TAMPINES MERIDIAN JUNIOR COLLEGE

JC2 PRELIMINARY EXAMINATION

CANDIDATE NAME: ____

CIVICS GROUP: ____

H2 MATHEMATICS 9758

Paper 1

14 SEPTEMBER 2021

3 hours

Candidates answer on the question paper.

Additional material: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name and Civics Group on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

For Examiners' Use	
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The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

This document consists of 26 printed pages and 2 blank pages.

[Turn Over

1 A curve C has equation $y = a(2x+1)^2 + bx + ce^{2x}$, where a, b and c are constants. It is given that C passes through the point $(1,\ 17-e^2)$ and the gradient of C at the point $(0,1)$ is 5. Find the equation of C. [5]

2 On the same axes, sketch the graphs of $y = \ln\left(\frac{4}{x-a}\right)$ and $y = \ln|x-a|$, where $a > 1$.

Hence, or otherwise, solve the inequality $\ln\left(\frac{4}{x-a}\right) \geq \ln|x-a|$. [7]

3 Two curves C_1 and C_2 have equations $y = -\sqrt{\left(1-\frac{(x-2)^2}{4}\right)}$ and $y = -\frac{1}{4}x^2$ respectively.

(i) Sketch C_1 and C_2 on the same diagram, stating the coordinates of any axial intercepts and points of intersections between C_1 and C_2. [3]

(ii) The region R is bounded by C_1 and C_2. Find the volume of the solid of revolution formed when R is rotated through 2π radians about the x-axis. [3]

4 A quartic (degree four) polynomial $\mathrm{P}(z) = z^4 + az^3 + bz^2 + cz + d$ has real coefficients. The equation $\mathrm{P}(z) = 0$ has a root $re^{i\theta}$, where $r > 0$ and $0 < \theta < \pi$.

(i) Write down a second root in terms of r and θ, and hence show that a quadratic factor of $\mathrm{P}(z)$ is $z^2 - (2r\cos\theta)z + r^2$. [3]

(ii) Given that $\sqrt{3}e^{\frac{i\pi}{6}}$ and $\sqrt{2}e^{\frac{i\pi}{4}}$ are two roots of the equation $\mathrm{P}(z) = 0$, find an expression for $\mathrm{P}(z)$ in the form $z^4 + az^3 + bz^2 + cz + d$ where a, b, c and d are constants to be determined. [4]

5 It is given that

$$f(x) = \begin{cases} x, & \text{for } -2 < x \le 3, \\ \sqrt{(x^2 + kx + 3)} + 3, & \text{for } 3 < x \le 5, \end{cases}$$

where k is a real constant such that f is continuous for $-2 < x \le 5$.

(i) Show that $k = -4$. [1]

(ii) Sketch the graph of $y = f(x)$. Hence, justify that f has an inverse. [3]

(iii) Find $f^{-1}(x)$ in similar form. Leave your answer in the exact form. [4]

(iv) Write down the solution set for $f^{-1}(x) = f(x)$. [1]

6 The *folium of Descartes* is a curve given by the equation $x^3 - 9xy + y^3 = 0$.

(i) Find the equation of the tangent to the curve at the point $(4,2)$. [4]

(ii) Find the exact coordinates of the point, other than the origin, where the tangent to the curve is parallel to the *y*-axis. [4]

7 A geometric progression has first term a and common ratio r, and an arithmetic progression has first term b and common difference d, where a, b, d and r are non-zero real numbers. The first, third and eighth term of the geometric progression are equal to the second, third and fifth term of the arithmetic progression respectively.

(i) Show that $r^7 - 3r^2 + 2 = 0$. [2]

(ii) Find the values of r, giving your answer correct to 5 decimal places. Hence explain why the sum to infinity of the geometric progression exists. [2]

It is now given that $a = 12$ and r is the positive value found in part **(ii)**.

