Since $\lvert r\rvert=\dfrac{1}{5}<1$, the geometric series is convergent. (Or $-1<r<1$) Sum to infinity of odd-numbered terms $=\dfrac{a}{1-\left(\frac{1}{5}\right)^2}$ $=\dfrac{25}{24}a$
	$\dfrac{n}{2}\left[2(3)+(n-1)(2)\right]>\dfrac{25}{24}(3072)$ $\dfrac{n}{2}(4+2n)>3200$ $n^2+2n-3200>0$ $n<-57.577$ or $n>55.577$ $\therefore$ least $n=56$ **<u>Alternative method (using GC)</u>** When n = 55, S_n = 3135 When n = 56, S_n = 3248 When n = 56, S_n = 3363 $\therefore$ least $n=56$

7	$x=2t,\qquad y=\dfrac{1}{1-t}$ $\dfrac{dx}{dt}=2,\qquad \dfrac{dy}{dt}=-(1-t)^{-2}(-1)=\dfrac{1}{(1-t)^2}$ $\dfrac{dy}{dx}=\dfrac{\frac{1}{(1-t)^2}}{2}=\dfrac{1}{2(1-t)^2}$ Gradient of normal = $-2(1-t)^2$ Equation of normal at point with parameter t: $y-\dfrac{1}{1-t}=-2(1-t)^2(x-2t)$ $(1-t)y-1=-2(1-t)^3(x-2t)$ $(1-t)y+2(1-t)^3x=1+4t(1-t)^3\quad$ (shown)

	Equation of normal at point where $t = 2$: $-y-2x=-7$ $y+2x=7$ Substitute $x=2t$ & $y=\dfrac{1}{1-t}$ into equation of normal: $\dfrac{1}{1-t}+4t=7$ $1+4t(1-t)=7(1-t)$ $1+4t-4t^2=7-7t$ $4t^2-11t+6=0$ $(t-2)(4t-3)=0$ $t=2$ or $t=\dfrac{3}{4}$ At Q, $t=\dfrac{3}{4}$ $x=\dfrac{3}{2}$, $y=\dfrac{1}{1-\frac{3}{4}}=4$ $\therefore$ coordinates of Q are $\left(\dfrac{3}{2}, 4\right)$.

8(i)	Let $y=1+\dfrac{2}{x^2-4}$. $x^2-4=\dfrac{2}{y-1}$ $x^2=4+\dfrac{2}{y-1}$ $x=\pm\sqrt{4+\dfrac{2}{y-1}}$ (rej. neg since $0\le x<2$) $\therefore \mathrm{f}^{-1}(x)=\sqrt{4+\dfrac{2}{x-1}}$ $\mathrm{D}_{\mathrm{f}^{-1}}=\mathrm{R}_{\mathrm{f}}=\left(-\infty,\dfrac{1}{2}\right]$
(ii)	



	
(iii)	$\mathrm{fg}(x)=\frac{2}{5}$ $1+\frac{2}{\left(\frac{1}{x}\right)^2-4}=\frac{2}{5}$ $1+\frac{2x^2}{1-4x^2}=\frac{2}{5}$ $D_{fg}=D_g=[1,\infty)$ From GC, $x\approx 1.2247=1.22\ \ (3\text{ s.f.})$ **<u>Alternative method</u>** $\mathrm{fg}(x)=\frac{2}{5}$ $\mathrm{g}(x)=\mathrm{f}^{-1}\left(\frac{2}{5}\right)$ $\frac{1}{x}=\sqrt{4+\frac{2}{\frac{2}{5}-1}}$ $=\sqrt{\frac{2}{3}}$ $x=\sqrt{\frac{3}{2}}$

9(i)	Required amount $=500(1.001)^{11}$ $\approx \$505.53$

(ii)

Month	Amount of Money in the Account at the Start of the month	Amount of Money in the Account at the End of the month
1	500	500(1.001)
2	500(1.001) + x	$500(1.001)^2 + 1.001x$
3	$500(1.001)^2 + 1.001x + x$	$500(1.001)^3 + (1.001)^2x + 1.001x$

Amount of money in the account on the last day of the nth month

$$= 500(1.001)^n + (1.001)^{n-1}x + (1.001)^{n-2}x \ \ldots\ + 1.001x$$

$$= 500(1.001)^n + \left[(1.001)^{n-1} + (1.001)^{n-2} + \ \ldots\ + 1.001\right]x$$

$$= 500(1.001)^n + \left[\frac{1.001\left((1.001)^{n-1} - 1\right)}{1.001 - 1}\right]x = 500(1.001)^n + 1001\left[(1.001)^{n-1} - 1\right]x \quad \text{(shown)}$$

__**Alternative method**__

The first $\$x$ put on 1 Feb 2021 will be in the bank for $(n-1)$ months and will become $1.001^{n-1}x$ by the end of nth month.

