Promo Practice Paper 5 [TMJC 2021] Solutions

Qn	Solution
1	**RI JC1 Promo 9758/2019/Q10**
	$zz^* + w^2 = 4 - 2\mathrm{i}$ $(-1+\mathrm{i}a)(-1-\mathrm{i}a) + (b+\mathrm{i})^2 = 4 - 2\mathrm{i}$ $1 + a^2 + b^2 + 2b\mathrm{i} - 1 = 4 - 2\mathrm{i}$ Comparing real and imaginary parts, $a^2 + b^2 = 4$ and $2b = -2 \Rightarrow b = -1$ $a^2 = 3$ $a = \pm\sqrt{3}$
2	**System of Linear Equations**
(i)	Let x, y and z be the price per kg for Mao Shan Wang, Red Prawn and D24 durians respectively. $4.5x + 4y + 4z = 218.5$ $3x + 4.5y + 2z = 162.5$ $6x + 3y + 3z = 219$ Using GC, $x = 21$ $y = 15$ $z = 16$ The total amount paid for Group D is 7.5(21) + 3(15) + 6(16) = $298.50. Hence k = 298.50
(ii)	Possible answers: • The average expenditure for each tourist is $44.93, which is less than that at the new durian shop. • The total expenditure for the four groups is $898.50. Total expenditure will be higher at $1000 if he engages the new durian shop. • The average cost for each group is $224.63, while the cost for the new durian shop is higher at $250 per group.

Qn	Solution
3	**Differentiation**
	$y = 3x^2 - k^2 \ln\left(\frac{x}{4}\right)$ $\frac{\mathrm{d}y}{\mathrm{d}x} = 6x - k^2\left(\frac{4}{x}\right)\left(\frac{1}{4}\right) = 6x - \frac{k^2}{x}$ $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ $6x - \frac{k^2}{x} = 0$ $6x = \frac{k^2}{x}$ $x^2 = \frac{k^2}{6}$ $x = \pm\frac{k}{\sqrt{6}} \left(\text{since } k > 0\right)$ $x = \frac{k}{\sqrt{6}}$ (since $x > 0$) $y = 3\left(\frac{k}{\sqrt{6}}\right)^2 - k^2 \ln\left(\frac{\frac{k}{\sqrt{6}}}{4}\right) = \frac{k^2}{2} - k^2 \ln\left(\frac{k}{4\sqrt{6}}\right) = k^2\left(\frac{1}{2} - \ln\left(\frac{\sqrt{6}k}{24}\right)\right)$ Coordinates of stationary point is $\left(\frac{\sqrt{6}k}{6}, k^2\left(\frac{1}{2} - \ln\sqrt{6}k + \ln 24\right)\right)$ $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6 + \frac{k^2}{x^2} > 0$ since $\frac{k^2}{x^2} > 0$ Hence the stationary point is a minimum point.

<table>
<tr><th>Qn</th><th>Solution</th></tr>
<tr><td>4</td><td>Equations and Inequalities</td></tr>
<tr><td>(i)</td><td>

$$\frac{7}{x^2-2x-6}\ge -1$$
$$\frac{7}{x^2-2x-6}+1\ge 0$$
$$\frac{7+x^2-2x-6}{x^2-2x-6}\ge 0$$
$$\frac{x^2-2x+1}{x^2-2x-6}\ge 0$$
$$\frac{(x-1)^2}{(x-1)^2-7}\ge 0$$
$$\frac{(x-1)^2}{\left(x-\left(1-\sqrt{7}\right)\right)\left(x-\left(1+\sqrt{7}\right)\right)}\ge 0$$

$$\begin{array}{ccccccc} + & | & - & | & - & | & + \\ & 1-\sqrt{7} & & 1 & & 1+\sqrt{7} & \quad x \end{array}$$

$$\left\{x\in\mathbb{R}:\ x<1-\sqrt{7}\quad \text{or}\quad x=1\quad \text{or}\quad x>1+\sqrt{7}\right\}$$

