2021 JC 1 H2 Mathematics Promotional Examination (Suggested solutions)

[Solution]	**Comments**
Q1(a) $y=\mathrm{f}(x)\xrightarrow{T} y=\mathrm{f}(x+1)$ $\xrightarrow{S} y=\mathrm{f}(2x+1)$ $\xrightarrow{R} y=\mathrm{f}(-2x+1)$ A sequence of transformations is: 1 A translation of 1 unit in the negative direction of the x-axis 2 A scaling parallel to the x-axis by factor $\frac{1}{2}$ 3 A reflection about the y-axis	- Use proper phrasing, eg translation instead of shift, reflection instead of flip, scaling…by factor $\frac{1}{2}$ instead of $\frac{1}{2}$ units. - Generally well done for students who did translation first, some errors made are those highlighted in yellow.
OR $y=\mathrm{f}(x)\xrightarrow{S} y=\mathrm{f}(2x)$ $\xrightarrow{T} y=\mathrm{f}\left(2\left(x+\frac{1}{2}\right)\right)=\mathrm{f}(2x+1)$ $\xrightarrow{R} y=\mathrm{f}(-2x+1)$ A sequence of transformations is: 1 A scaling parallel to the x-axis by factor $\frac{1}{2}$ 2 A translation of $\frac{1}{2}$ units in the negative direction of the x-axis 3 A reflection about the y-axis	- Students who did scaling first tend to get the translation in the second step wrong. Note that we only replace x as highlighted in green.
OR $y=\mathrm{f}(x)\xrightarrow{R} y=\mathrm{f}(-x)$ $\xrightarrow{T} y=\mathrm{f}(-(x-1))=\mathrm{f}(-x+1)$ $\xrightarrow{S} y=\mathrm{f}(-2x+1)$ A sequence of transformations is: 1 A reflection about the y-axis 2 A translation of 1 unit in the positive direction of the x-axis 3 A scaling parallel to the x-axis by factor $\frac{1}{2}$	- Students who did reflection first tend to get the translation in the second step wrong. Note that we only replace x as highlighted in green.

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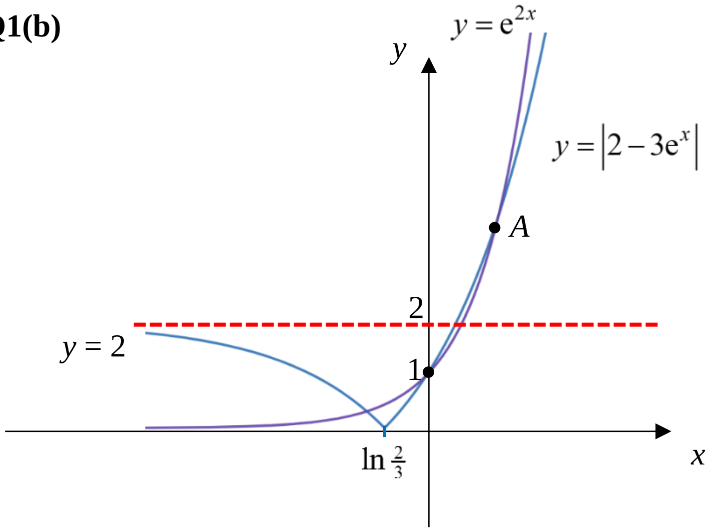
Q1(b)



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Let $x = 0$, $y = 1$
Let $y = \left|2-3\mathrm{e}^x\right| = 0 \Rightarrow x = \ln\frac{2}{3}$
Asymptote: $y = 2$
(As $x \to -\infty$, $\mathrm{e}^x \to 0 \therefore y \to 2$)

Things to take note:
1. Horizontal asymptote $y = 2$ must be drawn.
2. Sharp point at $(\ln\frac{2}{3}, 0)$.

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Add the graph of $y = \mathrm{e}^{2x}$ on the same diagram.
At the intersection point A, $-\left(2-3\mathrm{e}^x\right) = \mathrm{e}^{2x}$
$\mathrm{e}^{2x} - 3\mathrm{e}^x + 2 = 0$
$\left(\mathrm{e}^x - 2\right)\left(\mathrm{e}^x - 1\right) = 0$
$\mathrm{e}^x = 2$ or $\mathrm{e}^x = 1$
$x = \ln 2$ or $x = 0$
(rej. since $x > 0$ for A)

For $\left|2-3\mathrm{e}^x\right| > \mathrm{e}^{2x}$, where $x \geq 0$, the solution is $0 < x < \ln 2$

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- Hence means need to make use of the graph above to solve.
- Since $x \geq 0$, only need to solve for the intersection point A.
- Algebraic errors made in the process of solving for intersection, eg $3\mathrm{e}^x - \mathrm{e}^{2x} = \mathrm{e}^x\left(3-\mathrm{e}^2\right)$ ✗
- Note that the solution must be given in exact form.

