1 **(i)** Show that $\dfrac{1}{r(r+1)}-\dfrac{1}{(r+1)(r+2)}=\dfrac{A}{r(r+1)(r+2)}$, where A is a constant to be found. [1]

$\dfrac{1}{r(r+1)}-\dfrac{1}{(r+1)(r+2)}=\dfrac{r+2-r}{r(r+1)(r+2)}$ $=\dfrac{2}{r(r+1)(r+2)}$	

(ii) Hence find $\displaystyle\sum_{r=1}^{n}\frac{1}{r(r+1)(r+2)}$. (There is no need to express your answer as a single algebraic fraction.) [3]

$\displaystyle\sum_{r=1}^{n}\frac{1}{r(r+1)(r+2)}$ $=\dfrac{1}{2}\left[\dfrac{1}{1(2)}-\cancel{\dfrac{1}{2(3)}}\right.$ $+\cancel{\dfrac{1}{2(3)}}-\cancel{\dfrac{1}{3(4)}}$ $+\ \ldots$ $+\cancel{\dfrac{1}{(n-1)n}}-\cancel{\dfrac{1}{n(n+1)}}$ $\left.+\cancel{\dfrac{1}{n(n+1)}}-\dfrac{1}{(n+1)(n+2)}\right]$ $=\dfrac{1}{2}\left[\dfrac{1}{2}-\dfrac{1}{(n+1)(n+2)}\right]=\dfrac{1}{4}-\dfrac{1}{2(n+1)(n+2)}$	

(iii) Given a reason why the series $\displaystyle\sum_{r=1}^{\infty}\frac{1}{r(r+1)(r+2)}$ converges, and write down its value. [2]

$\displaystyle\sum_{r=1}^{n}\frac{1}{r(r+1)(r+2)}=\frac{1}{4}-\frac{1}{2(n+1)(n+2)}$ As $n\to\infty$, $\dfrac{1}{2(n+1)(n+2)}\to 0$. Hence $\displaystyle\sum_{r=1}^{n}\frac{1}{r(r+1)(r+2)}\to\frac{1}{4}$ Thus series $\displaystyle\sum_{r=1}^{\infty}\frac{1}{r(r+1)(r+2)}$ converges. $\displaystyle\sum_{r=1}^{\infty}\frac{1}{r(r+1)(r+2)}=\frac{1}{4}$	Learn how to present your argument logically. A lot of presentation errors are seen. Wrong to say: As $r\to\infty$, $\dfrac{1}{r(r+1)(r+2)}\to 0$ $\dfrac{1}{r(r+1)(r+2)}$ is the term, not the sum. Pay attention to the presentation. **<u>The following are correct</u>** ✓ $\displaystyle\sum_{r=1}^{\infty}\frac{1}{r(r+1)(r+2)}=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{r(r+1)(r+2)}$ ✓ $\displaystyle\sum_{r=1}^{\infty}\frac{1}{r(r+1)(r+2)}=\frac{1}{4}$

	✓ $\sum_{r=1}^{n}\frac{1}{r(r+1)(r+2)}\to\frac{1}{4}$ as $n\to\infty$ Since the question asks for the value, you must state the value of sum to infinity. Thus it is insufficient to stop at line 3 of the solution $\sum_{r=1}^{n}\frac{1}{r(r+1)(r+2)}\to\frac{1}{4}$ You must conclude and state that $\sum_{r=1}^{\infty}\frac{1}{r(r+1)(r+2)}=\frac{1}{4}$

2 **(i)** Given that $y=\sin^{-1}\left(x^2\right)$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x. [1]

$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin^{-1}x^2\right)=\frac{1}{\sqrt{1-x^4}}\times\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\right)$ $=\frac{2x}{\sqrt{1-x^4}}$	Remember chain rule

(ii) Evaluate $\int_2^3 \ln\left[\tan^{-1}x\right]\,\mathrm{d}x$, giving your answer correct to 3 decimal places. [1]

$\int_2^3 \ln\left(\tan^{-1}x\right)\mathrm{d}x=0.17022=0.170\quad(3\text{ d.p.})$	Use GC since only 1 mark is given and question does not require exact value.

(iii) Given that $k>0$ and $\int_0^k x^2\sin^{-1}\left(x^2\right)\,\mathrm{d}x=\int_2^3 \ln\left[\tan^{-1}x\right]\,\mathrm{d}x$, show that

$$\int_0^k \frac{x^4}{\sqrt{1-x^4}}\,\mathrm{d}x=ak^3\sin^{-1}k^2+b,$$

where a and b are constants to be determined. [4]

$\int_0^k x^2\sin^{-1}\left(x^2\right)\,\mathrm{d}x$ $=\left[\frac{x^3}{3}\sin^{-1}\left(x^2\right)\right]_0^k-\int_0^k\frac{x^3}{3}\cdot\frac{2x}{\sqrt{1-x^4}}\mathrm{d}x$ $=\frac{k^3}{3}\sin^{-1}k^2-\frac{2}{3}\int_0^k\frac{x^4}{\sqrt{1-x^4}}\mathrm{d}x$ $\therefore\frac{k^3}{3}\sin^{-1}k^2-\frac{2}{3}\int_0^k\frac{x^4}{\sqrt{1-x^4}}\mathrm{d}x=0.17022$ Hence $\int_0^k\frac{x^4}{\sqrt{1-x^4}}\mathrm{d}x=\frac{3}{2}\left[\frac{k^3}{3}\sin^{-1}k^2-0.17022\right]$ $=\frac{1}{2}k^3\sin^{-1}k^2-0.255$ **<u>Alternative</u>**	Let $u=\sin^{-1}\left(x^2\right)$ and $\frac{\mathrm{d}v}{\mathrm{d}x}=x^2$ $\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{2x}{\sqrt{1-x^4}}$ and $\frac{\mathrm{d}v}{\mathrm{d}x}=\frac{x^3}{3}$ You should substitute in the 5 d.p. answers from (ii) so as to avoid rounding off error if any. **<u>Alternative</u>**