Alternative Solution

$$\frac{7}{x^2-2x-6}\ge -1$$
$$\frac{7}{x^2-2x-6}+1\ge 0$$
$$\frac{7+x^2-2x-6}{x^2-2x-6}\ge 0$$
$$\frac{x^2-2x+1}{x^2-2x-6}\ge 0$$
$$\frac{(x-1)^2}{(x-1)^2-7}\ge 0$$

Since $(x-1)^2\ge 0$ for all real values of x,

$$(x-1)^2-7>0$$
$$\left(x-\left(1-\sqrt{7}\right)\right)\left(x-\left(1+\sqrt{7}\right)\right)>0$$
$$x<1-\sqrt{7}\ \text{ or }\ x>1+\sqrt{7}.$$
$$\left\{x\in\mathbb{R}:\ x<1-\sqrt{7}\quad \text{or}\quad x=1\quad \text{or}\quad x>1+\sqrt{7}\right\}$$

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<tr><td>(ii)</td><td>

Replacing x by $|x|$,

$|x|<1-\sqrt{7}$ (NA since $|x|\ge 0$)

or $|x|=1\ \Rightarrow x=\pm 1$

or $|x|>1+\sqrt{7}\Rightarrow x>1+\sqrt{7}\quad$ or $\quad x<-1-\sqrt{7}$

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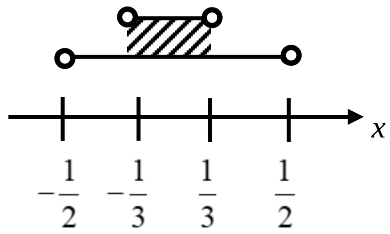
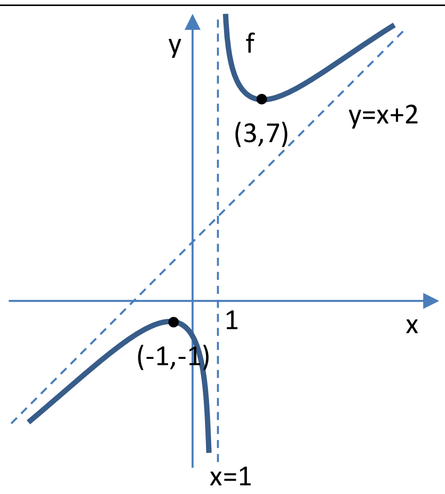
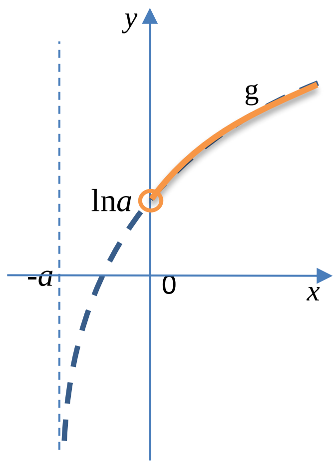
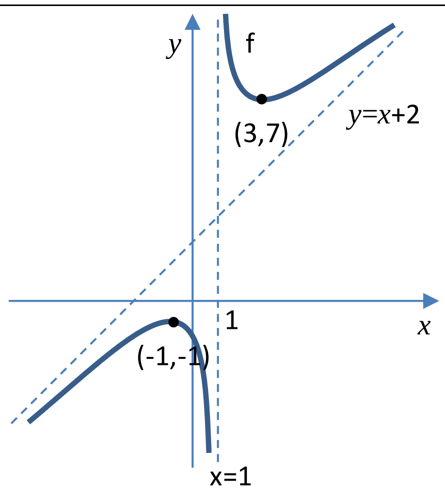
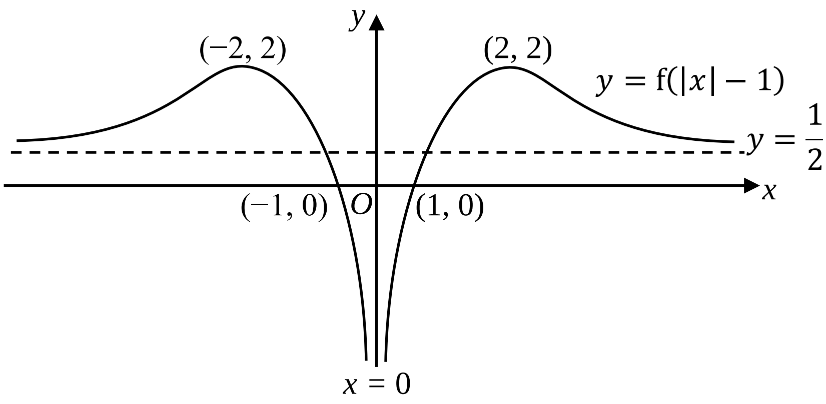
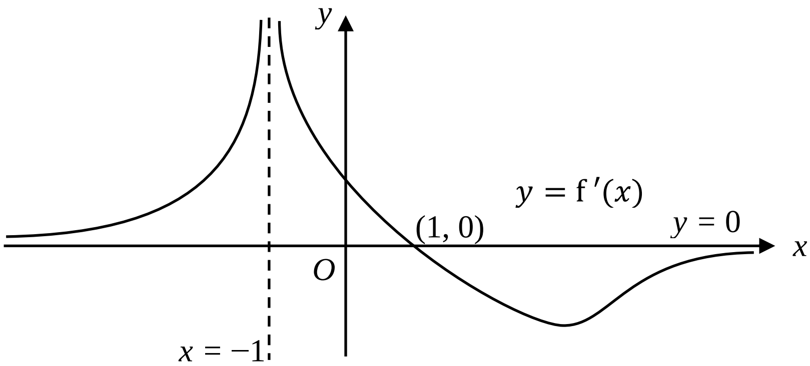
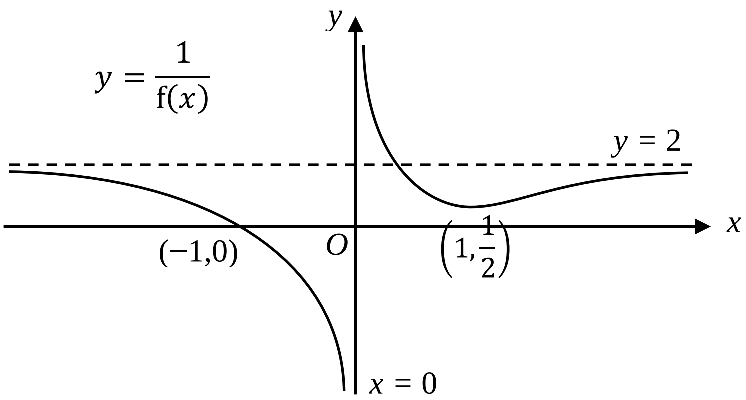
Qn	Solution
5	**Integration Techniques**