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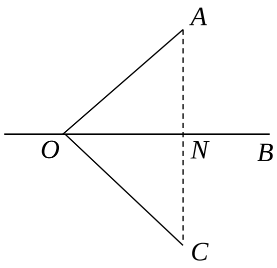
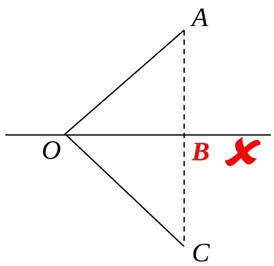
Q2	[Solution]	Comments
(i)	Let N be the foot of perpendicular from A on line OB. $\overrightarrow{ON} = \lambda \underset{\sim}{b}$ for some $\lambda \in \mathbb{R}$ Note that λ can be negative $\overrightarrow{AN} \cdot \underset{\sim}{b} = 0$ $(\lambda \underset{\sim}{b} - \underset{\sim}{a}) \cdot \underset{\sim}{b} = 0$ $\lambda \underset{\sim}{b} \cdot \underset{\sim}{b} - \underset{\sim}{a} \cdot \underset{\sim}{b} = 0$ $\underset{\sim}{b} \cdot \underset{\sim}{b} = \lvert \underset{\sim}{b} \rvert^2$ $\lambda = \dfrac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{\lvert \underset{\sim}{b} \rvert^2}$ $= \underset{\sim}{a} \cdot \underset{\sim}{b}$ since $\lvert \underset{\sim}{b} \rvert = 1$ Therefore, $\overrightarrow{ON} = (\underset{\sim}{a} \cdot \underset{\sim}{b}) \underset{\sim}{b}$  Since N is the mid-point of AC, $\overrightarrow{ON} = \dfrac{\overrightarrow{OA} + \overrightarrow{OC}}{2}$ $\underset{\sim}{c} = 2\overrightarrow{ON} - \overrightarrow{OA}$ $= 2(\underset{\sim}{a} \cdot \underset{\sim}{b}) \underset{\sim}{b} - \underset{\sim}{a}$	Use **correct notation:** Vector $\underset{\sim}{b}$ or $\overrightarrow{ON}$ ON without → is treated as the length of ON. Use the usual procedure to find **c** 1) first find the position vector of foot of perpendicular from A to OB 2) then use ratio theorem to find the point of reflection of A about OB As 'numbers' are not given, you **need to use vector algebra and notation well.**
	Alternative to find $\overrightarrow{ON}$: $\overrightarrow{ON}$ is the projection vector of $\underset{\sim}{a}$ on $\underset{\sim}{b}$ Since $\underset{\sim}{b}$ is a unit vector, $\overrightarrow{ON} = (\underset{\sim}{a} \cdot \underset{\sim}{b}) \underset{\sim}{b}$	Use the alternative method only if you know the direct formula for **projection vector** of $\underset{\sim}{a}$ on $\underset{\sim}{b}$ **(not to be confused with length of projection)**. Note this *formula is often poorly understood or wrongly quoted.*
	Note: 1) We cannot assume that point B is the foot of the perpendicular from A to the line OB as it is not stated in the question. In fact B is not the foot of the perpendicular. Instead we have to let N be the foot of the perpendicular and find it as given in the above solution.	

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2) Many students used
$\overrightarrow{ON} =$ (length of projection of $\underset{\sim}{a}$ on $\underset{\sim}{b}$) $\dfrac{\underset{\sim}{b}}{|\underset{\sim}{b}|}$
$= \dfrac{|\underset{\sim}{a}\cdot\underset{\sim}{b}|}{|\underset{\sim}{b}|}\ \dfrac{\underset{\sim}{b}}{|\underset{\sim}{b}|}$
$= |\underset{\sim}{a}\cdot\underset{\sim}{b}|\ \underset{\sim}{b}$ since $|\underset{\sim}{b}| = 1$
$\neq (\underset{\sim}{a}\cdot\underset{\sim}{b})\underset{\sim}{b}$ incorrect as $\underset{\sim}{a}\cdot\underset{\sim}{b}$ can be negative

Cannot assume that $\overrightarrow{ON}$ is in the same direction as $\overrightarrow{OB}$.
As $\overrightarrow{ON}$ can be in the opposite direction to OB, hence it is possible $\underset{\sim}{a}\cdot\underset{\sim}{b} < 0$ and $(\underset{\sim}{a}\cdot\underset{\sim}{b}) \neq |\underset{\sim}{a}\cdot\underset{\sim}{b}|$ and the above method is not complete.
The above method works only if angle AOB is acute and $\overrightarrow{ON}$ is strictly in the same direction as $\overrightarrow{OB}$.
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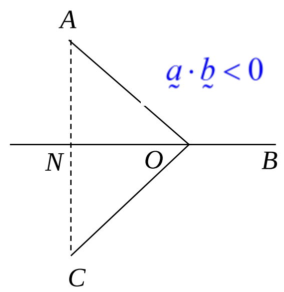

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From (i) $\underset{\sim}{c} = 2(\underset{\sim}{a}\cdot\underset{\sim}{b})\underset{\sim}{b} - \underset{\sim}{a}$
Rearranging $\underset{\sim}{a} = 2(\underset{\sim}{a}\cdot\underset{\sim}{b})\underset{\sim}{b} - \underset{\sim}{c}$
$\underset{\sim}{c}\times\underset{\sim}{a} = \underset{\sim}{c}\times\left[2(\underset{\sim}{a}\cdot\underset{\sim}{b})\underset{\sim}{b} - \underset{\sim}{c}\right]$
$= \underset{\sim}{c}\times\left[2(\underset{\sim}{a}\cdot\underset{\sim}{b})\underset{\sim}{b}\right] - \underset{\sim}{c}\times\underset{\sim}{c}$
$= 2(\underset{\sim}{a}\cdot\underset{\sim}{b})\ (\underset{\sim}{c}\times\underset{\sim}{b}) - \underset{\sim}{0}$ ($\underset{\sim}{0}$ is zero vector, not 0 (scalar))
$= 2(\underset{\sim}{a}\cdot\underset{\sim}{b})\ (\underset{\sim}{c}\times\underset{\sim}{b})$ (shown)

Alternative
$\underset{\sim}{c}\times\underset{\sim}{a} = \left[2(\underset{\sim}{a}\cdot\underset{\sim}{b})\underset{\sim}{b} - \underset{\sim}{a}\right]\times\underset{\sim}{a}$ from (i)
$= \left[2(\underset{\sim}{a}\cdot\underset{\sim}{b})\underset{\sim}{b}\right]\times\underset{\sim}{a} - \underset{\sim}{a}\times\underset{\sim}{a}$ expand
$= 2(\underset{\sim}{a}\cdot\underset{\sim}{b})\,(\underset{\sim}{b}\times\underset{\sim}{a}) - \underset{\sim}{0}$ ($\underset{\sim}{0}$ is zero vector, not 0 (scalar))
$= 2(\underset{\sim}{a}\cdot\underset{\sim}{b})\,(\underset{\sim}{b}\times\underset{\sim}{a})$ ----- (1)

