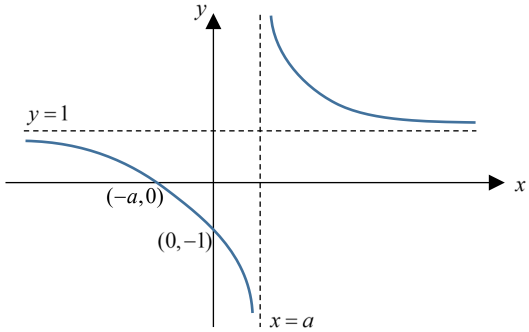
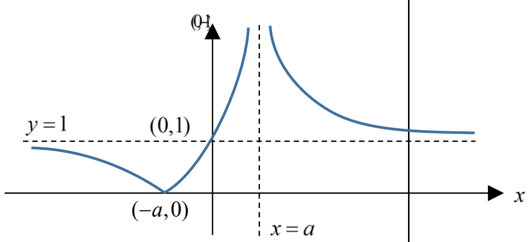
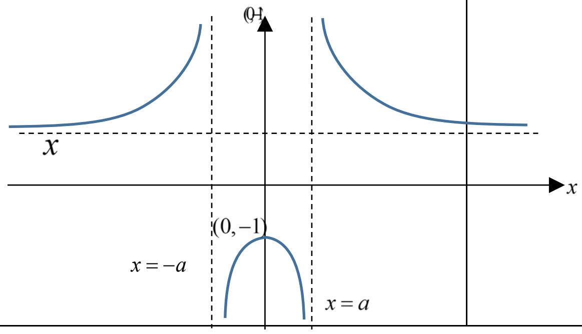
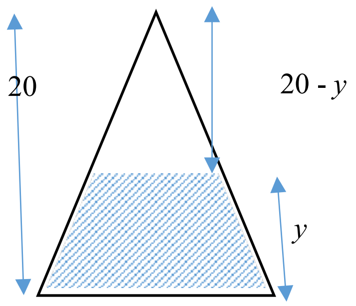
2021 JC2 H2 Mathematics Prelim P1 Solutions

1	$u=\sqrt{x}$ $\dfrac{\mathrm{d}u}{\mathrm{d}x}=\dfrac{1}{2\sqrt{x}}$ when $x=9,\ u=3$ when $x=1,\ u=1$ $\displaystyle\int_1^9 \frac{1}{x\left(2\sqrt{x}-1\right)}\,\mathrm{d}x=\int_1^3 \frac{1}{u^2\left(2u-1\right)}\,2u\ \mathrm{d}u$ $\displaystyle=\int_1^3 \frac{2}{u\left(2u-1\right)}\ \mathrm{d}u$ $\displaystyle=\int_1^3 \frac{-2}{u}+\frac{4}{2u-1}\ \mathrm{d}u$ $=\left[-2\ln\lvert u\rvert+2\ln\lvert 2u-1\rvert\right]_1^3$ $=-2\ln 3+2\ln 5$ $=2\ln\left(\dfrac{5}{3}\right)$	
2(i)		
2(ii) (a)	From the graph, $0<x<a$ or $x>a$ 	
2(ii) (b)	From the graph, $x<-a$ or $x>a$ 	