$\int_0^k \frac{x^4}{\sqrt{1-x^4}}\,\mathrm{d}x$ $=\int_0^k \frac{x^3}{2}\left(\frac{2x}{\sqrt{1-x^4}}\right)\mathrm{d}x$ $=\left[\frac{x^3}{2}\sin^{-1}\left(x^2\right)\right]_0^k-\int_0^k \frac{3}{2}x^2\sin^{-1}\left(x^2\right)\,\mathrm{d}x$ $=\frac{k^3}{2}\sin^{-1}\left(k^2\right)-\frac{3}{2}\int_2^3 \ln\left[\tan^{-1}x\right]\,\mathrm{d}x$ $=\frac{k^3}{2}\sin^{-1}\left(k^2\right)-\frac{3}{2}(0.17022)$ $=\frac{k^3}{2}\sin^{-1}\left(k^2\right)-0.255$	Let $u=\frac{x^3}{2}$ and $\frac{\mathrm{d}v}{\mathrm{d}x}=\frac{2x}{\sqrt{1-x^4}}$ $\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{3x^2}{2}$ and $v=\sin^{-1}\left(x^2\right)$

3 (a) Do not use a calculator in answering this question.

The equation $z^2+(-5+2\mathrm{i})z+(21-\mathrm{i})=0$ has a root $z=3+\mathrm{i}\,c$, where c is a real constant. Find the value of c and hence find the second root of the equation in Cartesian form, $a+\mathrm{i}b$, showing your working. [5]

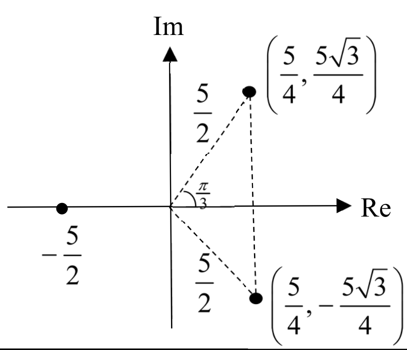
Since $3+\mathrm{i}\,c$ is a root, $(3+\mathrm{i}\,c)^2+(-5+2\mathrm{i})(3+\mathrm{i}\,c)+(21-\mathrm{i})=0$ $\left(9+6c\mathrm{i}-c^2\right)+(-15+6\mathrm{i}-5c\mathrm{i}-2c)+(21-\mathrm{i})=0$ $\left(15-2c-c^2\right)+\mathrm{i}(c+5)=0$ Comparing real and imaginary parts: $15-2c-c^2=0$ and $c+5=0$ $(c-3)(c+5)=0$ and $c+5=0$ $\therefore c=-5$ or $c=3$ and $c=-5$ Hence $c=-5$ Let $z=k$ be the second root. $z^2+(-5+2\mathrm{i})z+(21-\mathrm{i})=[z-(3-5\mathrm{i})][z-k]$ Comparing coefficients: z^0: $21-\mathrm{i}=(3-5\mathrm{i})k$ $k=\frac{21-\mathrm{i}}{3-5\mathrm{i}}\times\frac{3+5\mathrm{i}}{3+5\mathrm{i}}$ $=\frac{63+105\mathrm{i}-3\mathrm{i}+5}{9+25}$ $=2+3\mathrm{i}$ Hence the second root is $2+3\mathrm{i}$	Coefficients of the equation $z^2+(-5+2\mathrm{i})z+(21-\mathrm{i})=0$ are not all real. Hence, the conjugate $3-\mathrm{i}c$ is not a root. c has to satisfy both equations. Hence $c=3$ is rejected.

(b) The complex number z is such that $8z^3+125=0$.

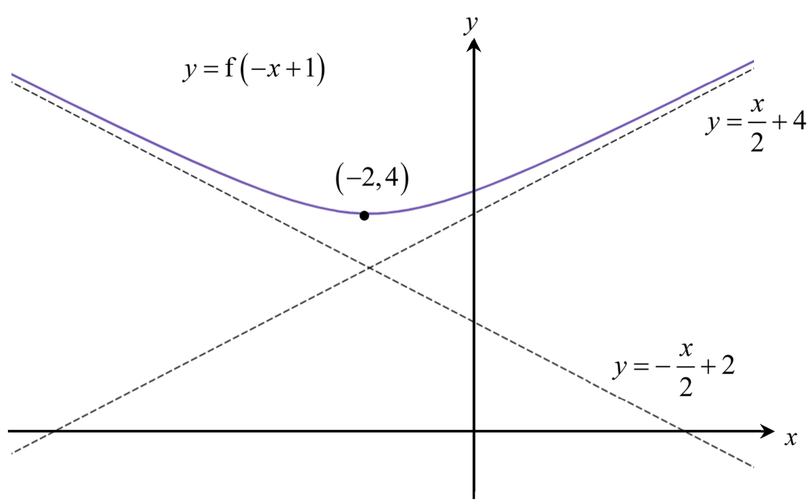
(i) Given that one possible value of z is $-\frac{5}{2}$, use a **non-calculator method** to find the other possible values of z. Give your answers in the form $a+\mathrm{i}b$, where a and b are exact values. [3]