(i)	$\frac{\sqrt{1-3x}}{2+4x}=(1-3x)^{\frac{1}{2}}\left[2^{-1}(1+2x)^{-1}\right]$ $\therefore (1-3x)^{\frac{1}{2}}\left[2^{-1}(1+2x)^{-1}\right]$ $=\frac{1}{2}\left[1+\frac{1}{2}(-3x)-\frac{1}{8}(-3x)^2+\cdots\right]\left[1+(-1)(2x)+\frac{(-1)(-2)}{2!}(2x)^2+\cdots\right]$ $=\frac{1}{2}\left(1-\frac{3}{2}x-\frac{9}{8}x^2+\cdots\right)\left(1-2x+4x^2+\ldots\right)$ $=\frac{1}{2}\left(1-\frac{7x}{2}+\frac{47}{8}x^2+\cdots\right)$ $=\frac{1}{2}-\frac{7}{4}x+\frac{47}{16}x^2+\cdots$
(ii)	For expansion to be valid, $\lvert -3x\rvert<1$ and $\lvert 2x\rvert<1$ $-\frac{1}{3}<x<\frac{1}{3}$ and $-\frac{1}{2}<x<\frac{1}{2}$  $\therefore -\frac{1}{3}<x<\frac{1}{3}$
(iii)	Since $\frac{\sqrt{1-3x}}{2+4x}\simeq\frac{1}{2}-\frac{7}{4}x+\frac{47}{16}x^2$, Let $x=-\frac{1}{4}$ $\frac{\sqrt{1-3\left(-\frac{1}{4}\right)}}{2+4\left(-\frac{1}{4}\right)}=\frac{1}{2}-\frac{7}{4}\left(-\frac{1}{4}\right)+\frac{47}{16}\left(-\frac{1}{4}\right)^2+\cdots$ $\sqrt{\frac{7}{4}}=\frac{1}{2}-\frac{7}{4}\left(-\frac{1}{4}\right)+\frac{47}{16}\left(-\frac{1}{4}\right)^2+\cdots\approx\frac{287}{256}$ $\sqrt{7}\approx\frac{287}{128}$ $p=287,\quad q=128$