(iii) Let E be the sum of the first n even-numbered terms of the arithmetic progression and S_∞ be the sum to infinity of the geometric progression. Find the largest value of n such that the difference between E and S_∞ is less than 1000. [6]

8 **You are not allowed to use a graphing calculator for this question.**

(i) By considering $z^3-8=(z-2)(z^2+2z+4)$, or otherwise, find the three roots of $z^3=8$ in **both** cartesian form $x+\mathrm{i}y$ and exponential form $r\mathrm{e}^{\mathrm{i}\theta}$ exactly, where $r>0$ and $-\pi<\theta\le\pi$. [3]

(ii) Hence, or otherwise, find the six roots of the equation $z^6=64$ in exact exponential form. [3]

(iii) Let n be a positive integer. It is given that w is a root of the equation $z^n=2^n$, where $w\ne 2$. Let

$$\mathrm{f}(w)=2^{n+1}+2^n w+2^{n-1}w^2+\ldots+2^2 w^{n-1}+2w^n+w^{n+1}.$$

Show that $\mathrm{f}(w)=2^{n+1}\left(1+\dfrac{w}{2}\right)$. [3]

(iv) It is given that all the roots of the equation $z^n=2^n$ lie on a circle on the Argand diagram. For all values of n, the complex number $\dfrac{\mathrm{f}(w)}{2^{n+1}}$ lie on another circle represented by the curve C. Find the cartesian equation of C. [2]

9 The line l passes through the point A with coordinates $(1,-2,3)$ and is parallel to the vector $\begin{pmatrix}4\\0\\-1\end{pmatrix}$.

The plane π_1 contains the point B with coordinates $(2,-1,3)$ and the line l.

(i) Show that the cartesian equation of π_1 is $x-y+4z=15$. [2]

Let F be the point on l which is closest to B.

(ii) Find the coordinates of F. [3]

Point C has coordinates $(8,\ 1,-3)$.

(iii) Find the exact shortest distance from C to π_1. [2]

(iv) Hence, find the exact volume of the tetrahedron $CABF$. [3]

[Volume of tetrahedron = $\dfrac{1}{3}\times$ base area $\times$ perpendicular height]

(v) D is a general point on plane π_2 such that $DABF$ has the same volume as tetrahedron $CABF$. Given that π_2 does not contain C, find an equation of π_2. [2]

10 Two students, Andy and Ben, conduct a research on the population of Kawaii otters in Otterland. Initially, there are one thousand Kawaii otters in Otterland. After t years, the number of Kawaii otters in Otterland becomes x (in thousands).

Andy observes that the birth rate of the Kawaii otters is inversely proportional to the population (in thousands) of Kawaii otters. At the same time, the death rate of the Kawaii otters is proportional to the population (in thousands) of Kawaii otters. Andy suggests that the number of Kawaii otters remains constant when the population reaches 2000.

(i) Based on Andy's findings, show that $\frac{\mathrm{d}x}{\mathrm{d}t} = k\left(\frac{4-x^2}{x}\right)$, where k is a positive real constant. [3]

(ii) Solve the differential equation in part **(i)** to find an expression for x in terms of t and k. Describe what happens to the number of Kawaii otters if this situation continues over many years. [6]

Ben proposes that the population growth of Kawaii otters can be modelled by the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{10}{4+t}\ln\left(1+\frac{1}{4}t\right) \text{ for } t > 0.$$

(iii) Based on Ben's proposal, find a general solution for this differential equation. [2]

(iv) Explain if Ben's model is appropriate in modelling the population of the Kawaii otters in the long run. [1]

11 Curved ceilings are used in buildings to achieve acoustics or aesthetic effects. Some architects use mathematical functions to aid them in the design of such curved ceilings. In a small room of an art gallery, an architect designs a curved ceiling for an art exhibition. The cross section of the ceiling can be modelled by curve C_1, which is defined parametrically by

$$x = 2t - \sin 2t,\ y = 5 + 2\sin^2 t \ \text{ for } \ 0 \le t \le \pi,$$

and x and y are given in metres.

(i) Sketch C_1. You are not required to label the turning point. [2]

(ii) Find $\int \sin^2 t\,(1 - \cos 2t)\, \mathrm{d}t$. [2]

(iii) The sketch in part **(i)** can be used to model the cross section of the small room, where C_1 is the curved ceiling, the x-axis is the floor and the lines $x = 0$ and $x = 2\pi$ are the walls. A curtain of negligible thickness will be hung from the curved ceiling as part of the art exhibition. Without using a calculator, find the exact smallest amount of curtain required to completely cover the cross section of the room. [4]

A surface of revolution is a surface created by rotating a curve around an axis of rotation. If the curve is rotated by k radians about the x-axis, the area of the surface formed can be found by the formula

$$k\int_0^{\pi} y\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t.$$

(iv) The surface area of the curved ceiling of the small room is equivalent to the area of the surface formed when the curve C_1 is rotated by $\frac{\pi}{4}$ radians about the x-axis. Without using a calculator, find the exact surface area of the curved ceiling of the small room. [5]

End of Paper