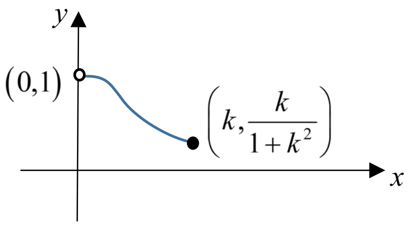
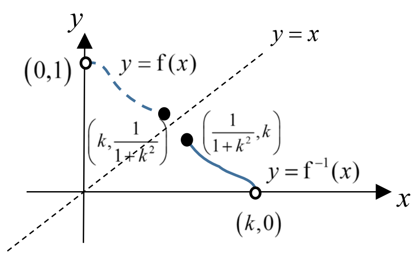
3(i)	$\frac{1}{(r-2)!}-\frac{1}{r!}=\frac{r(r-1)-1}{r!}=\frac{r^2-r-1}{r!}$ $\sum_{r=1}^{n}\frac{r^2-r-1}{r!}=\sum_{r=2}^{n}\left(\frac{1}{(r-2)!}-\frac{1}{r!}\right)+\frac{1-1-1}{1!}$ $=\Big(\frac{1}{0!}-\cancel{\frac{1}{2!}}$ $+\frac{1}{1!}-\cancel{\frac{1}{3!}}$ $+\cancel{\frac{1}{2!}}-\cancel{\frac{1}{4!}}$ $+\ldots$ $+\cancel{\frac{1}{(n-4)!}}-\cancel{\frac{1}{(n-2)!}}$ $+\cancel{\frac{1}{(n-3)!}}-\frac{1}{(n-1)!}$ $+\cancel{\frac{1}{(n-2)!}}-\frac{1}{n!}\Big)\quad -1$ $=1+1-\frac{1}{(n-1)!}-\frac{1}{n!}-1$ $=1-\frac{1}{(n-1)!}-\frac{1}{n!}$	
3(ii)	As $n\to\infty$, $-\frac{1}{(n-1)!}\to 0$ and $-\frac{1}{n!}\to 0$ $\therefore \sum_{r=1}^{n}\frac{r^2-r-1}{r!}\to 1$, a finite value $\therefore \sum_{r=1}^{\infty}\frac{r^2-r-1}{r!}$ converges sum to infinity $=\sum_{r=1}^{\infty}\frac{r^2-r-1}{r!}=1$	
3(iii)	$\sum_{r=1}^{n}\frac{r-2}{(r-1)!}=\sum_{r=1}^{n}\frac{r^2-2r}{r!}=\sum_{r=1}^{n}\frac{r^2-r-r}{r!}$ $<\sum_{r=1}^{n}\frac{r^2-r-1}{r!}\quad\left(\because \frac{r}{r!}>\frac{1}{r!}\text{ for } r>1\right)$ $=1-\frac{1}{(n-1)!}-\frac{1}{n!}<1$	

Note to tutors:
In recent years, A level phrasing of 3(ii) is "Give a reason why the series converges and write down the value of the sum to infinity". With such phrasing, Cambridge expects students to write "sum to infinity = ___" (See report below).
N07/II/2

(iii) Adequate answers to this part were rare. Many candidates seemed to be unable to think of convergence of series that were not geometric; answers such as 'it converges because $|r|<1$' were common. Nor is it correct to say that it converges because the terms decrease, as is disproved by the harmonic series. The correct statements are that $\frac{1}{(N+1)^2}$ tends to 0 (*not* '$\frac{1}{\infty}$')

4	Let the volume of water in the cone be V cm^3 at time t seconds, when the depth is y cm.  $\tan\theta = \frac{8}{h} \Rightarrow h = 20$ $V = \frac{1}{3}\pi(8^2)(20) - \frac{1}{3}\pi(x^2)(20-y),$ Since $\tan\theta = \frac{x}{20-y},$ $V = \frac{1280\pi}{3} - \frac{1}{3}\pi\left[0.4(20-y)\right]^2(20-y)$ $= \frac{1280\pi}{3} - \frac{0.16}{3}\pi(20-y)^3$ $\frac{d}{dt}: \quad \frac{dV}{dt} = 0.16\pi(20-y)^2\frac{dy}{dt}$ When $y = 10,\ \frac{dV}{dt} = 8,$ $8 = 0.16\pi(10)^2\frac{dy}{dt}$ $\frac{dy}{dt} = 0.159$ Depth of water increases at 0.159 cm per second.	
5(i)	$\mathbf{a}+\mathbf{b}+\mathbf{c}=\mathbf{0}$ $(\mathbf{a}+\mathbf{b}+\mathbf{c})\times\mathbf{b}=\mathbf{0}\times\mathbf{b}$ $\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{b}+\mathbf{c}\times\mathbf{b}=\mathbf{0}$ $\mathbf{a}\times\mathbf{b}+\mathbf{0}+\mathbf{c}\times\mathbf{b}=\mathbf{0}$ $\mathbf{a}\times\mathbf{b}=-(\mathbf{c}\times\mathbf{b})$ $=\mathbf{b}\times\mathbf{c}$	
5(ii)	A, B and C are collinear (lie on a straight line).	