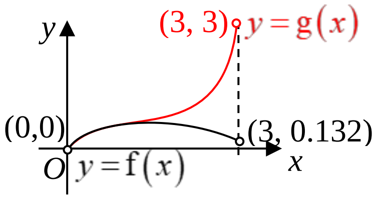
Qn	Solution
6	**Functions**
(i)	$f(x)=x+2+\frac{4}{x-1}$

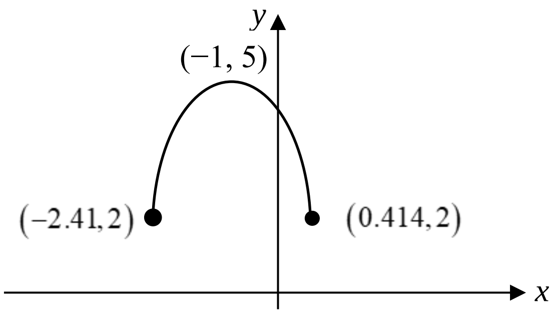
	 $R_f = (-\infty,-1]\cup[7,\infty) \not\subseteq D_g = (0,\infty)$. $\therefore$ gf does not exist regardless of the value of a.
(ii)	 For fg to exist, $R_g \subseteq D_f$. $R_g = (\ln a, \infty)$ and $D_f = (-\infty,1)\cup(1,\infty)$. $\therefore$ fg exists if $\ln a \geq 1 \Rightarrow a \geq e$
(iii)	$\text{fg} : x \mapsto \ln(x+a)+2+\dfrac{4}{\ln(x+a)-1}$ for $x\in\mathbb{R}, x>0$
(iv)	If $a > e^3$, $\ln a > 3$, $D_{fg} = D_g \rightarrow R_g = (\ln a,\infty) \rightarrow R_{fg} = \left(\ln a + 2 + \dfrac{4}{\ln a - 1},\infty\right)$

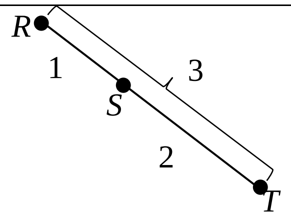


$$f(x) = x + 2 + \frac{4}{x-1}$$

Qn	Solution
7	**Transformation of Graphs**
(i)	
(ii)	
(iii)	

