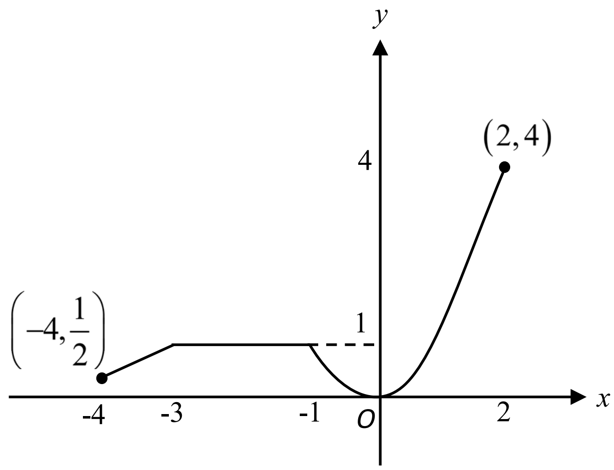
**Solutions 2021 JC2 Preliminary Examination Paper 1**

| 1 | Solution |
|---|---|
| **(a) (i)** | Total area of $n$ rectangles, $A$<br>$=\frac{1}{n}\ln\left(1+\frac{1}{n}\right)+\frac{1}{n}\ln\left(1+\frac{2}{n}\right)+\ldots+\frac{1}{n}\ln\left(1+\frac{n}{n}\right)$<br>$=\frac{1}{n}\ln\left(\frac{n+1}{n}\right)+\frac{1}{n}\ln\left(\frac{n+2}{n}\right)+\ldots+\frac{1}{n}\ln\left(\frac{2n}{n}\right)$<br>$=\frac{1}{n}\{[\ln(n+1)-\ln n]+[\ln(n+2)-\ln n]+\ldots+[\ln(2n)-\ln n]\}$<br>$=\frac{1}{n}[\ln(n+1)+\ln(n+2)+\ldots+\ln(2n)]-\frac{1}{n}(n\ln n)$<br>$=\frac{1}{n}\sum_{r=1}^{n}[\ln(n+r)]-\ln n$ (shown) |
| **(ii)** | $\lim_{n\to\infty} A=\int_1^2 \ln x \,\mathrm{d}x$<br>$=0.3863$ (4 d.p.) |

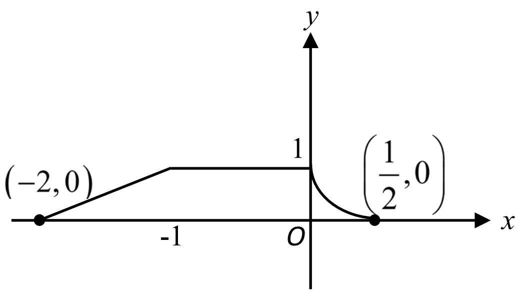
| 1 | Solution |
|---|---|
| **(b)** | At $(1,2)$,<br>$a+b+c=2------(1)$<br><br>$\mathrm{f}'(x)=\frac{-2a}{x^3}+b$<br>$-2a+b=0------(2)$<br><br>$\int_1^3 \mathrm{f}(x)\,\mathrm{d}x=\left[-\frac{a}{x}+b\frac{x^2}{2}+cx+d\right]_1^3$<br><br>$\left[-\frac{a}{3}+b\frac{9}{2}+3c+d\right]-\left[-a+b\frac{1}{2}+c+d\right]=\frac{20}{3}$<br>$\frac{2}{3}a+4b+2c=\frac{20}{3}------(3)$<br><br>From GC<br><br>$a=1, b=2, c=-1$<br>$y=\frac{1}{x^2}+2x-1$ |

| 2 | Solution |
|---|---|

**(i)**




**(ii)**




| 3 | Solution |
|---|---|

**(i)**

$\mathbf{a} \bullet \mathbf{p} = \mathbf{b} \bullet \mathbf{p}$

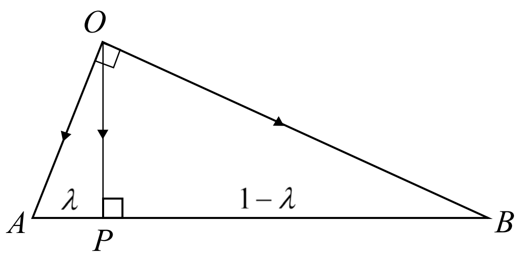
$\mathbf{b} \bullet \mathbf{p} - \mathbf{a} \bullet \mathbf{p} = 0$

$(\mathbf{b} - \mathbf{a}) \bullet \mathbf{p} = 0$

$\overrightarrow{AB} \bullet \overrightarrow{OP} = 0$

Hence, *AB* is perpendicular to *OP*.