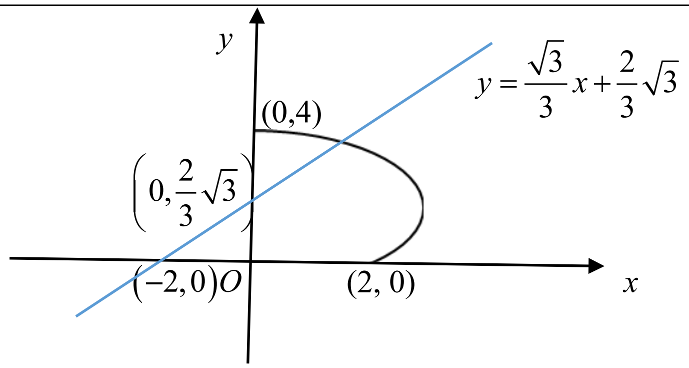
5(iii)	$\mathbf{a}\times\mathbf{b}=\mathbf{b}\times\mathbf{c}$ (from part (i)) $\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\sin\angle AOB=\lvert\mathbf{b}\rvert\lvert\mathbf{c}\rvert\sin\angle BOC$ $\lvert\mathbf{a}\rvert\sin\angle AOB=\lvert\mathbf{c}\rvert\sin(2\angle AOB)$ $\quad(\text{since } \lvert\mathbf{b}\rvert\neq 0)$ $=2\lvert\mathbf{c}\rvert\sin\angle AOB\cos\angle AOB$ $\lvert\mathbf{a}\rvert=2\lvert\mathbf{c}\rvert\cos\angle AOB$ $(\text{since } \sin\angle AOB\neq 0)$ $0=\lvert\mathbf{a}\rvert^2-4\lvert\mathbf{c}\rvert^2\cos^2\angle AOB$ $\lvert\mathbf{a}\rvert^2=4\lvert\mathbf{c}\rvert^2\cos^2\angle AOB$ $(\mathbf{a}+2\mathbf{c}).(\mathbf{a}-2\mathbf{c})$ $=\mathbf{a}.\mathbf{a}+(2\mathbf{c}).\mathbf{a}-\mathbf{a}.(2\mathbf{c})+(2\mathbf{c}).(2\mathbf{c})$ $=\lvert\mathbf{a}\rvert^2-4\lvert\mathbf{c}\rvert^2$ $=4\lvert\mathbf{c}\rvert^2\cos^2\angle AOB-4\lvert\mathbf{c}\rvert^2$ $=4\lvert\mathbf{c}\rvert^2(\cos^2\angle AOB-1)$ $=4\lvert\mathbf{c}\rvert^2(-\sin^2\angle AOB)<0$ $(\because \lvert\mathbf{c}\rvert^2\geq 0 \text{ and} \sin^2\angle AOB>0$	
5(iv)	$k=0$ since the origin O is in the plane p.	
6(a) (i)	$A=1, B=0 \Rightarrow u_{n+1}=u_n+C \Rightarrow$ AP $\sum_{r=11}^{30}u_r=\frac{20}{2}(u_{11}+u_{30})$ $=10(4+10C+4+29C)$ $=80+390C$	
(ii)	GP $\Rightarrow u_{n+1}=Au_n$ for all values of n $\therefore B=0, C=0$ $U_{20}>2000 \Rightarrow 4(A)^{19}>2000$ $\Rightarrow A>500^{\frac{1}{19}}$ $\Rightarrow A>1.39$	
(b)	$u_{n+1}=Au_n+Bn+C$ $n=1: \quad 16=4A+B+C$ $n=2: \quad 70=16A+2B+C$ $n=3: \quad 334=70A+3B+C$ From GC, $A=5,\ B=-6, C=2$ $\therefore u_5=5u_4-6(4)+2=1648$	