$2(\underset{\sim}{a}\cdot\underset{\sim}{b})\,(\underset{\sim}{c}\times\underset{\sim}{b}) = 2(\underset{\sim}{a}\cdot\underset{\sim}{b})\left(\left[2(\underset{\sim}{a}\cdot\underset{\sim}{b})\underset{\sim}{b} - \underset{\sim}{a}\right]\times\underset{\sim}{b}\right)$ from (i)
$= 2(\underset{\sim}{a}\cdot\underset{\sim}{b})\left(\left[2(\underset{\sim}{a}\cdot\underset{\sim}{b})\underset{\sim}{b}\times\underset{\sim}{b}\right] - \underset{\sim}{a}\times\underset{\sim}{b}\right)$ expand
$= 2(\underset{\sim}{a}\cdot\underset{\sim}{b})\,(\underset{\sim}{0} - \underset{\sim}{a}\times\underset{\sim}{b})$
$= 2(\underset{\sim}{a}\cdot\underset{\sim}{b})\,(\underset{\sim}{b}\times\underset{\sim}{a})$ ----- (2)

From (1) & (2),
$\underset{\sim}{c}\times\underset{\sim}{a} = 2(\underset{\sim}{a}\cdot\underset{\sim}{b})\,(\underset{\sim}{c}\times\underset{\sim}{b})$ (shown)
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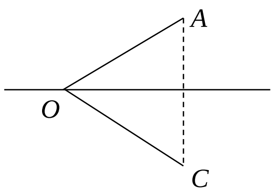
Cross product is a vector:
$\underset{\sim}{a}\times\underset{\sim}{a} = \underset{\sim}{0}$

Common mistakes:
1) Wrongly concluding that because $\underset{\sim}{c}$ is the reflection of $\underset{\sim}{a}$ about OB $\underset{\sim}{a} = -\underset{\sim}{c}$. As seen in diagram below, $\underset{\sim}{a} \neq -\underset{\sim}{c}$.


2) Wrongly equating vector to area (scalar):
$\dfrac{1}{2}(\underset{\sim}{c}\times\underset{\sim}{a}) \neq$ Area of triangle OAC
vector $\neq$ scalar

Formula should be
$\dfrac{1}{2}\left|\underset{\sim}{c}\times\underset{\sim}{a}\right|$ = Area of triangle OAC
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Q3	[Solution]	Comments

(i)

Using Pythagoras theorem, $x^2 + h^2 = (2r)^2$

$\Rightarrow\ h = \sqrt{4r^2 - x^2}$

$S = khx^2 = kx^2\sqrt{4r^2 - x^2}$ where $k > 0$, k is a real const

Comments: As this is a "show" question, do not skip steps.

(ii)

$S = k\sqrt{4r^2x^4 - x^6}$

$$\frac{dS}{dx} = \frac{1}{2}k\left(4r^2x^4 - x^6\right)^{-\frac{1}{2}}\left(16r^2x^3 - 6x^5\right)$$

Let $\dfrac{dS}{dx} = \dfrac{k\left(8r^2x^3 - 3x^5\right)}{\sqrt{4r^2x^4 - x^6}} = \dfrac{kx^3\left(8r^2 - 3x^2\right)}{\sqrt{4r^2x^4 - x^6}} = 0$

$$x^3\left(8r^2 - 3x^2\right) = 0$$

$x = 0$ (rej. since $x > 0$) or $x = \sqrt{\frac{8}{3}}\,r$

Alternative method

$S = kx^2\sqrt{4r^2 - x^2}$

$$\frac{dS}{dx} = 2xk\sqrt{4r^2 - x^2} + kx^2\,\frac{1}{2}\left(4r^2 - x^2\right)^{-\frac{1}{2}}(-2x)$$

$$= 2xk\sqrt{4r^2 - x^2} - \frac{kx^3}{\sqrt{4r^2 - x^2}}$$

$$= \frac{2xk\left(4r^2 - x^2\right) - kx^3}{\sqrt{4r^2 - x^2}}$$

$$= \frac{8kxr^2 - 3kx^3}{\sqrt{4r^2 - x^2}} = \frac{kx\left(8r^2 - 3x^2\right)}{\sqrt{4r^2 - x^2}}$$

$$x\left(8r^2 - 3x^2\right) = 0$$

$x = 0$ (rej. since $x > 0$) or $x = \sqrt{\frac{8}{3}}\,r$

x	$\left(\sqrt{\frac{8}{3}}\,r\right)^-$	$\sqrt{\frac{8}{3}}\,r$	$\left(\sqrt{\frac{8}{3}}\,r\right)^+$
$\dfrac{dS}{dx} = \dfrac{kx^3\left(8r^2 - 3x^2\right)}{\sqrt{4r^2x^4 - x^6}}$	**> 0** since $\left(8r^2 - 3x^2\right) > 0$, $x^3 > 0$ and $k > 0$	**0**	**< 0** since $\left(8r^2 - 3x^2\right) < 0$, $x^3 > 0$ and $k > 0$
shape	/	—	\

Hence S is max i.e the beam is the strongest when the width $x = \sqrt{\frac{8}{3}}\,r$

Comments:

Common mistakes:

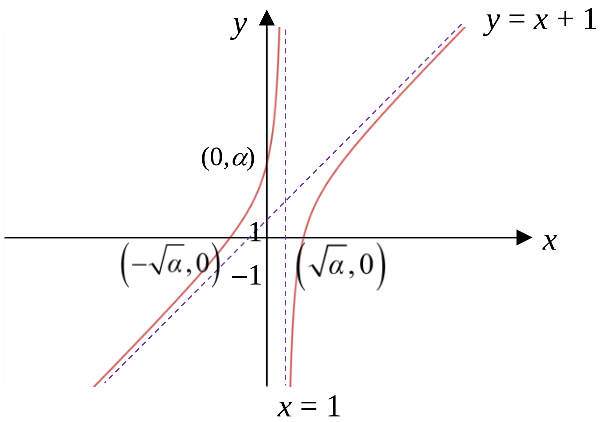
-Forget chain rule

-treat r as a variable instead of constant (see the word particular highlighted in blue)