$8z^3+125=0$ Since $z=-\frac{5}{2}$, then $2z+5$ is a factor. $8z^3+125=(2z+5)(Az^2+Bz+C)$ $=(2z+5)(4z^2+Bz+25)$ Comparing coefficient of z^2: $0=2B+20$ $\Rightarrow B=-10$ $\therefore (2z+5)(4z^2-10z+25)=0$ $z=-\frac{5}{2}$ or $z=\frac{10\pm\sqrt{100-4(4)(25)}}{8}$ $=\frac{10\pm\sqrt{-300}}{8}$ $=\frac{10\pm\sqrt{300}\,\mathrm{i}}{8}$ $=\frac{5\pm5\sqrt{3}\,\mathrm{i}}{4}$	By Fundamental Theorem of Algebra, $8z^3+125=0$ has 3 roots. To find all roots, we need to 1. Obtain the first root (in this case given by question), then write down the equivalent linear factor 2. The quadratic factor has to be found either through long division or comparing of coefficients. 3. Solve the quadratic equation $az^2+bz+c=0$ using quadratic formula i.e $z=\frac{-b\pm\sqrt{b^2-4(a)(c)}}{2a}$ **<u>GC CANNOT be used!</u>** There are many different methods to solve this question. Learn the most efficient method.

(ii) Write these values of z in modulus argument form and represent them on an Argand diagram. [4]

$z=-\frac{5}{2},\ \frac{5}{4}\pm\frac{5\sqrt{3}}{4}i$ $\sqrt{\left(\frac{5}{2}\right)^2+\left(\pm\frac{5\sqrt{3}}{4}\right)^2}=\frac{5}{2}$ $\tan\alpha=\sqrt{3}$ $\Rightarrow\alpha=\frac{\pi}{3}$ $\therefore$ The three roots are $z=\frac{5}{2}e^{\,i\pi},\ \frac{5}{2}e^{\,i\frac{\pi}{3}}$ and $\frac{5}{2}e^{-i\frac{\pi}{3}}$. Or $z=\frac{5}{2}(\cos\pi+i\sin\pi)$, $\frac{5}{2}\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$ and $\frac{5}{2}\left[\cos\left(-\frac{\pi}{3}\right)++i\sin\left(-\frac{\pi}{3}\right)\right]$ 	All the values of z need to be represented in one single Argand diagram. You need to give the values of z in modulus argument form.

4 The diagram below shows the graph of $y=f(-x+1)$ which has a minimum point at $(-2,4)$ and has lines $y=\frac{x}{2}+4$ and $y=-\frac{x}{2}+2$ as asymptotes.



(i) The diagram above shows a part of the curve with equation $\dfrac{(y+a)^2}{m}-\dfrac{(x+b)^2}{k}=1$. Find the values of a, b and k. [4]

$y=\dfrac{x}{2}+4 \quad ---(1)$

$y=-\dfrac{x}{2}+2 \quad ---(2)$

Solving (1) and (2): $x=-2$, $y=3$

Hence centre of the hyperbola is $(-2,3)$

$\therefore a=-3$ and $b=2$

$$\frac{(y-3)^2}{m}-\frac{(x+2)^2}{k}=1$$

Since $(-2,4)$ is a vertex on the hyperbola,

$3+\sqrt{m}=4$

$\Rightarrow m=1$

Asymptotes: $\dfrac{(y-3)^2}{1}=\dfrac{(x+2)^2}{k}$

$$(y-3)=\pm\left(\frac{1}{\sqrt{k}}\right)(x+2)$$

Comparing gradient: $\dfrac{1}{\sqrt{k}}=\dfrac{1}{2} \quad \Rightarrow k=4$

Given $\dfrac{(y+a)^2}{m}-\dfrac{(x+b)^2}{k}=1$

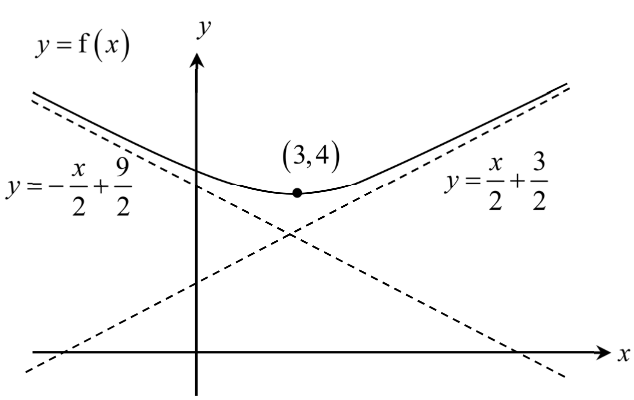
(1) Centre: $(-b,-a)$

(2) Vertices: $\left(-b,-a+\sqrt{m}\right)$ and $\left(-b,-a-\sqrt{m}\right)$

(3) Equations of asymptotes are

$$y=-a\pm\sqrt{\frac{m}{k}}(x+b)$$

(ii) Sketch the graph of $y=\mathrm{f}(x)$. [2]



$y=\mathrm{f}(-x+1)\xrightarrow{\text{replace } x \text{ with } -x} y=\mathrm{f}(x+1)$

Reflect the curve in the y-axis.