**(ii)**




<u>**Method 1**</u>

Let $AP : PB = \lambda : 1-\lambda$

By ratio theorem, $\mathbf{p} = \lambda\mathbf{b} + (1-\lambda)\mathbf{a}$

$\mathbf{a}\bullet\mathbf{p} = \mathbf{b}\bullet\mathbf{p}$

$\mathbf{a}\bullet\left(\lambda\mathbf{b} + (1-\lambda)\mathbf{a}\right) = \mathbf{b}\bullet\left(\lambda\mathbf{b} + (1-\lambda)\mathbf{a}\right)$

$\lambda\mathbf{a}\bullet\mathbf{b} + (1-\lambda)\mathbf{a}\bullet\mathbf{a} = \lambda\mathbf{b}\bullet\mathbf{b} + (1-\lambda)\mathbf{b}\bullet\mathbf{a}$

$|\mathbf{a}|^2 - \lambda|\mathbf{a}|^2 = \lambda|\mathbf{b}|^2 \qquad \left(\because \mathbf{a}\bullet\mathbf{a} = |\mathbf{a}|^2,\ \mathbf{b}\bullet\mathbf{b} = |\mathbf{b}|^2,\ \mathbf{a}\bullet\mathbf{b} = 0\right)$

$$\lambda = \frac{|\mathbf{a}|^2}{|\mathbf{a}|^2 + |\mathbf{b}|^2}$$

Hence, $\mathbf{p} = \dfrac{|\mathbf{a}|^2}{|\mathbf{a}|^2 + |\mathbf{b}|^2}\mathbf{b} + \left(1 - \dfrac{|\mathbf{a}|^2}{|\mathbf{a}|^2 + |\mathbf{b}|^2}\right)\mathbf{a}$

$$= \mathbf{a} + \frac{|\mathbf{a}|^2}{|\mathbf{a}|^2 + |\mathbf{b}|^2}(\mathbf{b} - \mathbf{a})$$

**Method 2**

The projection vector of $\overrightarrow{AO}$ onto $\overrightarrow{AB}$ is

$$\overrightarrow{AP} = \left(\frac{\overrightarrow{AO}\bullet\overrightarrow{AB}}{\left|\overrightarrow{AB}\right|}\right)\frac{\overrightarrow{AB}}{\left|\overrightarrow{AB}\right|}$$

$$= \left(\frac{-\mathbf{a}\bullet(\mathbf{b} - \mathbf{a})}{|\mathbf{b} - \mathbf{a}|}\right)\frac{(\mathbf{b} - \mathbf{a})}{|\mathbf{b} - \mathbf{a}|}$$

$$= \left(\frac{-\mathbf{a}\bullet\mathbf{b} + \mathbf{a}\bullet\mathbf{a}}{(\mathbf{b} - \mathbf{a})\bullet(\mathbf{b} - \mathbf{a})}\right)(\mathbf{b} - \mathbf{a})$$

$$= \left(\frac{-\mathbf{a}\bullet\mathbf{b} + |\mathbf{a}|^2}{|\mathbf{b}|^2 - 2\mathbf{a}\bullet\mathbf{b} + |\mathbf{a}|^2}\right)(\mathbf{b} - \mathbf{a}) \qquad \left(\because \mathbf{a}\bullet\mathbf{a} = |\mathbf{a}|^2,\ \mathbf{b}\bullet\mathbf{b} = |\mathbf{b}|^2\right)$$

$$= \frac{|\mathbf{a}|^2}{|\mathbf{a}|^2 + |\mathbf{b}|^2}(\mathbf{b} - \mathbf{a}) \qquad (\because \mathbf{a}\bullet\mathbf{b} = 0)$$

Hence, $\mathbf{p} = \overrightarrow{OA} + \overrightarrow{AP}$

$$= \mathbf{a} + \frac{|\mathbf{a}|^2}{|\mathbf{a}|^2 + |\mathbf{b}|^2}(\mathbf{b} - \mathbf{a})$$

**Method 3**

Note that *OAB* and *PAO* are similar triangles,

$$\therefore \frac{|\mathbf{a}|}{AP} = \frac{\sqrt{|\mathbf{a}|^2 + |\mathbf{b}|^2}}{|\mathbf{a}|}$$

$$AP = \frac{|\mathbf{a}|^2}{\sqrt{|\mathbf{a}|^2 + |\mathbf{b}|^2}}$$

$$\overrightarrow{AP} = AP\left(\frac{\overrightarrow{AB}}{\left|\overrightarrow{AB}\right|}\right)$$