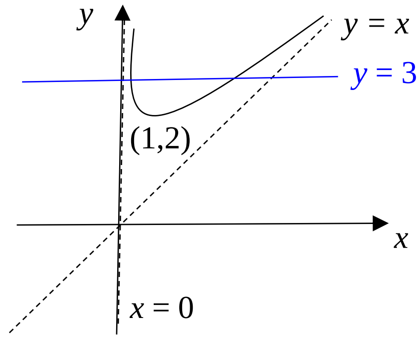
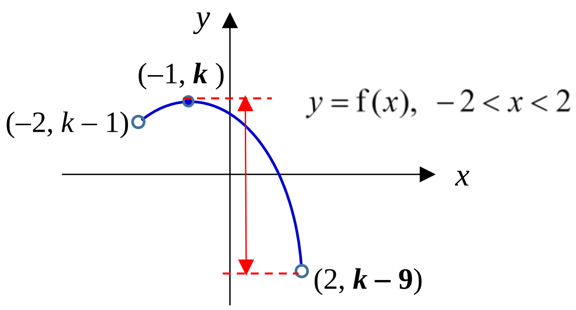
-careless mistakes in algebraic manipulation

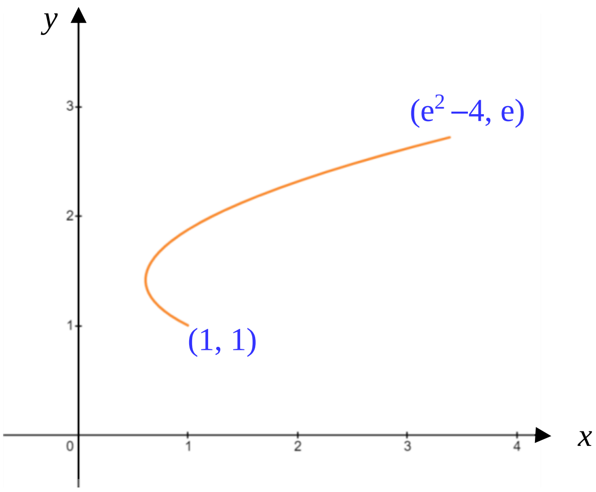
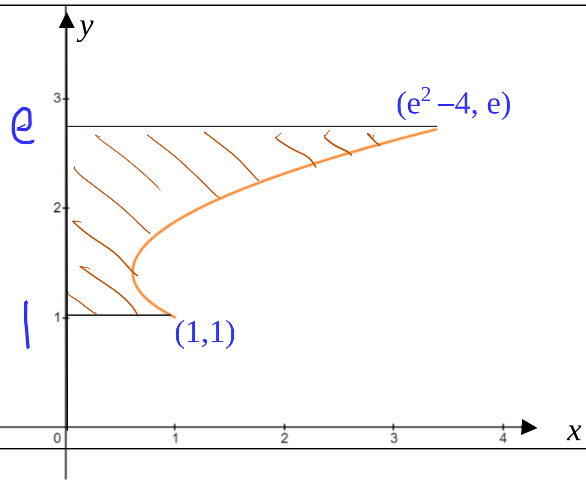
-Not enough explanation is given for using the first derivative test

Q4	[Solution]	Comments
(i)	Using sine rule, $\dfrac{y}{\sin\theta} = \dfrac{x}{\sin\left(\frac{5\pi}{6}-\theta\right)} = \dfrac{1}{\sin\frac{\pi}{6}}$. (Note that $\sin\frac{\pi}{6}=\frac{1}{2}$) $y = 2\sin\theta$ $x = 2\sin\left(\frac{5\pi}{6}-\theta\right)$	Note: Sum of angles in a triangle is π not 2π Use **sine rule** to find x and y, choose appropriate equality: $\dfrac{y}{\sin\theta}=2$ $\dfrac{x}{\sin\left(\frac{5\pi}{6}-\theta\right)}=2$
(ii)	$S=\frac{1}{2}(x)(1)\sin\theta=\frac{1}{2}\left(2\sin\left(\frac{5\pi}{6}-\theta\right)\right)\sin\theta=\sin\left(\frac{5\pi}{6}-\theta\right)\sin\theta$	There is no necessary to further breakdown $\sin\left(\frac{5\pi}{6}-\theta\right)$.
(iii)	At the instant when $\theta=\frac{\pi}{4}$, $\frac{dy}{dt}=-0.2$ $\dfrac{dy}{d\theta}=2\cos\theta$ Using $\dfrac{dy}{dt}=\dfrac{dy}{d\theta}\times\dfrac{d\theta}{dt}$, $-0.2=2\cos\frac{\pi}{4}\times\frac{d\theta}{dt}$ $\therefore \dfrac{d\theta}{dt}=-\dfrac{1}{5\sqrt{2}}$ or -0.141 $\dfrac{dS}{d\theta}=-\sin\theta\cos\left(\frac{5\pi}{6}-\theta\right)+\sin\left(\frac{5\pi}{6}-\theta\right)\cos\theta$ Using $\dfrac{dS}{dt}=\dfrac{dS}{d\theta}\times\dfrac{d\theta}{dt}$, $\dfrac{dS}{dt}=\left[-\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{7\pi}{12}\right)+\sin\left(\frac{7\pi}{12}\right)\cos\left(\frac{\pi}{4}\right)\right]\times\left(-\frac{1}{5\sqrt{2}}\right)$ $=-0.122$ Thus the rate of change of S is $-0.122\ \text{ms}^{-1}$	Given B is moving towards O $\Rightarrow$ y is decreasing. Therefore, $\frac{dy}{dt}$ is negative *Thinking:* We need to find $\frac{dS}{dt}$ and we have S in terms of θ. Therefore, we need to use $\dfrac{dS}{dt}=\dfrac{dS}{d\theta}\times\dfrac{d\theta}{dt}$ Hence, we must get $\frac{d\theta}{dt}$ first. Since we have $\frac{dy}{dt}$ and $\frac{dy}{d\theta}$, we can use find $\frac{d\theta}{dt}$ from $\dfrac{dy}{dt}=\dfrac{dy}{d\theta}\times\dfrac{d\theta}{dt}$.