7(i)	 **Every** horizontal line $y = k$ cuts the graph of $y = \mathrm{f}(x)$ at most **once**. Hence **f is one-one** and the inverse of f exists.	
7 (ii)	$y = \mathrm{f}(x) = \dfrac{1}{1+x^2}$ $\dfrac{1}{y} = 1 + x^2$ $x^2 = \dfrac{1}{y} - 1$ $x = \sqrt{\dfrac{1}{y} - 1}$ $\quad \because$ reject $-\sqrt{\dfrac{1}{y} - 1}$ since $x > 0$ $\mathrm{f}^{-1}(x) = \sqrt{\dfrac{1}{x} - 1}$ domain of $\mathrm{f}^{-1} = \left[\dfrac{1}{1+k^2}, 1\right)$	
7 (iii)		
7(iv)	Since $y = \mathrm{f}^{-1}(x)$ do not intersect the line $y = x$, it will not intersect $y = \mathrm{f}(x)$ too, hence no solutions for $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$. **Alternative:** Since the domains of f and f^{-1} are different, there is no solution for $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$	

8(a)(i)	$\arg(z) = \arg\left(1 - \mathrm{i}\sqrt{3}\right) + \arg(w)$ $= -\dfrac{\pi}{3} + \theta$ $\lvert z\rvert = \left\lvert 1 - \mathrm{i}\sqrt{3}\right\rvert \lvert w\rvert$ $= 2r$
8(a)(ii)	$\arg\left(\dfrac{z^2}{z^*}\right)^2 = 4\arg(z) - 2\arg\left(z^*\right)$ $= 4\arg(z) + 2\arg\left(z\right)$ $= 6\theta + 2k\pi \quad k \in \mathbb{Z}$ $\arg\left(36\mathrm{i}\right) = \dfrac{\pi}{2}$ $\lvert 36\mathrm{i}\rvert = 36$ $\left\lvert\left(\dfrac{z^2}{z^*}\right)^2\right\rvert = \dfrac{\lvert z\rvert^4}{\lvert z\rvert^2}$ $= 4r^2$ Hence, $4r^2 = 36 \Rightarrow r = 3$. $6\theta + 2k\pi = \dfrac{\pi}{2}$ $\theta = \dfrac{\pi - 4k\pi}{12}$ For $0 \le \theta \le \dfrac{\pi}{2}$, we have $k = -1,\ 0$, giving us $\theta = \dfrac{5\pi}{12}, \dfrac{\pi}{12}$
8(b)	Since p is a real value, by conjugate root theorem, $2 - q\mathrm{i}$ is also a root. $\left(z - \left(2 + q\mathrm{i}\right)\right)\left(z - \left(2 - q\mathrm{i}\right)\right) = z^2 - 4z + 4 + q^2$ Let the third root be v. We know that v is real. $z^3 + pz^2 + 17z - 13 = \left(z^2 - 4z + 4 + q^2\right)\left(z - v\right)$ Coefficient of z^0: $-(4 + q^2)v = -13$ ----(1) Coefficient of z: $4v + 4 + q^2 = 17 \Rightarrow q^2 = 13 - 4v$ ---(2) Sub (2) into (1), we have $\left(17 - 4v\right)v = 13$ $4v^2 - 17v + 13 = 0$ $(4v - 13)(v - 1) = 0$ $v = \dfrac{13}{4}$ or $v = 1$ However, if $v = \dfrac{13}{4}$, sub into (2), gives us $q = 0$. Hence, $v \neq \dfrac{13}{4}$ and $v = 1$. $\therefore z^3 + pz^2 + 17z - 13 = \left(z^2 - 4z + 4 + q^2\right)\left(z - 1\right)$ $z^3 + pz^2 + 17z - 13 = z^3 - 5z^2 + (q^2 + 4)z - 4 - q^2$ Hence, $p = -5$ and $q = 3$.
9 (i)	Angle $AFE = 180^\circ - 60^\circ = 120^\circ$ Area of sector FEA = $\dfrac{120}{360}\pi x^2 = \dfrac{1}{3}\pi x^2$