Qn	Solution
8	**Maclaurin Series**
	$y=\ln\left(1+\sin\left(x\right)\right),\Rightarrow \mathrm{e}^{y}=1+\sin\left(x\right)$ Differentiate with respect to x: $\mathrm{e}^{y}\dfrac{\mathrm{d}y}{\mathrm{d}x}=\cos\left(x\right)$ Differentiate with respect to x: $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}\mathrm{e}^{y}+\dfrac{\mathrm{d}y}{\mathrm{d}x}\mathrm{e}^{y}\dfrac{\mathrm{d}y}{\mathrm{d}x}=-\sin\left(x\right)$ $\mathrm{e}^{y}\left[\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]=-\sin\left(x\right)\quad\left(\text{shown}\right)$
(i)	Differentiate with respect to x: $\mathrm{e}^{y}\left[\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}+2\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)\left(\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)\right]+\mathrm{e}^{y}\dfrac{\mathrm{d}y}{\mathrm{d}x}\left[\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]=-\cos\left(x\right)$ $\mathrm{e}^{y}\left[\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}+3\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)\left(\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)+\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^3\right]=-\cos\left(x\right)$ When $x=0,\ y=0,\ \dfrac{\mathrm{d}y}{\mathrm{d}x}=1,\ \dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=-1,\ \dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}=1$ Maclaurin series for y is $y\approx x-\dfrac{1}{2}x^2+\dfrac{1}{6}x^3$
(ii)	Using GC,  The graphs of $y=\mathrm{g}\left(x\right)=x-\dfrac{1}{2}x^2+\dfrac{1}{6}x^3$ and $y=\mathrm{f}\left(x\right)=\ln\left(1+\sin x\right)$ **deviate/differ significantly at *x* = 2**, where $\mathrm{g}\left(2\right)=\dfrac{4}{3}\approx 1.33$ and $\mathrm{f}\left(2\right)=0.647$. Hence the approximation is not accurate. **<u>OR</u>** The value of $x=2$ is **not close to zero**, hence the approximation is not very good.

Qn	Solution
9	**Parametric Equations (Cross topical with Differentiation)**
(i)	$x=\sqrt{2}\sin\theta-1,\ \ y=2+3\cos\theta$ $\dfrac{\mathrm{d}x}{\mathrm{d}\theta}=\sqrt{2}\cos\theta,\quad \dfrac{\mathrm{d}y}{\mathrm{d}\theta}=-3\sin\theta$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{-3\sin\theta}{\sqrt{2}\cos\theta}$ For tangent to be parallel to y-axis, $\sqrt{2}\cos\theta=0$ Since $-\dfrac{\pi}{2}\le\theta\le\dfrac{\pi}{2}$, $\theta=-\dfrac{\pi}{2}$ or $\dfrac{\pi}{2}$ At $\theta=\dfrac{\pi}{2}$, $x=\sqrt{2}\sin\left(\dfrac{\pi}{2}\right)-1=\sqrt{2}-1,\ \ y=2+3\cos\left(\dfrac{\pi}{2}\right)=2$ Coordinates of point is $(\sqrt{2}-1,2)$ or (-2.41,2). At $\theta=-\dfrac{\pi}{2}$, $x=\sqrt{2}\sin\left(-\dfrac{\pi}{2}\right)-1=-\sqrt{2}-1,\ \ y=2+3\cos\left(-\dfrac{\pi}{2}\right)=2$ Coordinates of point is $(-\sqrt{2}-1,2)$ or (0.414,2).
(ii)	
(iii)	$x=\sqrt{2}\sin\theta-1 \qquad y=2+3\cos\theta$ $\sin\theta=\dfrac{x+1}{\sqrt{2}} \qquad \cos\theta=\dfrac{y-2}{3}$ Since $\sin^2\theta+\cos^2\theta=1$, $\left(\dfrac{x+1}{\sqrt{2}}\right)^2+\left(\dfrac{y-2}{3}\right)^2=1$ $\dfrac{(x+1)^2}{2}+\dfrac{(y-2)^2}{9}=1$ $(y-2)^2=9-\dfrac{9(x+1)^2}{2}$ $y=2\pm\sqrt{9-\dfrac{9(x+1)^2}{2}}$ Since $y\ge 2, y=2+\sqrt{9-\dfrac{9(x+1)^2}{2}}$