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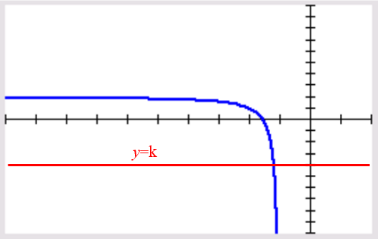
| | $$\mathbf{p}-\mathbf{a}=\frac{|\mathbf{a}|^2}{\sqrt{|\mathbf{a}|^2+|\mathbf{b}|^2}}\left(\frac{\mathbf{b}-\mathbf{a}}{\sqrt{|\mathbf{a}|^2+|\mathbf{b}|^2}}\right)$$ $$\mathbf{p}=\mathbf{a}+\frac{|\mathbf{a}|^2}{|\mathbf{a}|^2+|\mathbf{b}|^2}(\mathbf{b}-\mathbf{a})$$ |

| 4 | Solution |
|---|---|
| **(i)** |  Any horizontal line $y=k$, where $k\in\mathbb{R}$, cuts the graph of f **<u>at most once</u>**, therefore f is one-one and f has an inverse. |
| **(ii)** | Let $y=2+\frac{3}{1-x^2}$ <br> $\frac{3}{1-x^2}=y-2$ <br> $1-x^2=\frac{3}{y-2}$ <br> $x^2=1-\frac{3}{y-2}$ <br> $x=\pm\sqrt{1-\frac{3}{y-2}}$ <br> $x=\sqrt{1-\frac{3}{y-2}}$ or $x=-\sqrt{1-\frac{3}{y-2}}$ <br> (rej. since $x<-1$) <br> $\therefore \mathrm{f}^{-1}(x)=-\sqrt{1-\frac{3}{x-2}}$, $x<2$ |
| **(iii)** | $\mathrm{g}(x)=-x^2+6x+a$ <br> $\mathrm{g}(x)=-(x-3)^2+a+9$ <br> $D_{\mathrm{f}}=(-\infty,-1)$, $R_{\mathrm{g}}=(-\infty,a+9]$ <br> Since $a+9>-1$ for $a>0$, $R_{\mathrm{g}}\not\subseteq D_{\mathrm{f}}$ <br> Therefore, fg does not exist. |

| 5 | **Solution** |
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| | Volume generated<br>$=\frac{1}{3}\pi\left(3e^3\right)^2(3)-\pi\int_0^3 x^2e^{2x}dx=9\pi e^6-\pi\left\{\left[\frac{1}{2}x^2e^{2x}\right]_0^3-\int_0^3 xe^{2x}dx\right\}$    $u=x^2$, $\frac{dv}{dx}=e^{2x}$, $\frac{du}{dx}=2x$, $v=\frac{1}{2}e^{2x}$<br>$=9\pi e^6-\pi\left\{\frac{9}{2}e^6-\left[\frac{x}{2}e^{2x}\right]_0^3+\int_0^3\frac{1}{2}e^{2x}dx\right\}$    $u=x$, $\frac{dv}{dx}=e^{2x}$, $\frac{du}{dx}=1$, $v=\frac{1}{2}e^{2x}$<br>$=9\pi e^6-\pi\left(\frac{9}{2}e^6-\frac{3}{2}e^6+\left[\frac{e^{2x}}{4}\right]_0^3\right)$<br>$=9\pi e^6-\pi\left(3e^6+\frac{e^6}{4}-\frac{1}{4}\right)$<br>$=9\pi e^6-\frac{\pi}{4}(13e^6-1)$<br>$=\frac{\pi}{4}(23e^6+1)$ |

| 6 | **Solution** |
|---|---|
| **(i)** | Since the coefficients of the equation are real, $1-ai$ is also a root of the equation.<br><br>Quadratic factor<br>$=\left(z-(1+ai)\right)\left(z-(1-ai)\right)$<br>$=\left(z-1-ai\right)\left(z-1+ai\right)$<br>$=\left(z-1\right)^2-\left(ai\right)^2$<br>$=z^2-2z+1+a^2$<br><br>By comparison,<br>$2z^3-9z^2+30z+b=\left(z^2-2z+1+a^2\right)\left(cz+d\right)$<br><br>Comparing coefficient of $z^3 \Rightarrow c=2$<br>Comparing coefficient of $z^2 \Rightarrow -9=d-2(2) \Rightarrow d=-5$<br>Comparing coefficient of $z \Rightarrow 30=-2(-5)+\left(1+a^2\right)(2)$<br>$\Rightarrow a^2=9$<br>$\Rightarrow a=3$ or $a=-3$<br>Comparing constant $\Rightarrow b=-5(1+9)=-50$<br><br>$\therefore\ 2z^3-9z^2+30z-50=\left(z^2-2z+10\right)\left(2z-5\right)=0$<br><br>The roots are $1+3i$, $1-3i$ and $\frac{5}{2}$. |