The second $\$x$ put on 1 Mar 2021 will be in the bank for $(n-2)$ months and will become $1.001^{n-2}x$ by the end of nth month.

The third $\$x$ put on 1 Apr 2021 will be in the bank for $(n-3)$ months and will become $1.001^{n-3}x$ by the end of nth month.

…

The $(n-1)$ th $\$x$ put on the nth month will be in the bank for 1 month and will become $1.001x$ by the end of nth month.

Also, the initial $500 put on 1 Jan 2021 will be in the bank for n months and will become $1.001^n(500)$ by the end of nth month.

Hence, total amount at the end of nth month

$$= 500\left(1.001^n\right) + 1.001x + \ldots. + 1.001^{n-2}x + 1.001^{n-1}x$$

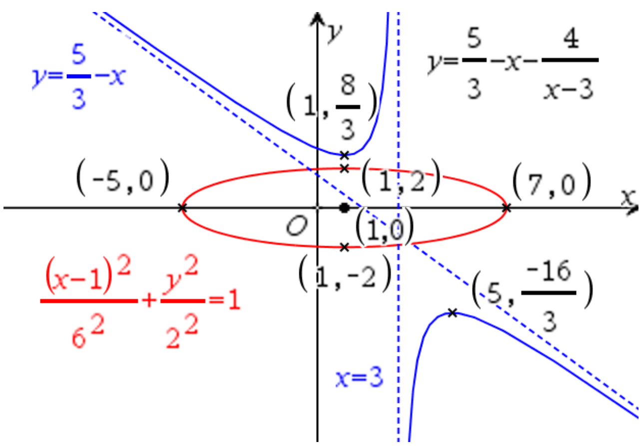
$$= 500\left(1.001^n\right) + x\left(1.001 + \ldots. + 1.001^{n-2} + 1.001^{n-1}\right)$$

$$= 500\left(1.001^n\right) + x\left[\frac{1.001\left(1.001^{n-1} - 1\right)}{1.001 - 1}\right]$$

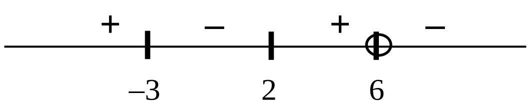
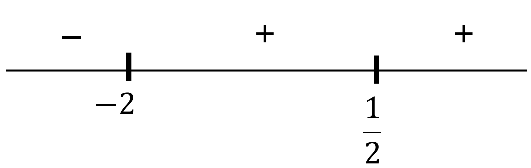
$$= 500\left(1.001^n\right) + 1001\left(1.001^{n-1} - 1\right)x \quad \text{(shown)}$$

(iii)	$500(1.001)^6 + 1001\left[(1.001)^5 - 1\right]x - 500 - 5x > 30$ $\Rightarrow x > 1797.10$ least integer value of x is 1798
(iv)	On 31 Dec 2025, $n = 48$ $500\left(1.001^{48}\right) + 1001(300)\left(1.001^{47} - 1\right) \approx 14968.22$ Jerald has \$14968.22 in his account on 31 Dec 2025. Since $14968.22 + 300 = 15268.22 > 15000$, The amount will first exceed \$15000 on 1st Jan 2026.

10(i)	$y = \frac{5}{3} - x - \frac{4}{x-3}$ $(x-3)y = (x-3)\left(\frac{5}{3} - x\right) - 4$ $xy - 3y = \frac{5}{3}x - x^2 - 5 + 3x - 4$ $3xy - 9y = -3x^2 + 14x - 27$ $3x^2 + (3y-14)x + (27-9y) = 0$ The equation has real roots $\Rightarrow b^2 - 4ac \geq 0$ $(3y-14)^2 - 4(3)(27-9y) \geq 0$ $9y^2 - 84y + 196 - 324 + 108y \geq 0$ $9y^2 + 24y - 128 \geq 0$