Qn	Solution
10	**Vector Algebra**
(i)	$\alpha+\beta+1=0 \Rightarrow \alpha=-\beta-1$ Sub into $\alpha\mathbf{p}+\beta\mathbf{q}+\mathbf{r}=\mathbf{0}$: This is $\underset{\sim}{0}$ (a vector). Important to use the correct notation. $(-\beta-1)\mathbf{p}+\beta\mathbf{q}+\mathbf{r}=\mathbf{0}$ $\beta(\mathbf{q}-\mathbf{p})+(\mathbf{r}-\mathbf{p})=\mathbf{0}$ $\beta\overrightarrow{PQ}+\overrightarrow{PR}=\mathbf{0}$ $\overrightarrow{PQ}=k\overrightarrow{PR},\quad k=-\frac{1}{\beta},\ \beta\neq 0$ Since $\overrightarrow{PQ}$ is parallel to $\overrightarrow{PR}$, and P is a common point, P, Q and R are collinear.
(ii)	$\overrightarrow{OS}=\frac{\overrightarrow{OT}+2\overrightarrow{OR}}{3}$ $\mathbf{s}=\frac{\overrightarrow{OT}+2\mathbf{r}}{3}$ $\overrightarrow{OT}=3\mathbf{s}-2\mathbf{r}$ 
(iii)	$\frac{\lvert(\mathbf{p}-\mathbf{r})\cdot(\mathbf{s}-\mathbf{r})\rvert}{\lvert\mathbf{s}-\mathbf{r}\rvert}=\left\lvert(\mathbf{p}-\mathbf{r})\cdot\frac{(\mathbf{s}-\mathbf{r})}{\lvert\mathbf{s}-\mathbf{r}\rvert}\right\rvert$ is the length of projection of $\overrightarrow{RP}$ on $\overrightarrow{RS}$.
(iv)	Area of triangle $PRT=\frac{1}{2}\left\lvert\overrightarrow{RP}\times\overrightarrow{RT}\right\rvert$ $=\frac{1}{2}\lvert(\mathbf{p}-\mathbf{r})\times(3\mathbf{s}-2\mathbf{r}-\mathbf{r})\rvert$ $=\frac{1}{2}\lvert(\mathbf{p}-\mathbf{r})\times(3\mathbf{s}-3\mathbf{r})\rvert$ $=\frac{3}{2}\lvert\mathbf{p}\times\mathbf{s}-\mathbf{p}\times\mathbf{r}-\mathbf{r}\times\mathbf{s}+\mathbf{r}\times\mathbf{r}\rvert$ $=\frac{3}{2}\lvert\mathbf{p}\times\mathbf{s}-\mathbf{p}\times\mathbf{r}-\mathbf{r}\times\mathbf{s}\rvert\quad(\because \mathbf{r}\times\mathbf{r}=\mathbf{0})$ $\therefore \gamma=\frac{3}{2}$ To be stated explicitly

Qn	Solution
11	**ACJC Promo 9758/2021/Q12**
i	$U_n > 6550$ $3000+(n-1)100 > 6550$ $n > 36.5$ $\therefore 37^{\text{th}}$ month
ii	$S_n = \frac{n}{2}\left[6000+(n-1)100\right]$ $\frac{n}{2}\left[6000+(n-1)100\right] \geq 200,000$ <u>Method 1</u> By GC, When $n = 40, y = 198,000 < 200,000$ When $n = 41, y = 205,000 > 200,000$ <u>Method 2</u> $\frac{n}{2}\left[6000+(n-1)100\right] \geq 200,000$ $100n^2 + 5900n - 400,000 \geq 0$ $(n-40.287)(n+99.287) \geq 0$ $n \leq -99.3$ (rej) or $n \geq 40.3$ $\therefore$ 41 months $S_{40} = \frac{40}{2}\left[6000+(40-1)100\right] = \$198,000$ $\$200,000 - \$198,000 = \$2000$
iii	see below

n	**End of the month**
1	$1.003(500000-x)$
2	$1.003^2(500000-x)-(1.003)x$
3	$1.003^3(500000-x)-(1.003)^2x-(1.003)x$
$\vdots$	$\vdots$
n	