9 (ii)	total area = $\frac{1}{2}x^2\sin 60^\circ + BC(2x) + \frac{1}{3}\pi x^2$ $2x(BC) = 1000 - \frac{\sqrt{3}}{4}x^2 - \frac{1}{3}\pi x^2$ $BC = \frac{500}{x} - \frac{\sqrt{3}}{8}x - \frac{\pi}{6}x$ $P = x + 2\left(\frac{500}{x} - \frac{\sqrt{3}}{8}x - \frac{\pi}{6}x\right) + 2x + \frac{1}{3}(2\pi x)$ $= \frac{1000}{x} + \frac{x}{12}\left(4\pi + 36 - 3\sqrt{3}\right)$
9 (iii)	$\frac{\mathrm{d}P}{\mathrm{d}x} = \frac{-1000}{x^2} + \frac{1}{12}\left(36 - 3\sqrt{3} + 4\pi\right)$ $\frac{\mathrm{d}P}{\mathrm{d}x} = 0 \Rightarrow \frac{1000}{x^2} = \frac{1}{12}\left(36 - 3\sqrt{3} + 4\pi\right)$ $x = 16.634$ $P = 120.2$ $\frac{\mathrm{d}^2P}{\mathrm{d}x^2} = \frac{2000}{x^3}$ When $x = 16.634$, $\frac{\mathrm{d}^2P}{\mathrm{d}x^2} = \frac{2000}{16.634^3} > 0$ $\therefore$ minimum $P = 120.2$
10(i)	$\int \frac{1}{2 + x - x^2}\,\mathrm{d}x$ $= \int \frac{1}{\frac{9}{4} - \left(x - \frac{1}{2}\right)^2}\,\mathrm{d}x$ $= \frac{1}{2\left(\frac{3}{2}\right)} \ln\left\lvert \frac{\frac{3}{2} + \left(x - \frac{1}{2}\right)}{\frac{3}{2} - \left(x - \frac{1}{2}\right)} \right\rvert + C$ $= \frac{1}{3}\ln\left(\frac{1+x}{2-x}\right) + C \quad \because x \le \frac{1}{2}$

10(ii) (a)	Given $x=\frac{1}{2},\ \frac{\mathrm{d}x}{\mathrm{d}t}=-\frac{1}{4}$ $\therefore -\frac{1}{9}=k\left(2+\frac{1}{2}-\frac{1}{4}\right)$ $\Rightarrow \frac{9}{4}k=-\frac{1}{4}\Rightarrow k=-\frac{1}{9}$ $\frac{\mathrm{d}x}{\mathrm{d}t}=-\frac{1}{9}(2+x-x^2)$ $\int \frac{1}{2+x-x^2}\,\mathrm{d}x=\int -\frac{1}{9}\,\mathrm{d}t$ $-\frac{1}{9}t=\frac{1}{3}\ln\left(\frac{1+x}{2-x}\right)+C$ $t=-3\ln\left(\frac{1+x}{2-x}\right)+C$ When $t=0,\ x=\frac{1}{2},\ \therefore 0=-3\ln 1+C\Rightarrow C=0$ $t=-3\ln\left(\frac{1+x}{2-x}\right)$
10(b) (ii)	At $x=0,\ t=-3\ln\left(\frac{1}{2}\right)=3\ln 2$
10(b) (iv)	$\frac{\mathrm{d}^2y}{\mathrm{d}t^2}=\frac{1}{t+1}$ $\frac{\mathrm{d}y}{\mathrm{d}t}=\ln\lvert t+1\rvert+C$ $=\ln(t+1)+C\quad \because t+1\geq 1>0$ When $t=0,\ \frac{\mathrm{d}y}{\mathrm{d}t}=0\Rightarrow 0=\ln(0+1)+C\Rightarrow C=0$ $\frac{\mathrm{d}y}{\mathrm{d}t}=\ln(t+1)$ $y=\int \ln(t+1)\ \mathrm{d}t$ $=t\ln(t+1)-\int \frac{t}{t+1}\ \mathrm{d}t$ $=t\ln(t+1)-\int 1-\frac{1}{t+1}\ \mathrm{d}t$ $=t\ln(t+1)-t+\ln(t+1)+D$ When $t=0,\ y=1,\ D=0$ Hence, $y=(t+1)\ln(t+1)-t$

11(i)	(i) $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{3\cos t-3\cos 3t}{-3\sin t+3\sin 3t}$ When $t=\dfrac{\pi}{3}, x=\dfrac{5}{2}, y=\dfrac{3\sqrt{3}}{2}, \dfrac{\mathrm{d}y}{\mathrm{d}x}=-\sqrt{3}$ Equation of normal: $y-\dfrac{3\sqrt{3}}{2}=\dfrac{\sqrt{3}}{3}\left(x-\dfrac{5}{2}\right)\Rightarrow y=\dfrac{\sqrt{3}}{3}x+\dfrac{2}{3}\sqrt{3}$
11(ii)	(ii) 