Q5	[Solution]	Comments
(i)	$y=\dfrac{\alpha-x^2}{1-x}=\underline{x+1}+\dfrac{\alpha-1}{1-x}$ Equations of asymptotes: $x=1$, $\underline{y=x+1}$	Perform long division till the degree of the remainder is <u>less than</u> the degree of the divisor.
(ii)	$\dfrac{\mathrm{d}y}{\mathrm{d}x}=1+\dfrac{\alpha-1}{(1-x)^2}$ Given C has positive gradient for all $x\in\mathbb{R}$, $\dfrac{\mathrm{d}y}{\mathrm{d}x}>0$ $1+\dfrac{\alpha-1}{(1-x)^2}>0$ $\dfrac{(1-x)^2+\alpha-1}{(1-x)^2}>0$ $\dfrac{x^2-2x+\alpha}{(1-x)^2}>0$ Since $(1-x)^2>0$, $x^2-2x+\alpha>0$ for all $x\in\mathbb{R}$, $x\neq 1$ Discriminant = $2^2-4(1)(\alpha)<0$ $\therefore\ \alpha>1$	Students must differentiate y w.r.t. x correctly to obtain the inequality $x^2-2x+\alpha>0$ for all $x\in\mathbb{R}$, $x\neq 1$ Remember the objective is to find the range of value of α, not x.
(iii)		Note: Since the question requested for ‘the **coordinates** of the axial intercepts’, it is required to labelled the axial intercepts with the correct coordinates form. Also it must be shown that the graph approaches to the asymptotes appropriately.

Q6	[Solution]	Comments
(a)	$\int \sin px \sin qx \, \mathrm{d}x$ $= -\frac{1}{2}\int \cos(p+q)x - \cos(p-q)x \, \mathrm{d}x$ $= -\frac{1}{2}\left[\frac{\sin(p+q)x}{p+q} - \frac{\sin(p-q)x}{p-q}\right] + c$	• Use **Factor Formula** $\sin A \sin B = -\frac{1}{2}[\cos(A+B) - \cos(A-B)]$ • Be familiar with integration techniques $\int \cos(ax+b)\,\mathrm{d}x = \frac{1}{a}\sin(ax+b) + c$
(b)	$\int_{-\ln 2}^{0}\left(\frac{\mathrm{e}^{2x}}{\mathrm{e}^{2x}+1}\right)\mathrm{d}x = \frac{1}{2}\int_{-\ln 2}^{0}\left(\frac{2\mathrm{e}^{2x}}{\mathrm{e}^{2x}+1}\right)\mathrm{d}x$ $= \frac{1}{2}\left[\ln\left(\mathrm{e}^{2x}+1\right)\right]_{-\ln 2}^{0}$ $= \frac{1}{2}\left[\ln\left(\mathrm{e}^{0}+1\right) - \left(\mathrm{e}^{-2\ln 2}+1\right)\right]$ $= \frac{1}{2}\left(\ln 2 - \ln\left(\frac{1}{4}+1\right)\right)$ $= \frac{1}{2}\ln\frac{8}{5}$	$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{2x}+1\right) = 2\mathrm{e}^{2x}$ Use $\boxed{\int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)}\,\mathrm{d}x = \ln\lvert \mathrm{f}(x)\rvert + c}$ $\ln\lvert \mathrm{e}^{2x}+1\rvert = \ln\left(\mathrm{e}^{2x}+1\right)$ since $\mathrm{e}^{2x}+1>0$ Use result $\boxed{\mathrm{e}^{\ln x} = x}$ $\therefore\ \mathrm{e}^{-2\ln 2} = \mathrm{e}^{\ln(2^{-2})} = 2^{-2} = \frac{1}{4}$
(c)	$x-1 = A(1-2x) + B = -2Ax + (A+B)$ Equating coefficients, $-2A = 1 \Rightarrow A = -\frac{1}{2}$ $A+B = -1 \Rightarrow B = -1-A = -\frac{1}{2}$ $\therefore\ x-1 = -\frac{1}{2}(1-2x) - \frac{1}{2}$ $\int \frac{x-1}{\sqrt{6+x-x^2}}\,\mathrm{d}x$ $= \int \frac{-\frac{1}{2}(1-2x) - \frac{1}{2}}{\sqrt{6+x-x^2}}\,\mathrm{d}x$ $= -\frac{1}{2}\int (1-2x)\left(6+x-x^2\right)^{-\frac{1}{2}}\mathrm{d}x - \frac{1}{2}\int \frac{1}{\sqrt{6+x-x^2}}\,\mathrm{d}x$ $= -\frac{1}{2}\cdot\frac{\left(6+x-x^2\right)^{\frac{1}{2}}}{\frac{1}{2}} - \frac{1}{2}\int \frac{1}{\sqrt{\left(\frac{5}{2}\right)^2 - \left(x-\frac{1}{2}\right)^2}}\,\mathrm{d}x$ $= -\sqrt{6+x-x^2} - \frac{1}{2}\sin^{-1}\left(\frac{2x-1}{5}\right) + C$	Split into 2 integrals **using the suggested form** $\boxed{\int \mathrm{f}'(x)[\mathrm{f}(x)]^n\,\mathrm{d}x = \frac{[\mathrm{f}(x)]^{n+1}}{n+1} + c}$ Note that $\sqrt{6+x-x^2} = \sqrt{-1}\sqrt{x^2-x-6}$ **BUT** $\sqrt{-1}$ **is undefined.** $\boxed{\int \frac{1}{\sqrt{a^2-(px+q)^2}}\,\mathrm{d}x = \frac{1}{p}\sin\left(\frac{px+q}{a}\right) + c}$