At the end of the nth month, the outstanding amount would be

$1.003^n(500000-x)-(1.003)^{n-1}x-\ldots-(1.003)x$

$=1.003^n(500000)-(1.003^n)x-\ldots-(1.003)x$

$=1.003^n(500000)-x\left[1.003+1.003^2+\ldots+1.003^n\right]$

$=1.003^n(500000)-x\left[\frac{1.003(1.003^n-1)}{1.003-1}\right]$

$=1.003^n(500000)-\frac{1003}{3}x(1.003^n-1)$

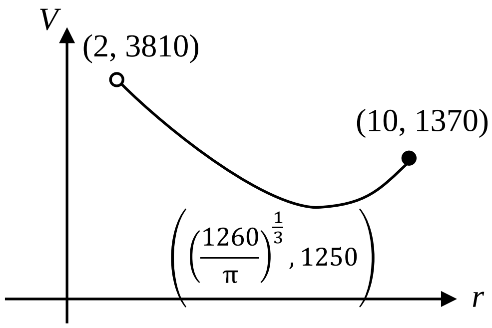
$\therefore A = 500000, B = \frac{1003}{3}$

iv	$1.003^{96}\left(500000\right)-\frac{1003}{3}x\left(1.003^{96}-1\right)=0$ Using GC, $x=\$5984.09$.
v	Total paid: $\$5984.09\times 12\times 8=\$574,472.64$ Interest: $\$574,472.64-\$500,000=\$74,472.64$

Qn	Solution
12	**Applications of Differentiation**
(i)	Volume from external wall $=\frac{1}{3}\pi\left(r+3\right)^2\left(h+5\right)+\pi\left(r+3\right)^2\left(4h\right)$ $=\frac{13}{3}\pi\left(r+3\right)^2 h+\frac{5}{3}\pi\left(r+3\right)^2$ Volume from internal wall $=\frac{1}{3}\pi r^2 h+\pi r^2\left(4h\right)=k$ $\Rightarrow h=\frac{3k}{13\pi r^2}$ $V=\frac{13}{3}\pi\left(r+3\right)^2 h+\frac{5}{3}\pi\left(r+3\right)^2-k$ $V=\frac{13}{3}\pi\left(r+3\right)^2\left(\frac{3k}{13\pi r^2}\right)+\frac{5}{3}\pi\left(r+3\right)^2-k$ $=\left(r+3\right)^2\left(\frac{k}{r^2}\right)+\frac{5}{3}\pi\left(r+3\right)^2-k \quad \text{(Shown)}$
(ii)	$V=\left(r+3\right)^2\left(\frac{k}{r^2}\right)+\frac{5}{3}\pi\left(r+3\right)^2-k$ $\frac{\mathrm{d}V}{\mathrm{d}r}=2\left(r+3\right)\left(\frac{k}{r^2}\right)+\left(r+3\right)^2\left(\frac{-2k}{r^3}\right)+\frac{10}{3}\pi\left(r+3\right)$ To find stationary point, $\frac{\mathrm{d}V}{\mathrm{d}r}=0$ $2\left(r+3\right)\left(\frac{k}{r^2}\right)+\left(r+3\right)^2\left(\frac{-2k}{r^3}\right)+\frac{10}{3}\pi\left(r+3\right)=0$ $\left(r+3\right)\left[\frac{2k}{r^2}-\frac{2k}{r^3}\left(r+3\right)+\frac{10}{3}\pi\right]=0$ $\left(r+3\right)\left[\frac{2k}{r^2}-\frac{2k}{r^2}-\frac{6k}{r^3}+\frac{10}{3}\pi\right]=0$ $\left(r+3\right)\left[-\frac{6k}{r^3}+\frac{10}{3}\pi\right]=0$ $-\frac{6k}{r^3}+\frac{10}{3}\pi=0 \qquad \text{or} \qquad r=-3\left(\text{Reject, } \because r>0\right)$ $r^3=\frac{9k}{5\pi}$ $r=\left(\frac{9k}{5\pi}\right)^{\frac{1}{3}}$

(iii)

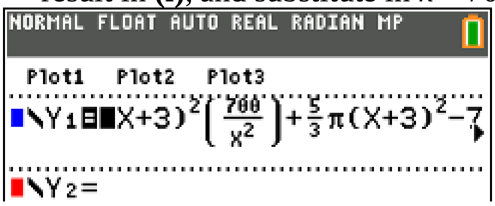
When k = 700, $r = \left(\dfrac{9(700)}{5\pi}\right)^{\frac{1}{3}} = 7.3746$