$y=\mathrm{f}(x+1)\xrightarrow{\text{replace } x \text{ with } x-1} y=\mathrm{f}(x)$

Translate 1 unit in the positive x-direction

Note that asymptotes will also undergo the same transformations as above. Hence need to find the equations of the new asymptotes.

E.g:

$$y=\frac{x}{2}+4\rightarrow y=\frac{-x}{2}+4\rightarrow y=\frac{-(x-1)}{2}+4$$

i.e $y=\dfrac{-x}{2}+\dfrac{9}{2}$

which is a line with negative gradient

5 Given that $f(\theta)=\tan\left(\frac{1}{2}\theta\right)$, show that $f'(\theta)=\frac{1}{2}\left\{1+\left[f(\theta)\right]^2\right\}$. [2]

$f(\theta)=\tan\left(\frac{1}{2}\theta\right)$ $f'(\theta)=\frac{1}{2}\sec^2\left(\frac{1}{2}\theta\right)$ $=\frac{1}{2}\left\{1+\tan^2\left(\frac{1}{2}\theta\right)\right\}$ $=\frac{1}{2}\left\{1+\left[f(\theta)\right]^2\right\}$	

By repeated differentiation, find the Maclaurin series for $f(\theta)$, up to and including the term in θ^3.

[4]

$f''(\theta)=\frac{1}{2}\left[2f(\theta)\times\frac{d}{d\theta}f(\theta)\right]$ $=f(\theta)f'(\theta)$	Remember chain rule.
$f'''(\theta)=f(\theta)\times\frac{d}{d\theta}f'(\theta)+f'(\theta)\times\frac{d}{d\theta}f(\theta)$ $=f(\theta)f''(\theta)+\left[f'(\theta)\right]^2$ When $\theta=0$, $f(0)=\tan\left(\frac{1}{2}\times0\right)=0$ $f'(0)=\frac{1}{2}\left(1+0^2\right)=\frac{1}{2}$ $f''(0)=(0)\left(\frac{1}{2}\right)=0$ $f'''(0)=\left(\frac{1}{2}\right)^2+0=\frac{1}{4}$ The Maclaurin series for $f(\theta)$ is $f(\theta)=0+\frac{1}{2}(\theta)+0\left(\frac{\theta^2}{2!}\right)+\frac{1}{4}\left(\frac{\theta^3}{3!}\right)+...$ $=\frac{1}{2}\theta+\frac{1}{24}\theta^3+...$	Easier to carry out implicit differentiation.

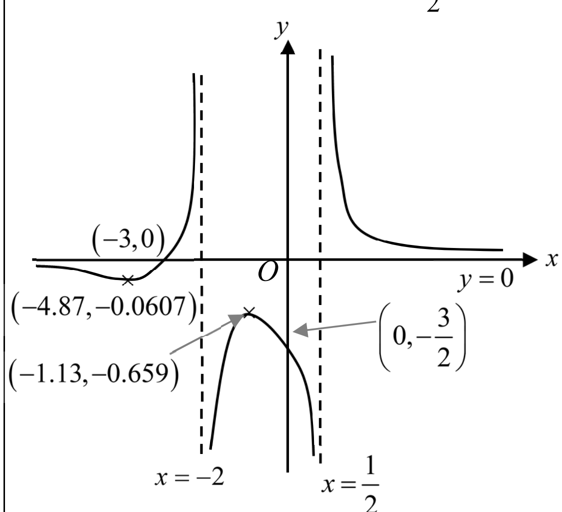
Explain why a Maclaurin series for $g(\theta)=\cot\left(\frac{1}{2}\theta\right)$ cannot be found. [1]

$g(0)=\cot(0)=\frac{1}{\tan(0)}$ is undefined. So a Maclaurin series for $g(\theta)$ cannot be found.	

The equation $\cot\left(\frac{\pi}{2}-\frac{1}{2}\theta\right)-\frac{5\sqrt{35}}{59}\theta=0$ has a positive root close to zero. Use the expansion above to obtain the approximate value of the root, correct to 4 decimal places. [3]

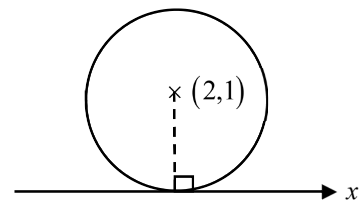
$\cot\left(\frac{\pi}{2}-\frac{1}{2}\theta\right)-\frac{5\sqrt{35}}{59}\theta=0$ $\Rightarrow \tan\left(\frac{1}{2}\theta\right)-\frac{5\sqrt{35}}{59}\theta=0$ $\Rightarrow \frac{1}{2}\theta+\frac{1}{24}\theta^3-\frac{5\sqrt{35}}{59}\theta=0$ $\Rightarrow \frac{1}{24}\theta^3+\left(\frac{1}{2}-\frac{5\sqrt{35}}{59}\right)\theta=0$ From the GC, $\theta=0.1808$ or -0.1808 or 0 For the positive root close to zero, $\theta=0.1808$ (to 4 d.p)	Since we are prompted to use expansion in previous part (which is expanded up to θ^3) hence small angle approximation cannot be used here. Recall the following identity: $\sin\left(\frac{\pi}{2}-x\right)=\cos x$ $\cos\left(\frac{\pi}{2}-x\right)=\sin x$ $\therefore \cot\left(\frac{\pi}{2}-x\right)=\tan x$