11 (iii)

(iii) Point of intersection $\left(\frac{5}{2}, \frac{3\sqrt{3}}{2}\right)$

Area

$$= \int_0^{5/2} y \, \mathrm{d}x - \text{area of trapizum}$$

$$= \int_{\pi/2}^{\pi/3} \left(3\sin t - \sin(3t)\right) 3\left(-\sin t + \sin 3t\right) \, \mathrm{d}t - \frac{1}{2}\left(\frac{5}{2}\right)\left(\frac{2}{3}\sqrt{3} + \frac{3}{2}\sqrt{3}\right)$$

$$= 3\int_{\pi/2}^{\pi/3} \left(-3\sin^2 t + 4\sin t \sin(3t) - \sin^2(3t)\right) \, \mathrm{d}t - \frac{1}{2}\left(\frac{5}{2}\right)\left(\frac{13}{6}\sqrt{3}\right)$$

$$= 3\int_{\pi/2}^{\pi/3} \left(3\left(\frac{\cos 2t - 1}{2}\right) - 2\left(\cos 4t - \cos 2t\right) + \frac{\cos 6t - 1}{2}\right) \, \mathrm{d}t - \frac{65}{24}\sqrt{3}$$

$$= 3\int_{\pi/2}^{\pi/3} \left(-2 + \frac{7}{2}\cos 2t - 2\cos 4t + \frac{\cos 6t}{2}\right) \, \mathrm{d}t - \frac{65}{24}\sqrt{3}$$

$$= 3\left[-2t + \frac{7}{4}\sin 2t - \frac{2\sin 4t}{4} + \frac{\sin 6t}{12}\right]_{\pi/2}^{\pi/3} - \frac{65}{24}\sqrt{3}$$

$$= 3\left[-\frac{2\pi}{3} + \frac{7}{4}\left(\frac{\sqrt{3}}{2}\right) + \frac{\sqrt{3}}{4} - (-\pi)\right] - \frac{65}{24}\sqrt{3}$$

$$= \pi + \frac{2}{3}\sqrt{3}$$

or

Area

$$= \int_{3\sqrt{3}/2}^{4} x \, \mathrm{d}y + \text{area of triangle}$$

$$= \int_{\pi/3}^{\pi/2} \left(3\cos t - \cos(3t)\right) 3\left(\cos t - \cos 3t\right) \, \mathrm{d}t + \frac{1}{2}\left(\frac{5}{2}\right)\left(\frac{3}{3}\sqrt{3} - \frac{2}{3}\sqrt{3}\right)$$

$$= 3\int_{\pi/3}^{\pi/2} \left(3\cos^2 t - 4\cos t \cos(3t) + 3\cos^2(3t)\right) \, \mathrm{d}t + \frac{25}{24}\sqrt{3}$$

$$= 3\int_{\pi/3}^{\pi/2} \left(3\left(\frac{\cos 2t + 1}{2}\right) - 2\left(\cos 4t + \cos 2t\right) + \frac{\cos 6t + 1}{2}\right) \, \mathrm{d}t + \frac{25}{24}\sqrt{3}$$

$$= 3\int_{\pi/3}^{\pi/2} \left(2 - \frac{1}{2}\cos 2t - 2\cos 4t + \frac{\cos 6t}{2}\right) \, \mathrm{d}t + \frac{25}{24}\sqrt{3}$$

$$= 3\left[2t - \frac{1}{4}\sin 2t - \frac{2\sin 4t}{4} + \frac{\sin 6t}{12}\right]_{\pi/3}^{\pi/2} + \frac{25}{24}\sqrt{3}$$

$$= 3\left[\pi - \left(\frac{2\pi}{3} - \frac{1}{4}\left(\frac{\sqrt{3}}{2}\right) + \frac{\sqrt{3}}{4}\right)\right] + \frac{25}{24}\sqrt{3}$$

$$= \pi + \frac{2}{3}\sqrt{3}$$