Q7	[Solution]	Comments
(a)(i)	 Since the **horizontal line** $y = 3$ cuts the **graph of** $y = \mathrm{g}(x)$ twice, **g is not one-one**. Thus g does not have an inverse.	Your sketch should indicate the coordinates of the min point (1, 2). Give **proper argument**! Give **equation of a particular line** eg, $y = 3$ to show that g is not 1-1.
(a)(ii)	For g to be 1-1, $x \geq 1$, **Minimum value** of p is 1. Let $y = x + \frac{1}{x}$ $x^2 - yx + 1 = 0$ $x = \frac{y \pm \sqrt{y^2 - 4}}{2}$ $\boxed{\text{Since } x \geq 1, \quad x = \frac{y + \sqrt{y^2-4}}{2}}$ $\mathrm{g}^{-1}(x) = \frac{x + \sqrt{x^2-4}}{2}, \quad \mathrm{D}_{\mathrm{g}^{-1}} = \mathrm{R}_{\mathrm{g}} = [2, \infty)$	<u>Need to answer to the question</u> on what is the <u>min value</u> of p. Use **quadratic formula** to express x in terms of y. Always **state the reason** why we rejected one of the answers.
(b)	For hf to exist, $\mathrm{R_f} \subseteq \mathrm{D_h}$.  $\boxed{\mathrm{R_f} = (k-9, k]}$ $\mathrm{D_h} = (0, \infty)$ Hence, the **minimum value** of k is 9.	**Sketch the graph of f to obtain the range of f** (for the smallest and largest value of y) in the given domain. Unless the function is strictly increasing or decreasing, the range <u>should not</u> be obtained by substituting the min/max x-values. <u>Need to answer to the question</u> on what is the <u>min value</u> of k.

Q8	[Solution]	Comments
(a)	$\int xe^{\frac{1}{2}x}\,dx = x\left(2e^{\frac{1}{2}x}\right) - \int 2e^{\frac{1}{2}x}\,dx$ $= 2xe^{\frac{1}{2}x} - 4e^{\frac{1}{2}x} + c$ or $2e^{\frac{1}{2}x}(x-2)+c$	Integration by parts: ensure you know how to decide which term to integrate and which term to differentiate. Be careful of careless mistakes such as 1) writing '+' instead of '–' 2) forgetting constant of integration ' $+c$ '
(b)(i)		Ensure you answer the question by giving the **coordinates** of the **end-points** in **exact form.**
(ii)	$x = e^t - 2t$, $y = e^{\frac{1}{2}t}$ Tangent is parallel to y-axis: $\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\frac{1}{2}e^{\frac{1}{2}t}}{e^t - 2} = \frac{e^{\frac{1}{2}t}}{2(e^t-2)}$ is undefined $2(e^t - 2) = 0$ $e^t = 2$ $t = \ln 2$ $\therefore\ x = e^{\ln 2} - 2\ln 2 = 2 - 2\ln 2$	When the tangent is parallel to the y-axis, we have a 'vertical line', i.e. the gradient approaches $\pm\infty$. i.e. put denominator of $\frac{dy}{dx}$ to 0 to solve for t. Do not confuse with tangents parallel to the x-axis where we put $\frac{dy}{dx} = 0$.
(iii)		

	When $y = \mathrm{e}$, $\mathrm{e} = \mathrm{e}^{\frac{1}{2}t} \Rightarrow 1 = \frac{1}{2}t \Rightarrow t = 2$ When $y = 1$, $1 = \mathrm{e}^{\frac{1}{2}t} \Rightarrow \ln 1 = \frac{1}{2}t \Rightarrow t = 0$ Area $= \int_1^{\mathrm{e}} x \, \mathrm{d}y$ $= \int_0^2 \left(\mathrm{e}^t - 2t\right)\left(\frac{1}{2}\mathrm{e}^{\frac{1}{2}t}\right)\mathrm{d}t$ $= \int_0^2 \left(\frac{1}{2}\mathrm{e}^{\frac{3}{2}t} - t\,\mathrm{e}^{\frac{1}{2}t}\right)\mathrm{d}t$ $= \frac{1}{2} \times \frac{2}{3}\left[\mathrm{e}^{\frac{3}{2}t}\right]_0^2 - \left[2\mathrm{e}^{\frac{1}{2}t}(t-2)\right]_0^2$ using **(a)** $= \frac{1}{3}\left(\mathrm{e}^3 - 1\right) - 2\left[0 - (-2)\right]$ $= \frac{1}{3}\left(\mathrm{e}^3 - 13\right)$ units^2	Students to be careful to **use the correct formula** $\int x \, \mathrm{d}y$ or $\int y \, \mathrm{d}x$ as using the wrong formula from the start may mean not being awarded any mark at all. For areas involving parametric equations, use **integration by substitution** to find the integral. There is no need to find the cartesian equation. DO NOT FORGET to change the limits to values of t as well. Students can **make use of the result in (a)** without having to integrate by parts again.

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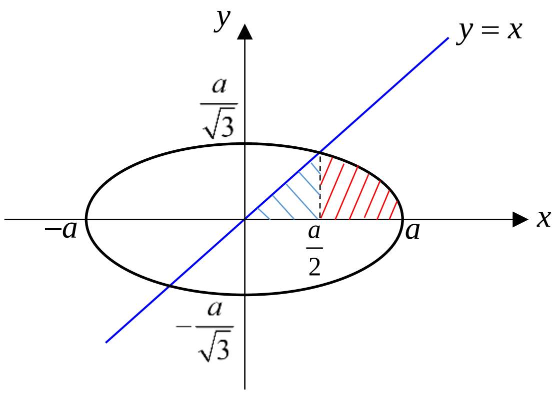
Q9(i)



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Note: the line $y = x$ is for part (ii) only.
Remember to always label all vertices (calculate the coordinates carefully!), and try to draw a symmetrical diagram.