**Alternative Method (Not recommended)**

$(1+ai)^3$
$=1+3(ai)+3(ai)^2+(ai)^3$
$=1+3ai-3a^2-a^3i$
$=\left(1-3a^2\right)+i\left(3a-a^3\right)$

Since $1+ai$ is a root of the equation,

$$2\left(1+ai\right)^3-9\left(1+ai\right)^2+30\left(1+ai\right)+b=0$$
$$2\left[\left(1-3a^2\right)+i\left(3a-a^3\right)\right]-9\left(1+2ai-a^2\right)+30\left(1+ai\right)+b=0$$
$$\left(23+b+3a^2\right)+i\left(18a-2a^3\right)=0$$

Comparing the real part: $23+b+3a^2=0$ --- (1)
Comparing the imaginary part: $18a-2a^3=0$ --- (2)

From (2): $a=0$ (rej $\because a\neq 0$ ) or $a=3$ or $a=-3$

From (1): $23+b+3(-3)^2=0 \Rightarrow b=-50$

Since the coefficients of the equation are real, $1-3i$ and $1+3i$ are roots of the equation.

Quadratic factor
$=\left(z-(1+3i)\right)\left(z-(1-3i)\right)$
$=\left(z-1-3i\right)\left(z-1+3i\right)$
$=\left(z-1\right)^2-\left(3i\right)^2$
$=z^2-2z+10$

By comparison,
$2z^3-9z+30z-50=\left(z^2-2z+10\right)\left(cz+d\right)$

Comparing coefficient of $z^3 \Rightarrow c=2$
Comparing constant $\Rightarrow 10d=-50 \quad \Rightarrow d=-5$

$\therefore\ 2z^3-9z^2+30z-50=\left(z^2-2z+10\right)\left(2z-5\right)=0$

The roots are $1+3i$, $1-3i$ and $\frac{5}{2}$.

**(ii)**

$bz^3+30z^2-9z+2=0$
Divide throughout by $z^3$,
$$b+\frac{30}{z}-\frac{9}{z^2}+\frac{2}{z^3}=0$$
$$2\left(\frac{1}{z}\right)^3-9\left(\frac{1}{z}\right)^2+30\left(\frac{1}{z}\right)+b=0$$

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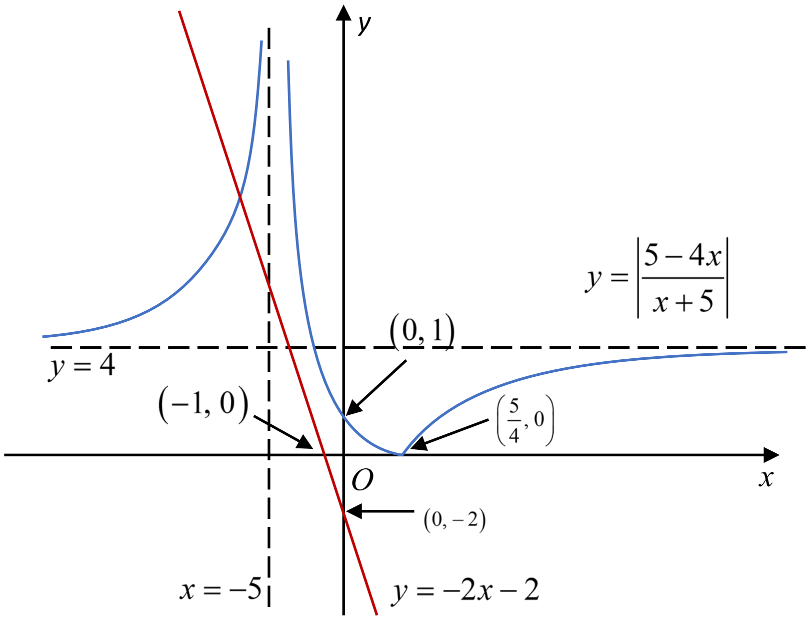
| | Replace $z$ with $\frac{1}{z}$,<br>$\frac{1}{z}=\frac{5}{2}$ or $\frac{1}{z}=1+3\mathrm{i}$ or $\frac{1}{z}=1-3\mathrm{i}$<br>$z=\frac{2}{5}$ or $z=\frac{1}{1+3\mathrm{i}}=\frac{1}{1+3\mathrm{i}}\times\frac{1-3\mathrm{i}}{1-3\mathrm{i}}=\frac{1}{10}-\frac{3}{10}\mathrm{i}$<br>or $z=\frac{1}{1-3\mathrm{i}}\times\frac{1+3\mathrm{i}}{1+3\mathrm{i}}=\frac{1}{10}+\frac{3}{10}\mathrm{i}$ |