	Consider $9y^2+24y-128=0$ $y=\dfrac{-24\pm\sqrt{24^2-4(9)(-128)}}{18}$ $=\dfrac{8}{3}$ *or* $-\dfrac{16}{3}$ For $9y^2+24y-128\geq 0$, + − + ; $-\dfrac{16}{3}$, $\dfrac{8}{3}$ The range of values that C can take is $y\leq-\dfrac{16}{3}$ or $y\geq\dfrac{8}{3}$
(ii)	
(iii)	For C and D to intersect at more than one point, $k>\dfrac{8}{3}$.
(iv)	The line $y+\dfrac{4}{3}=m(x-3)$ has gradient m and passes through the point $\left(3,-\dfrac{4}{3}\right)$ which is the point of intersection of the vertical and oblique asymptotes. From the graph, the line does not intersect C when $m\geq-1$.

11(a)	Let \$$x$, \$$y$ and \$$z$ be the usual retail price of a bottle of cooking oil, a packet of biscuits and rice respectively. $5x+4y+2z=73.45 \qquad \cdots(1)$

	$2x+12y+z=53.30 \qquad \cdots(2)$ $4x+2y+5z=\frac{100}{95}(103.93)=109.4 \qquad \cdots(3)$ From GC, $x=6.85$, $y=2$ and $z=15.6$ The usual retail prices of 1 bottle of cooking oil, 1 packet of biscuits and 1 packet of rice are \$6.85, \$2 and \$15.60 respectively.
(b)(i)	$\frac{x^2}{6-x} \geqslant 1$ $\frac{x^2}{6-x}-1 \ge 0$ $\frac{x^2-(6-x)}{6-x} \ge 0$ $\frac{(x-2)(x+3)}{6-x} \ge 0$  $\therefore \quad x \le -3 \quad \text{or} \quad 2 \le x < 6$
(b)(ii)	$x^2-4 \ge (x+2)(4x^2-3x-1)$ $(x+2)(x-2)-(x+2)(4x^2-3x-1) \ge 0$ $(x+2)\left[x-2-(4x^2-3x-1)\right] \ge 0$ $(x+2)(-4x^2+4x-1) \ge 0$ $(x+2)(4x^2-4x+1) \le 0$ $(x+2)(2x-1)^2 \le 0$  $\therefore \quad x \le -2 \quad \text{or} \quad x=\frac{1}{2}$

12 **(i)**	$V=\pi r^2 h=20\pi$ $h=\frac{20}{r^2}$

	$C = 0.50 \times 2\left(\pi r^2\right) + 0.30\left(2\pi rh\right) + 0.8 \times 2 \times \left(2\pi r\right) + 0.6h$ $= \pi r^2 + 0.6\pi r\left(\dfrac{20}{r^2}\right) + 3.2\pi r + 0.6\left(\dfrac{20}{r^2}\right)$ $= \pi r^2 + \dfrac{12\pi}{r} + 3.2\pi r + \dfrac{12}{r^2}$
12 (ii)	$\dfrac{\mathrm{d}C}{\mathrm{d}r} = 2\pi r - \dfrac{12\pi}{r^2} + 3.2\pi - \dfrac{24}{r^3}$ Putting $\dfrac{\mathrm{d}C}{\mathrm{d}r} = 0,$ $2\pi r - \dfrac{12\pi}{r^2} + 3.2\pi - \dfrac{24}{r^3} = 0$ $2\pi r^4 + 3.2\pi r^3 - 12\pi r - 24 = 0$ [*optional*] Using GC, $r \approx 1.613596$ or $r \approx -0.685758$ (Rej since $r > 0$) $h = \dfrac{20}{1.613596^2} \approx 7.6814$ $\therefore$ $r = 1.61$ (3 s.f.) and $h = 7.68$ (3 s.f.) $\dfrac{\mathrm{d}^2C}{\mathrm{d}r^2} = 2\pi + \dfrac{24\pi}{r^3} + \dfrac{72}{r^4}$ When $r \approx 1.613596,$ since $r > 0,$ $\dfrac{\mathrm{d}^2C}{\mathrm{d}r^2} > 0$ (or $\dfrac{\mathrm{d}^2C}{\mathrm{d}r^2} \approx 34.85 > 0$) Thus C is minimum. The least cost is \$52.37.
12 (iii)	When $0.5 \le r \le 2$, From GC, $52.37 \le C \le 129.21$ (2 d.p.)