$$x^2+3y^2=a^2 \Rightarrow y=\pm\frac{\sqrt{a^2-x^2}}{\sqrt{3}}$$

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(ii) Area $= \frac{1}{2}\left(\frac{a}{2}\right)\left(\frac{a}{2}\right)+\frac{1}{\sqrt{3}}\int_{\frac{a}{2}}^{a}\sqrt{a^2-x^2}\,\mathrm{d}x$

$$= \frac{a^2}{8}+\frac{1}{\sqrt{3}}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\sqrt{a^2-a^2\sin^2\theta}\,(a\cos\theta)\,\mathrm{d}\theta$$

$$= \frac{a^2}{8}+\frac{1}{\sqrt{3}}a^2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\cos^2\theta\ \mathrm{d}\theta$$

$$= \frac{a^2}{8}+\frac{a^2}{2\sqrt{3}}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}(\cos 2\theta+1)\ \mathrm{d}\theta$$

$$= \frac{a^2}{8}+\frac{a^2}{2\sqrt{3}}\left[\frac{1}{2}\sin 2\theta+\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \frac{a^2}{8}+\frac{a^2}{2\sqrt{3}}\left[\frac{\pi}{2}-\left(\frac{\sqrt{3}}{4}+\frac{\pi}{6}\right)\right]$$

$$= \frac{\pi a^2}{6\sqrt{3}}$$

Thus, $k=\frac{1}{6\sqrt{3}}$

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At intersection: sub $y = x$ into $x^3 + 3y^2 = a^2 \Rightarrow 4x^2 = a^2$.
Since $x > 0$, $x=\frac{1}{2}a$ and $y=\frac{1}{2}a$
Use the diagram in (i) to see that the first portion on the left is a triangle (use $\frac{1}{2}\times$ base $\times$ height), and to ensure that the limits of integration are correct.

When $x=\frac{1}{2}a$, $\frac{1}{2}=\sin\theta \Rightarrow \theta=\frac{\pi}{6}$
When $x=a$, $1=\sin\theta \Rightarrow \theta=\frac{\pi}{2}$

A lot of careless seen here with the integral of $\cos 2\theta$ and also using +/− signs.

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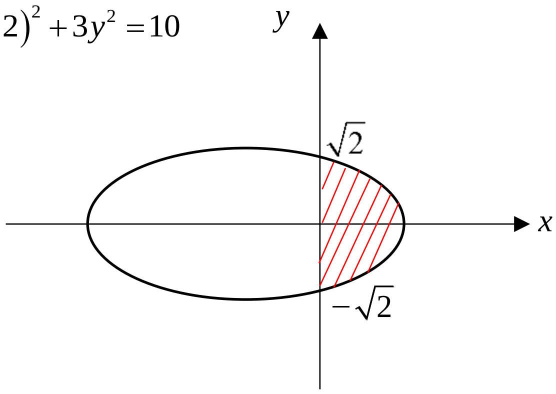
(iii) $(x+2)^2+3y^2=10$



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It is given that $a=\sqrt{10}$, so do not continue using a in the equation.

$(x+2)^2+3y^2=10$
$\Rightarrow x=-2\pm\sqrt{10-3y^2}$
Since $x > 0$, $x=-2+\sqrt{10-3y^2}$
When $y = 0$, $x=\pm\sqrt{2}$

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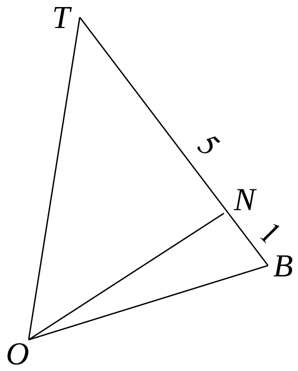
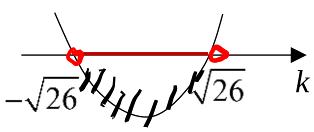
Volume of solid formed (= Volume of solid formed when the region between $(x+2)^2+3y^2=10$ and the <u>*y*-axis</u> is rotated about the <u>*y*-axis</u>) $= \pi \int_{-\sqrt{2}}^{\sqrt{2}} x^2 \mathrm{d}y$ or $2\times\pi\int_0^{\sqrt{2}} x^2 \mathrm{d}y$ $= 2\pi\int_0^{\sqrt{2}} \left(\left(-2+\sqrt{10-3y^2}\right)^2\right) \mathrm{d}y$ = 6.80 units^3	

Q10	[Solution]	Comments
(i)	$\overrightarrow{OB}=\begin{pmatrix}2\\-1\\0\end{pmatrix}$, $\overrightarrow{OA}=\begin{pmatrix}-2\\-1\\0\end{pmatrix}$ Since $\overrightarrow{AB}$ is in the direction of $\mathbf{i}$, a vector equation of the line AB is $\underset{\sim}{r}=\begin{pmatrix}2\\-1\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\0\\0\end{pmatrix}$, $\lambda\in\mathbb{R}$	Observe the direction of $\mathbf{i}$ and $\mathbf{j}$ carefully: For example, the y-coordinate of B is negative not positive. Also, $\overrightarrow{AB}$ is in the direction of $\mathbf{i}$ and not $-\mathbf{i}$.
(ii)	$\overrightarrow{ON}=\dfrac{5\overrightarrow{OB}+\overrightarrow{OT}}{6}$ $=\dfrac{1}{6}\left[5\begin{pmatrix}2\\-1\\0\end{pmatrix}+\begin{pmatrix}0\\0\\5\end{pmatrix}\right]$ $=\dfrac{1}{6}\begin{pmatrix}10\\-5\\5\end{pmatrix}$  Since P lies on line AB, $\overrightarrow{OP}=\begin{pmatrix}2\\-1\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\0\\0\end{pmatrix}$ for some $\lambda\in\mathbb{R}$ $\overrightarrow{PN}=\dfrac{1}{6}\begin{pmatrix}10\\-5\\5\end{pmatrix}-\left[\begin{pmatrix}2\\-1\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\0\\0\end{pmatrix}\right]=\begin{pmatrix}-\frac{1}{3}\\\frac{1}{6}\\\frac{5}{6}\end{pmatrix}-\lambda\begin{pmatrix}1\\0\\0\end{pmatrix}$ $\overrightarrow{TB}=\begin{pmatrix}2\\-1\\0\end{pmatrix}-\begin{pmatrix}0\\0\\5\end{pmatrix}=\begin{pmatrix}2\\-1\\-5\end{pmatrix}$	Students must learn to use **ratio theorem** to find $\overrightarrow{ON}$ as this can help to save a lot of time during exam. DO NOT use $\overrightarrow{OP}=\begin{pmatrix}x\\y\\z\end{pmatrix}$ unless you recognise that in this case the y-coordinate of P is -1 and z-coordinate of P is 0, i.e. $\overrightarrow{OP}=\begin{pmatrix}x\\-1\\0\end{pmatrix}$.