| 7 | **Solution** |
|---|---|
| **(i)** | <br> |
| **(ii)** | From the graph, the point of intersection occurs when $x<-5$.<br>$x<-5\Rightarrow 5-4x>0,\ x+5<0$<br>$\therefore \frac{5-4x}{x+5}<0$ and $\left\|\frac{5-4x}{x+5}\right\|=-\frac{5-4x}{x+5}$<br>$-\frac{5-4x}{x+5}=-2x-2$<br>$-5+4x=(-2x-2)(x+5)$<br>$-5+4x=-2x^2-12x-10$<br>$2x^2+16x+5=0$<br>$x=\frac{-16\pm\sqrt{16^2-4(2)(5)}}{2(2)}$<br>$=\frac{-16\pm\sqrt{216}}{4}$<br>$=\frac{-16\pm6\sqrt{6}}{4}$<br>$=-4\pm\frac{3}{2}\sqrt{6}$<br>Since $x<-5$, hence $x=-4-\frac{3}{2}\sqrt{6}$.<br>$-4-\frac{3}{2}\sqrt{6}<x<-5$ or $x>-5$ |

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| | **Alternative Method (strongly discouraged):**<br>$\left(\frac{5-4x}{x+5}\right)^2=(-2x-2)^2$<br>$(5-4x)^2=(-2x-2)^2(x+5)^2$<br>$(5-4x)^2=[(-2x-2)(x+5)]^2$<br><br>$(5-4x)^2-[(-2x-2)(x+5)]^2=0$<br>Since $a^2-b^2=(a-b)(a+b)$,<br>$[(5-4x)-(-2x-2)(x+5)][(5-4x)+(-2x-2)(x+5)]=0$<br>$\left[(5-4x)-\left(-2x^2-12x-10\right)\right]\left[(5-4x)+\left(-2x^2-12x-10\right)\right]=0$<br>$\left[2x^2+8x+15\right]\left[-2x^2-16x-5\right]=0$<br>$2x^2+8x+15=0$ or $-2x^2-16x-5=0$<br><br>For $2x^2+8x+15=0=0$<br>$x=\frac{-8\pm\sqrt{8^2-4(2)(15)}}{2(2)}$<br>$=\frac{-8\pm\sqrt{-56}}{4}$<br>There are no real roots.<br><br>For $-2x^2-16x-5=0$<br>$x=\frac{16\pm\sqrt{16^2-4(-2)(-5)}}{2(-2)}$<br>$=\frac{16\pm\sqrt{216}}{-4}$<br>$=\frac{16\pm6\sqrt{6}}{-4}$<br>$=-4\pm\frac{3}{2}\sqrt{6}$<br>Since $x<-5$, hence $x=-4-\frac{3}{2}\sqrt{6}$.<br>$-4-\frac{3}{2}\sqrt{6}<x<-5$ or $x>-5$ |
| **(iii)** | $\left\|\frac{10-4x}{x+10}\right\|>-x-2$<br>$\left\|\frac{2(5-2x)}{2(0.5x+5)}\right\|>-x-2$ |

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| | $\left\lvert\dfrac{5-4\left(\frac{x}{2}\right)}{\left(\frac{x}{2}\right)+5}\right\rvert > -2\left(\frac{x}{2}\right)-2$<br>Hence replace $x$ with $\frac{x}{2}$,<br>$-4-\frac{3}{2}\sqrt{6} < \frac{x}{2} < -5$ or $\frac{x}{2} > -5$<br>$-8-3\sqrt{6} < x < -10$ or $x > -10$ |