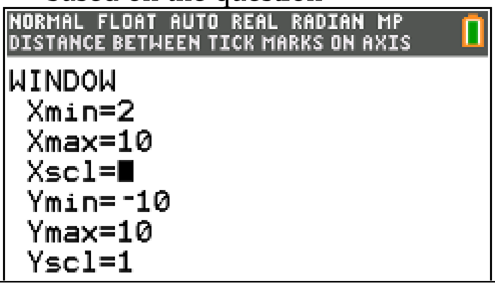
GC Skill:

This part can be done using the GC

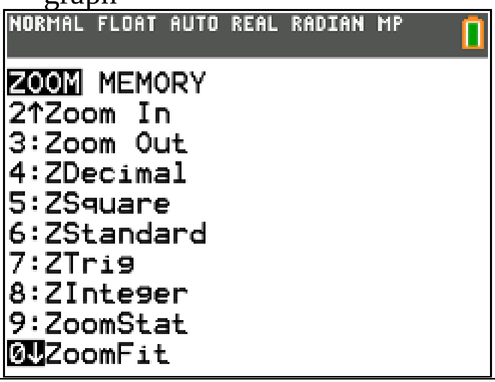
1) Type in the equation of V from show result in **(i)**, and substitute in k = 700



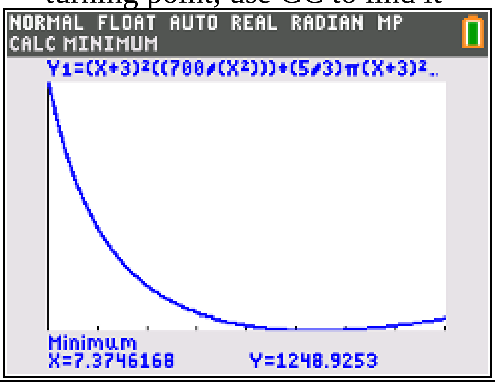
2) Press @ and set the Xmin and Xmax based on the question



3) Press # and select **0: ZoomFit** to see the graph



4) From the graph, you can clearly see a turning point, use GC to find it



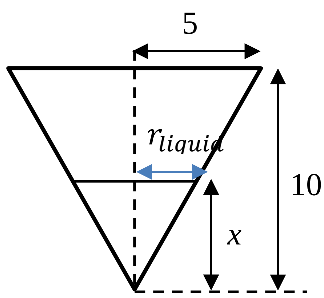
(iv)

Volume of cone section $= \dfrac{1}{3}\pi\left(5^2\right)(10) = 261.80$

Hence when volume of liquid is 200 m^3 it is still in the cone section

By similar triangles,

$$\frac{r_{liquid}}{x} = \frac{5}{10} \quad \Rightarrow r_{liquid} = \frac{1}{2}x$$



Let volume of the liquid at time t minutes be W

$$W = \frac{1}{3}\pi\left(r_{liquid}\right)^2 x$$

$$= \frac{1}{3}\pi\left(\frac{1}{2}x\right)^2 x = \frac{1}{12}\pi x^3$$

$$\frac{\mathrm{d}W}{\mathrm{d}x} = \frac{1}{4}\pi x^2$$

	When $W = 200$, $200 = \frac{1}{12}\pi x^3 \quad \Rightarrow x = \left(\frac{2400}{\pi}\right)^{\frac{1}{3}}$ $\frac{\mathrm{d}W}{\mathrm{d}t} = \frac{\mathrm{d}W}{\mathrm{d}x}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)$ When $x = \left(\frac{2400}{\pi}\right)^{\frac{1}{3}}$, $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{0.1}{\left(\frac{1}{4}\pi\left(\frac{2400}{\pi}\right)^{\frac{2}{3}}\right)} = 0.00152 \quad (3\text{ s.f.})$ The rate of change of depth of the acid in the storage well is 0.00152 m/min