6 **(a)** Sketch the graph of $y=\frac{x+3}{(2x-1)(x+2)}$. Give the equations of the asymptotes, the coordinates of the point(s) where the curve crosses either axis and the coordinates of the two stationary points. [4]

$y=\frac{x+3}{(2x-1)(x+2)}$ Intercepts: $(-3,0)$ and $\left(0,-\frac{3}{2}\right)$ Asymptotes: $y=0$, $x=-2$ and $x=\frac{1}{2}$ 	When drawing graphs with asymptotes, we need to make sure that the graph tends gradually to the asymptotes. Question asks for: (a) Equations of asymptotes (b) Coordinates of stationary points (c) <u>Coordinates</u> of axial intercepts Hence (a), (b) and (c) must be clearly labelled in our graph.

(b) The curve C has equation $\dfrac{(3y-3)^2}{a^2}+\dfrac{(4-2x)^2}{b^2}=1$, where a and b are positive constants.

Given that C is a circle and the x-axis is a tangent to C, state the values of a and b. [2]

$\dfrac{(3y-3)^2}{a^2}+\dfrac{(4-2x)^2}{b^2}=1$ $\dfrac{3^2(y-1)^2}{a^2}+\dfrac{(-2)^2(x-2)^2}{b^2}=1$ Given C is a circle with centre at $(2,1)$. Since x-axis is a tangent to circle, radius = 1 Hence $a=3$, $b=2$	Recall circle property: tangent is perpendicular to radius 

Describe a sequence of transformations that transforms the graph of C to $\dfrac{(y-3)^2}{a^2}+\dfrac{(5-2x)^2}{b^2}=1$. [2]

$\dfrac{(3y-3)^2}{a^2}+\dfrac{(4-2x)^2}{b^2}=1$ Replace y with $\dfrac{1}{3}y$: $\dfrac{(y-3)^2}{a^2}+\dfrac{(4-2x)^2}{b^2}=1$ Replace x with $\left(x-\dfrac{1}{2}\right)$: $\dfrac{(y-3)^2}{a^2}+\dfrac{\left(4-2\left(x-\frac{1}{2}\right)\right)^2}{b^2}=1$ $\Rightarrow \dfrac{(y-3)^2}{a^2}+\dfrac{(5-2x)^2}{b^2}=1$ Description: (1) Stretch C parallel to y-axis by factor 3, with x-axis invariant, **then** (2) Translate resultant curve by $\dfrac{1}{2}$ units in the positive x direction.	

[Turn Over

7 A curve is such that $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{x}-\frac{1}{2\sqrt{x+8}}$ and P is a point that moves along the curve. The y-coordinate of P is decreasing at 0.3 units per second at $(1,0)$.

(i) Find the rate of decrease of the x-coordinate of P at $(1,0)$. [2]

At $(1,0)$, $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{1}-\frac{1}{2\sqrt{1+8}}=\frac{5}{6}$, $\frac{\mathrm{d}y}{\mathrm{d}t}=-0.3$ $\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{\mathrm{d}x}{\mathrm{d}y}\times\frac{\mathrm{d}y}{\mathrm{d}t}$ $=\frac{6}{5}\times(-0.3)$ $=-0.36$ $\therefore$ rate of decrease of the x-coordinate at P is 0.36 units per second.	"y-coordinate of P is **decreasing** at 0.3 units per second" means $\frac{\mathrm{d}y}{\mathrm{d}t}=-0.3$. Remember to include –ve sign.

(ii) Find the equation of the curve, leaving your answer in the form $y=\mathrm{f}(x)$. [3]

$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{x}-\frac{1}{2\sqrt{x+8}}$ $y=\int\frac{1}{x}-\frac{1}{2\sqrt{x+8}}\,\mathrm{d}x$ $=\ln\lvert x\rvert-\sqrt{x+8}+c$	$\int\frac{1}{x}\,\mathrm{d}x=\ln\lvert x\rvert+C$. You cannot drop the modulus sign.
(1, 0) is a point on the curve $\Rightarrow 0=\ln 1-3+c$ $\Rightarrow c=3$ $\therefore y=\ln\lvert x\rvert-\sqrt{x+8}+3$	Since (1, 0) is on the curve, it satisfies the equation of curve, You can use it to find C.

Mr Tan owns a company that manufactures and sells chipsets for smartphones. He models the average profit from one chipset, \$$y$, $y\geqslant 0$, when the selling price of each chipset is \$$x$, with the equation $y=\mathrm{f}(x)$.