| **8** | **Solution** |
|---|---|
| **(i)** | $y=\sin\left(\ln(1+\mathrm{e}x)\right)$<br>$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{e}}{1+\mathrm{e}x}\cos\left(\ln(1+\mathrm{e}x)\right)$<br>$\left(1+\mathrm{e}x\right)\frac{\mathrm{d}y}{\mathrm{d}x}=\mathrm{e}\cos\left(\ln(1+\mathrm{e}x)\right)$<br>$\left(1+\mathrm{e}x\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\mathrm{e}\frac{\mathrm{d}y}{\mathrm{d}x}=\mathrm{e}\left(-\sin\left(\ln(1+\mathrm{e}x)\right)\right)\frac{\mathrm{e}}{\left(1+\mathrm{e}x\right)}$<br>$\left(1+\mathrm{e}x\right)^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\mathrm{e}\left(1+\mathrm{e}x\right)\frac{\mathrm{d}y}{\mathrm{d}x}=-\mathrm{e}^2y$ (Shown) |
| **(ii)** | $\left(1+\mathrm{e}x\right)^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\mathrm{e}\left(1+\mathrm{e}x\right)\frac{\mathrm{d}y}{\mathrm{d}x}=-\mathrm{e}^2y$<br>$\left(1+\mathrm{e}x\right)^2\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+2\mathrm{e}\left(1+\mathrm{e}x\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\mathrm{e}\left(1+\mathrm{e}x\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\mathrm{e}^2\frac{\mathrm{d}y}{\mathrm{d}x}=-\mathrm{e}^2\frac{\mathrm{d}y}{\mathrm{d}x}$<br>$\left(1+\mathrm{e}x\right)^2\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+3\mathrm{e}\left(1+\mathrm{e}x\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2\mathrm{e}^2\frac{\mathrm{d}y}{\mathrm{d}x}=0$<br>When $x=0, y=0, \frac{\mathrm{d}y}{\mathrm{d}x}=\mathrm{e}, \frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\mathrm{e}^2, \frac{\mathrm{d}^3y}{\mathrm{d}x^3}=\mathrm{e}^3$<br>$\mathrm{f}\left(x\right)=\mathrm{f}\left(0\right)+\mathrm{f}'\left(0\right)x+\frac{\mathrm{f}''\left(0\right)}{2!}x^2+\frac{\mathrm{f}'''\left(0\right)}{3!}x^3+\ldots$<br>$y=\mathrm{e}x-\frac{\mathrm{e}^2}{2}x^2+\frac{\mathrm{e}^3}{6}x^3+\ldots$ |
| **(iii)** | $y=\sin\left(\mathrm{e}x-\frac{\left(\mathrm{e}x\right)^2}{2}+\frac{\left(\mathrm{e}x\right)^3}{3}+\ldots\right)$<br>$=\left(\mathrm{e}x-\frac{\mathrm{e}^2x^2}{2}+\frac{\mathrm{e}^3x^3}{3}+\ldots\right)-\frac{\left(\mathrm{e}x-\frac{\mathrm{e}^2x^2}{2}+\frac{\mathrm{e}^3x^3}{3}+\ldots\right)^3}{6}+\ldots$ |

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| | $=\mathrm{e}x-\frac{\mathrm{e}^2x^2}{2}+\frac{\mathrm{e}^3x^3}{3}-\frac{\mathrm{e}^3x^3}{6}+\ldots$<br>$=\mathrm{e}x-\frac{\mathrm{e}^2x^2}{2}+\frac{\mathrm{e}^3x^3}{6}+\ldots$ |

| **9** | **Solution** |
|---|---|
| **(i)** | $6x^2-4y^2=3xy^2$<br>$12x-8y\frac{\mathrm{d}y}{\mathrm{d}x}=3y^2+6xy\frac{\mathrm{d}y}{\mathrm{d}x}$<br>$\frac{\mathrm{d}y}{\mathrm{d}x}(8y+6xy)=12x-3y^2$<br>$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{12x-3y^2}{8y+6xy}$ |
| **(ii)** | When $x=2$,<br>$6x^2-4y^2-3xy^2=0$<br>$24-4y^2-6y^2=0$<br>$10y^2=24$<br>$y=\pm\sqrt{\frac{12}{5}}$<br>At $\left(2,\sqrt{\frac{12}{5}}\right)$<br>$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{24-3\left(\frac{12}{5}\right)}{20\left(\sqrt{\frac{12}{5}}\right)}=\frac{21}{25}\sqrt{\frac{5}{12}}$<br>$y-\sqrt{\frac{12}{5}}=\frac{21}{25}\sqrt{\frac{5}{12}}(x-2)---(1)$<br>At $\left(2,-\sqrt{\frac{12}{5}}\right)$<br>$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{24-3\left(\frac{12}{5}\right)}{20\left(-\sqrt{\frac{12}{5}}\right)}=-\frac{21}{25}\sqrt{\frac{5}{12}}$<br>$y+\sqrt{\frac{12}{5}}=-\frac{21}{25}\sqrt{\frac{5}{12}}(x-2)---(2)$<br>(1)+ (2): $y=0$ |