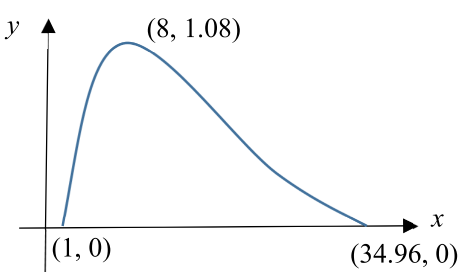
(iii) Find algebraically, the selling price of each chipset which gives the maximum average profit. [3]

$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{x}-\frac{1}{2\sqrt{x+8}}=0$ $\Rightarrow\frac{2\sqrt{x+8}-x}{2x\sqrt{x+8}}=0$ $\Rightarrow 2\sqrt{x+8}=x$ ----(*) $\Rightarrow 4x+32=x^2$ $\Rightarrow x^2-4x-32=0$ $\Rightarrow(x+4)(x-8)=0$ $\therefore x=8,\quad -4$ (reject since selling price is positive)	You are expected to solve this question "algebraically". You should either use factorisation or quadratic

$\frac{dy}{dx}=\frac{1}{x}-\frac{1}{2\sqrt{x+8}}$ $\Rightarrow \frac{d^2y}{dx^2}=-\frac{1}{x^2}+\frac{1}{4(x+8)^{\frac{3}{2}}}$ When $x=8$, $\frac{d^2y}{dx^2}=-0.0117$ (or $-\frac{3}{256}$) <0 $\therefore$ Selling price which gives maximum profit = $8. **OR** (see table below) $\therefore$ Selling price which gives maximum profit = $8.	formula to solve for x. You should reject x = -4. When choosing 2 values to do 1st derivative test, you must choose values which are close to 8. 7 and 9 are not considered close.

x	7.99	8	8.01
$\frac{dy}{dx}$	1.1736×10^{-4}	0	-1.1701×10^{-4}
Sign of $\frac{dy}{dx}$	+		-

(iv) Sketch the graph showing the average profit from one chipset as the selling price of each chip set varies. State a suitable range of values that Mr Tan should set for the selling price of a chipset. [2]

 $1\le x\le 34.96$	Note that $y\geqslant 0$. Do not include negative y portion in your sketch. You should label the maximum point (found in part (iii)) and the endpoints. Coordinates of these points can be found using GC. y value should be given in 2 decimal places.

8 **(i)** Show that $\frac{2x^2+5}{\left(x^2+1\right)^2}=\frac{A}{x^2+1}+\frac{B}{\left(x^2+1\right)^2}$, where A and B are constants to be determined. [1]

$\frac{2x^2+5}{\left(x^2+1\right)^2}=\frac{A}{x^2+1}+\frac{B}{\left(x^2+1\right)^2}$ $2x^2+5=A\left(x^2+1\right)+B$ Comparing coefficients $A=2$ and $B=3$	B1: A=2, B=3

(ii) Using the substitution $x = \tan u$, show that $\displaystyle\int \frac{1}{\left(x^2+1\right)^2}\ \mathrm{d}x = \frac{ax}{x^2+1} + b\tan^{-1} x + C$, where a and b are constants to be found. [5]

$x = \tan u \quad \Rightarrow \dfrac{\mathrm{d}x}{\mathrm{d}u} = \sec^2 u$ $\displaystyle\int \frac{1}{\left(x^2+1\right)^2}\ \mathrm{d}x = \int \frac{1}{\left(\tan^2 u+1\right)^2}\ \sec^2 u\ \mathrm{d}u$ $\displaystyle= \int \frac{1}{\sec^2 u}\ \mathrm{d}u$ $\displaystyle= \int \cos^2 u\ \mathrm{d}u$ $\displaystyle= \int \frac{\cos 2u + 1}{2}\ \mathrm{d}u$ $\displaystyle= \frac{1}{2}\left(\frac{\sin 2u}{2} + u\right) + C$ $\displaystyle= \frac{1}{2}\sin u \cos u + \frac{1}{2}u + C$ $\displaystyle= \frac{1}{2}\left(\frac{x}{\sqrt{x^2+1}}\right)\left(\frac{1}{\sqrt{x^2+1}}\right) + \frac{1}{2}\tan^{-1} x + C$ $\displaystyle= \frac{x}{2\left(x^2+1\right)} + \frac{1}{2}\tan^{-1} x + C$	Since $\tan u = \dfrac{x}{1} = \dfrac{opp}{adj}$, $\sin u = \dfrac{x}{\sqrt{x^2+1}}$ and $\cos u = \dfrac{1}{\sqrt{x^2+1}}$

(iii) Hence, find $\displaystyle\int \frac{2x^2+7x+5}{\left(x^2+1\right)^2}\ \mathrm{d}x$. [4]

$\displaystyle\int \frac{2x^2+7x+5}{\left(x^2+1\right)^2}\ \mathrm{d}x$ $\displaystyle= \int \frac{2x^2+5}{\left(x^2+1\right)^2}\ \mathrm{d}x + \int \frac{7x}{\left(x^2+1\right)^2}\ \mathrm{d}x$ $\displaystyle= \int \left(\frac{2}{x^2+1} + \frac{3}{\left(x^2+1\right)^2}\right)\ \mathrm{d}x + \frac{7}{2}\int 2x\left(x^2+1\right)^{-2}\ \mathrm{d}x$ $\displaystyle= 2\tan^{-1} x + 3\left(\frac{x}{2\left(x^2+1\right)} + \frac{1}{2}\tan^{-1} x\right) - \frac{7}{2\left(x^2+1\right)} + C$ $\displaystyle= \frac{7}{2}\tan^{-1} x + \frac{3x-7}{2\left(x^2+1\right)} + C$	Use standard form: $\displaystyle\int \mathrm{f}'(x)\left[\mathrm{f}(x)\right]^n\ \mathrm{d}x = \frac{1}{n+1}\left[\mathrm{f}(x)\right]^{n+1} + C$