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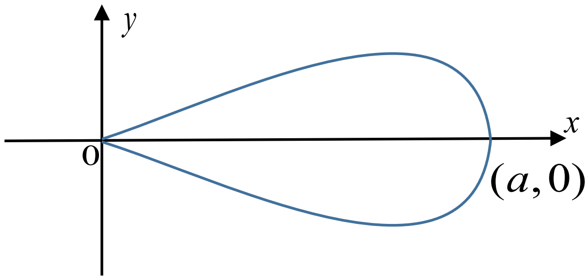
| | $\sqrt{\frac{12}{5}}=-\frac{21}{25}\sqrt{\frac{5}{12}}(x-2)$<br>$x-2=\sqrt{\frac{12}{5}}\left(-\frac{25}{21}\sqrt{\frac{12}{5}}\right)$<br>$x=-\frac{20}{7}+2=-\frac{6}{7}$<br><br>Coordinates are $\left(-\frac{6}{7},0\right)$ |

| 10 | Solution |
|---|---|
| (i) | <br>Line of symmetry: $y=0$ |
| (ii) | GC:<br>When $y=0$,<br>$a\cos^2\theta\sin\theta=0$<br>$\cos^2\theta=0$ or $\sin\theta=0$<br>$\theta=\frac{\pi}{2}$ or $-\frac{\pi}{2}$ $\theta=0$<br>$\therefore\theta=0$ or $\frac{\pi}{2}$ or $-\frac{\pi}{2}$ |
| (iii) | $x=a\cos^2\theta$<br>$\frac{dx}{d\theta}=-2a\cos\theta\sin\theta$<br>When $x=0,\theta=\frac{\pi}{2}$<br>When $x=a,\theta=0$<br><br>Area<br>$=\int_0^a y\,dx$<br>$=\int_{\frac{\pi}{2}}^{0}\left(a\cos^2\theta\sin\theta\right)\left(-2a\cos\theta\sin\theta\right)d\theta$<br>$=\int_0^{\frac{\pi}{2}}2a^2\cos^3\theta\sin^2\theta\,d\theta$ (Shown) |
| (iv) | Total area enclosed by $C$<br>$=2\int_0^a y\,dx$ |

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| | $=4a^2\int_0^{\frac{\pi}{2}}\cos^3\theta\sin^2\theta\,\mathrm{d}\theta$<br>$=4a^2\int_0^{\frac{\pi}{2}}\cos\theta(1-\sin^2\theta)(\sin^2\theta)\,\mathrm{d}\theta$<br>$=4a^2\int_0^{\frac{\pi}{2}}\cos\theta(\sin^2\theta-\sin^4\theta)\,\mathrm{d}\theta$<br>$=4a^2\int_0^{\frac{\pi}{2}}\cos\theta\sin^2\theta-\cos\theta\sin^4\theta\,\mathrm{d}\theta$<br>$=4a^2\left[\frac{\sin^3\theta}{3}-\frac{\sin^5\theta}{5}\right]_0^{\frac{\pi}{2}}$<br>$=4a^2\left[\frac{1}{3}-\frac{1}{5}\right]=4a^2\left(\frac{2}{15}\right)$<br>$=\frac{8}{15}a^2$ |
| **(v)** | Midpoint of $OP=\left(\frac{a\cos^2 p}{2},\frac{a\cos^2 p\sin p}{2}\right)$<br>$x=\frac{a\cos^2 p}{2},\ y=\frac{a\cos^2 p\sin p}{2}$<br>$y=\frac{a\cos^2 p\sin p}{2}=x\sin p$<br>$\therefore \sin p=\frac{y}{x}$<br>$x=\frac{a\cos^2 p}{2}=\frac{a(1-\sin^2 p)}{2}=\frac{a}{2}\left(1-\frac{y^2}{x^2}\right)$<br>$\frac{2x}{a}=\left(1-\frac{y^2}{x^2}\right)$<br>$2x^3=a\left(x^2-y^2\right)$ |

| **11** | **Solution** |
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| **(i)** | $\frac{\mathrm{d}x}{\mathrm{d}t}=ax-bx^2$<br>$\frac{\mathrm{d}x}{\mathrm{d}t}=0.49,\qquad x=1$<br>$0.49=a-b----(1)$<br>$\frac{\mathrm{d}x}{\mathrm{d}t}=0.96,\qquad x=2$<br>$0.96=2a-4b----(2)$<br>Solving,<br>$a=0.5,\ b=0.01$<br>$\frac{\mathrm{d}x}{\mathrm{d}t}=0.5x-0.01x^2$ |