9 A curve C has parametric equations

$$x = \frac{1}{t} + t \quad , \qquad y = \frac{1}{t} - t \,, \qquad \text{where } t \neq 0.$$

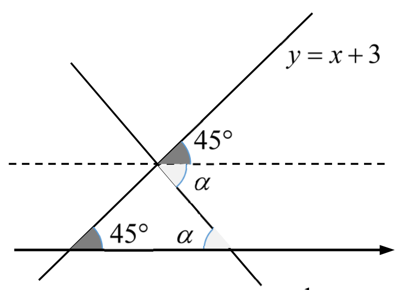
(i) Show that the gradient of the normal to the curve at $P\left(\frac{1}{p} + p, \frac{1}{p} - p\right)$ is given by $\frac{p^2 - 1}{p^2 + 1}$.

[2]

$x = \frac{1}{t} + t \quad , \qquad y = \frac{1}{t} - t$ $\frac{\mathrm{d}x}{\mathrm{d}t} = 1 - \frac{1}{t^2} = \frac{t^2 - 1}{t^2}$ $\frac{\mathrm{d}y}{\mathrm{d}t} = -1 - \frac{1}{t^2} = \frac{-t^2 - 1}{t^2}$ $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-t^2 - 1}{t^2} \times \frac{t^2}{t^2 - 1} = \frac{t^2 + 1}{1 - t^2}$ Hence at $P\left(\frac{1}{p} + p, \frac{1}{p} - p\right)$, $t = p$ Gradient of normal $= -\frac{1}{\frac{p^2 + 1}{1 - p^2}} = \frac{p^2 - 1}{p^2 + 1}$	This is a "show" question. You are expected to show working clearly.

(ii) Given that $p = \frac{1}{\sqrt{3}}$, determine the acute angle between the normal and the line $y = x + 3$.

[3]

When $p = \frac{1}{\sqrt{3}}$, gradient of normal $= \frac{\frac{1}{3} - 1}{\frac{1}{3} + 1} = -\frac{1}{2}$  $\alpha = \tan^{-1} \frac{1}{2}$ $= 26.565°$ Thus, acute angle between the 2 lines is $45° + 26.565° = 71.6°$ (1 d.p)	Note you are required to give the **acute** angle between the 2 lines.

(iii) Point Q on C has parameter q. Show that the gradient of PQ is $\dfrac{1+pq}{1-pq}$. [2]

$x=\dfrac{1}{t}+t \quad , \qquad y=\dfrac{1}{t}-t$ $P\left(\dfrac{1}{p}+p, \dfrac{1}{p}-p\right)$ $Q\left(\dfrac{1}{q}+q, \dfrac{1}{q}-q\right)$ Gradient of PQ $=\left(\dfrac{1}{p}-p-\dfrac{1}{q}+q\right)\div\left(\dfrac{1}{p}+p-\dfrac{1}{q}-q\right)$ $=\dfrac{q-p^2q-p+pq^2}{pq}\times\dfrac{pq}{q+p^2q-p-pq^2}$ $=\dfrac{(q-p)+pq(q-p)}{(q-p)-pq(q-p)}$ $=\dfrac{1+pq}{1-pq}$	This is a "show" question. You are expected to show working clearly.

(iv) The normal at P cuts the curve again at point R with parameter r. Show that $r=-\dfrac{1}{p^3}$. [3]

The normal at P cuts the curve again at point R, Hence gradient of normal at P = gradient of PR $\dfrac{p^2-1}{p^2+1}=\dfrac{1+pr}{1-pr}$ $(p^2-1)(1-pr)=(p^2+1)(1+pr)$ $p^2-1-p^3r+pr=p^2+1+p^3r+pr$ $2p^3r=-2$ $r=-\dfrac{1}{p^3}$	Note that $p\neq\dfrac{1}{\sqrt{3}}$ here.

10 Crypto currency Miners are rewarded certain amounts of the currency at varying time intervals when they carry out validation tasks of transactions using their personal computers.

A crypto currency, Shockcoin (SHC), has a reward payment schedule such that the first reward payout is 20 days after a Miner commences mining and the second reward payout is 28 days after the first. The duration of each subsequent payout is 8 days more than the duration between the two preceding payouts. For example, if the duration between the last and current payout is 100 days, then it would take 108 days to the next reward payout. Albert plans to commence mining on 1 January 2022.

(i) After receiving his first reward payout on 21 January 2022, on what day and month will he receive his second reward payout? [1]

18 February 2022	There are 31 days in the month of January.

(ii) Show that from the start of mining, the number of days Albert takes to receive his n^{th} reward payout is given by $4n^2+16n$. [2]

Number of days to n^{th} payout $=\frac{n}{2}\left[2(20)+(n-1)(8)\right]=4n^2+16n$	

[You may assume that there are 365 days in a year]

(iii) Find the number of reward payouts Albert have received as at 31 December 2023. [2]

$4n^2+16n=730$ $n=-15.66$ (reject) or $n=11.66$ Albert would have received 11 reward payouts by 31 December 2023.	From 1 January 2022 (i.e. start of mining) to 31 December 2023 is $365\times 2=730$ days.

The rewards for Miners are paid in Shockcoin (SHC).

The first reward for a Miner is SHC 120 000. For all subsequent reward payouts, the reward amount is 85% of the previous payout.