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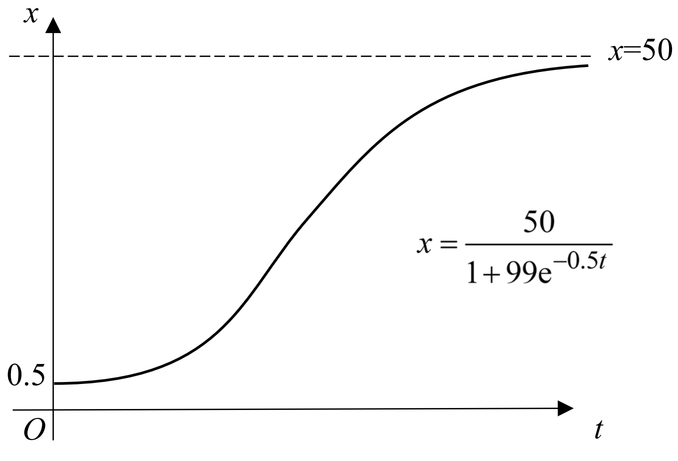
| | $\dfrac{dx}{dt}=\dfrac{1}{100}x\left(50-x\right)$ |
| **(ii)** | $\int \dfrac{1}{x(50-x)}\ dx=\int 0.01\ dt$<br>$\dfrac{1}{50}\int \dfrac{1}{50-x}+\dfrac{1}{x}\ dx=\int 0.01\ dt$<br>$\int \dfrac{1}{50-x}+\dfrac{1}{x}\ dx=\int 0.5\ dt$<br>$\ln\left\|\dfrac{x}{50-x}\right\|=0.5t+C$<br>$\dfrac{x}{50-x}=\pm e^{0.5t+C}=Ae^{0.5t}$<br>When $t=0,\ x=0.5$<br>$A=\dfrac{1}{99}$<br>$\dfrac{x}{50-x}=\dfrac{1}{99}e^{0.5t}$<br>$\dfrac{50-x}{x}=99e^{-0.5t}$<br>$\dfrac{50}{x}=1+99e^{-0.5t}$<br>$x=\dfrac{50}{1+99e^{-0.5t}}$<br><br>$45=\dfrac{50}{1+99e^{-0.5t}}$<br>From GC, it takes 13.6 days to reach 45 m. |
| **(iii)** |  |

| 12 | Solution |
|---|---|
| **(i)** | Total distance swam by Carol in 5 minutes<br>$=\dfrac{80(1-0.99^5)}{1-0.99}$<br>$=392.08$<br>Total distance pushed back by the waves in 5 minutes<br>$=\dfrac{5}{2}\left[2(4)+(5-1)(-0.05)\right]$<br>$=19.5$<br>Distance between Carol and the shore after the first 5 minutes<br>$=392.08-19.5$<br>$=372.58=373$ m (3 s.f.) |
| **(ii)** | $U_n=2$<br>$4+(n-1)(-0.05)=2$<br>$-0.05(n-1)=-2$<br>$n-1=40$<br>$n=41$ |
| **(iii)** | Distance between Carol and the shore in the first 40 minutes<br>$=\dfrac{80(1-0.99^{40})}{1-0.99}-\dfrac{40}{2}\left[2(4)+(40-1)(-0.05)\right]$<br>$=2527.2$<br>Remaining distance from 5 km $=5000-2527.2$<br>$=2472.8$<br>Let $t$ be the number of minutes after the 40th minute.<br>$\dfrac{80\left(0.99^{40}\right)\left(1-0.99^t\right)}{1-0.99}-2t\geq 2472.8$<br>From GC, $66.387\leq t\leq 1429.5$<br>$\therefore t\geq 67$<br>Therefore, minimum time (in nearest minute) is<br>40 + 67 = <u>107 minutes</u> |
| **(iv)** | Consider $d=\dfrac{80\left(0.99^{40}\right)\left(1-0.99^t\right)}{1-0.99}-2t+2527.2$<br>GC:<br><table><tr><td>$t$</td><td>Distance, $d$</td></tr><tr><td>327</td><td>7024.9187</td></tr><tr><td>328</td><td>7024.9196 (largest)</td></tr><tr><td>329</td><td>7024.9004</td></tr></table>$328+40=368$<br>Greatest distance is 7020 (3s.f.) metres, occurs at 368 minutes.<br>The distance Carol swam per minute drops below 2 metres per minute. |