[Give non-exact numerical answers correct to 2 decimal places.]

(iv) Find the amount, in SHC Albert is expected to receive in his last payout in 2023. [2]

Reward for last payment in 2023 $=(120000)\left(0.85^{10}\right)$ $=23624.93$	Give non-exact numerical answers correct to 2 decimal places.

(v) Write an expression for the total amount, in SHC, Albert would have received after the n^{th} reward payout. [2]

Total reward received after the n^{th} payout $=\dfrac{120000(1-0.85^n)}{1-0.85}$ $=800000(1-0.85^n)$	

It is estimated that it would cost Albert 25 SHC per day for the electricity to keep the computer on to mine SHC.

(vi) Show that the total net gain for Albert immediately after receiving n reward payouts is given by $a\left(8000\left(1-0.85^n\right)-bn^2-cn\right)$ where a, b and c are constants to be found. [2]

Total net gain after receiving n reward payout $=800000\left(1-0.85^n\right)-25(4n^2+16n)$ $=100\left(8000\left(1-0.85^n\right)-n^2-4n\right)$	

(v) In order to maximise his total net gain, Albert should stop mining after the kth payout. Find the value of k and the maximum total net gain. [2]

n	$100\left(8000\left(1-0.85^n\right)-n^2-4n\right)$
20	720992.38
21	721143.52
22	720396.99

Quit mining after collecting the 21st reward payout so as maximise his total net gain.

max net gain = SHC 721143.52

Give non-exact numerical answers correct to 2 decimal places.

11 **(i)** The complex number w can be expressed as $\mathrm{e}^{\mathrm{i}\theta}$.

(a) Find $w^n+\dfrac{1}{w^n}$ and $w^n-\dfrac{1}{w^n}$ in simplified trigonometric form. [3]

$w=\mathrm{e}^{i\theta} \Rightarrow w^n=\mathrm{e}^{in\theta}, \dfrac{1}{w^n}=\mathrm{e}^{-in\theta}$

$$\begin{aligned}
w^n+\frac{1}{w^n} &= \mathrm{e}^{in\theta}+\mathrm{e}^{-in\theta}\\
&= \cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta)\\
&= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta\\
&= 2\cos n\theta
\end{aligned}$$

$$\begin{aligned}
w^n-\frac{1}{w^n} &= \mathrm{e}^{in\theta}-\mathrm{e}^{-in\theta}\\
&= \cos n\theta + i\sin n\theta - \cos(-n\theta) - i\sin(-n\theta)\\
&= \cos n\theta + i\sin n\theta - \cos n\theta + i\sin n\theta\\
&= 2i\sin n\theta
\end{aligned}$$

(b) By considering the binomial expansion of $\left[\left(w+\frac{1}{w}\right)\left(w-\frac{1}{w}\right)\right]^3$ and the results from **part (a)**, show that $\sin^3\theta\cos^3\theta = \frac{3}{32}\sin 2\theta - \frac{1}{32}\sin 6\theta$. [4]

$\left[\left(w+\frac{1}{w}\right)\left(w-\frac{1}{w}\right)\right]^3$ $=\left(w^2-\frac{1}{w^2}\right)^3$ $=\left(w^2\right)^3+3\left(w^2\right)^2\left(-\frac{1}{w^2}\right)+3\left(w^2\right)\left(-\frac{1}{w^2}\right)^2+\left(-\frac{1}{w^2}\right)^3$ $=w^6-\frac{1}{w^6}-3\left(w^2-\frac{1}{w^2}\right)$ $=2i\sin 6\theta-6i\sin 2\theta$ $\left[\left(w+\frac{1}{w}\right)\left(w-\frac{1}{w}\right)\right]^3=\left[(2\cos\theta)(2i\sin\theta)\right]^3$ $=-64i\sin^3\theta\cos^3\theta$ $\therefore 2i\sin 6\theta-6i\sin 2\theta=-64i\sin^3\theta\cos^3\theta$ $\Rightarrow \sin^3\theta\cos^3\theta=\frac{2i}{-64i}\sin 6\theta-\frac{6i}{-64i}\sin 2\theta$ $\Rightarrow \sin^3\theta\cos^3\theta=\frac{3}{32}\sin 2\theta-\frac{1}{32}\sin 6\theta$ (shown)	It is easier to expand $\left(w^2-\frac{1}{w^2}\right)^3$ than $\left(w+\frac{1}{w}\right)^3\times\left(w-\frac{1}{w}\right)^3$.

(ii) Hence find the first two non-zero terms of the Maclaurin series for $\sin^3\theta\cos^3\theta$. [2]

$\sin^3\theta\cos^3\theta$ $=\frac{3}{32}\sin 2\theta-\frac{1}{32}\sin 6\theta$ $=\frac{3}{32}\left(2\theta-\frac{(2\theta)^3}{3!}+\frac{(2\theta)^5}{5!}+\ldots\right)-\frac{1}{32}\left(6\theta-\frac{(6\theta)^3}{3!}+\frac{(6\theta)^5}{5!}+\ldots\right)$ $=-\frac{1}{8}\theta^3+\frac{9}{8}\theta^3+\frac{1}{40}\theta^5-\frac{81}{40}\theta^5+\ldots$ $=\theta^3-2\theta^5+\ldots$	It is much more efficient to use standard series of $\sin\theta$ than maclaurin series formula with repeated differentiation.