	$\overrightarrow{PN} \perp \overrightarrow{TB} \Rightarrow \overrightarrow{PN} \cdot \overrightarrow{TB} = 0$ $\Rightarrow \left[\begin{pmatrix} -\frac{1}{3} \\ \frac{1}{6} \\ \frac{5}{6} \end{pmatrix} - \lambda \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right] \cdot \begin{pmatrix} 2 \\ -1 \\ -5 \end{pmatrix} = 0$ $-5 - 2\lambda = 0 \Rightarrow \lambda = -\frac{5}{2}$ $\overrightarrow{OP} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} - \frac{5}{2}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ -1 \\ 0 \end{pmatrix}$ or $-\frac{1}{2}\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ **Alternative method:** Since P lies on line AB (y-coordinate is -1) and is on the horizontal plane $ABCD$ (so z-coordinate is 0), we can let $\overrightarrow{OP} = \begin{pmatrix} x \\ -1 \\ 0 \end{pmatrix}$. $\overrightarrow{PN} = \frac{1}{6}\begin{pmatrix} 10 \\ -5 \\ 5 \end{pmatrix} - \begin{pmatrix} x \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{5}{3} - x \\ \frac{1}{6} \\ \frac{5}{6} \end{pmatrix}$ $\overrightarrow{PN} \perp \overrightarrow{TB} \Rightarrow \overrightarrow{PN} \cdot \overrightarrow{TB} = 0$ $\Rightarrow \begin{pmatrix} \frac{5}{3} - x \\ \frac{1}{6} \\ \frac{5}{6} \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ -5 \end{pmatrix} = 0$ $\frac{10}{3} - 2x - \frac{1}{6} - \frac{25}{6} = 0$ $x = -\frac{1}{2}$, $\therefore \overrightarrow{OP} = \begin{pmatrix} -\frac{1}{2} \\ -1 \\ 0 \end{pmatrix}$	

(iii)	Since $\overrightarrow{BC}$ // $\mathbf{j}$, A normal to plane TBC is $\overrightarrow{BC}\times\overrightarrow{TB}=\begin{pmatrix}0\\1\\0\end{pmatrix}\times\begin{pmatrix}2\\-1\\-5\end{pmatrix}=\begin{pmatrix}5\\0\\2\end{pmatrix}$ $\underset{\sim}{r}\cdot\begin{pmatrix}5\\0\\2\end{pmatrix}=\begin{pmatrix}0\\0\\5\end{pmatrix}\cdot\begin{pmatrix}5\\0\\2\end{pmatrix}=10$ A Cartesian equation of the plane TBC is $5x+2z=10$	**Remember to simplify your Cartesian equation (remove common factor).**
(iv)	$l:\ \dfrac{x+2}{5}=\dfrac{y}{1}=\dfrac{z}{k}$ $\boxed{\text{You must define the symbol } \theta \text{ or } \phi \text{ that you use.}}$ Let θ be the acute angle between l and the horizontal plane with equation $\mathbf{r}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}=0$. $\sin\theta=\dfrac{\left\|\begin{pmatrix}5\\1\\k\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}\right\|}{\sqrt{25+1+k^2}\sqrt{1}}=\dfrac{k}{\sqrt{26+k^2}}$ Since sine function is an **increasing function for acute angles**, $0^\circ<\theta<45^\circ \Rightarrow 0<\sin\theta<\sin 45^\circ$ $0<\dfrac{k}{\sqrt{26+k^2}}<\dfrac{\sqrt{2}}{2}$ $2k<\sqrt{2}\sqrt{26+k^2}$ $4k^2<52+2k^2$ $k^2<\dfrac{52}{2}=26$ $\left(k-\sqrt{26}\right)\left(k+\sqrt{26}\right)<0$  $-\sqrt{26}<k<\sqrt{26}$ Since $k>0$, $\therefore 0<k<\sqrt{26}$	Note that when $\dfrac{x+2}{5}=y=\dfrac{z}{k}=\mu$, we have $x=-2+5\mu$ $y=0+1\mu$ $z=0+k\mu$ So direction vector of line l is $\begin{pmatrix}5\\1\\k\end{pmatrix}$. It is NOT $\begin{pmatrix}5\\0\\k\end{pmatrix}$. Note that $k<\pm\sqrt{26}$ is WRONG!

Alternatively method 1

Let ϕ be the acute angle between l and the normal $\mathbf{k}$ to the horizontal plane.

$$\cos\phi = \frac{\left|\begin{pmatrix}5\\1\\k\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}\right|}{\sqrt{25+1+k^2}\sqrt{1}} = \frac{k}{\sqrt{26+k^2}}$$

Since cosine function is a **decreasing function for acute angles,**

$45^\circ < \phi < 90^\circ \Rightarrow\ 0 < \cos\phi < \cos 45^\circ$

$$0 < \frac{k}{\sqrt{26+k^2}} < \frac{\sqrt{2}}{2}$$

$$2k < \sqrt{2}\sqrt{26+k^2}$$

$$4k^2 < 52 + 2k^2$$

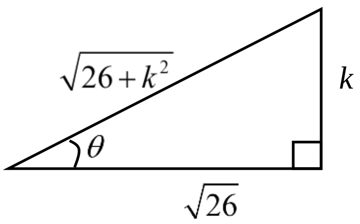
$$k^2 < \frac{52}{2} = 26$$

$$\left(k-\sqrt{26}\right)\left(k+\sqrt{26}\right) < 0$$

Since $k > 0, \quad 0 < k < \sqrt{26}$

Alternative Method 2

Since $\sin\theta = \dfrac{k}{\sqrt{26+k^2}}$, using the right-angled triangle



Since **tangent function is an increasing function for acute angles**.

$$0^\circ < \theta < 45^\circ$$

$$\Rightarrow 0 < \tan\theta < \tan 45^\circ$$

$$\therefore\quad 0 < \frac{k}{\sqrt{26}} < 1$$

$$\Rightarrow 0 < k < \sqrt{26}$$

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Q11(a)(i) n^{th} term, $\boxed{u_n = S_n - S_{n-1}}$

$$= \frac{2}{3}\left(1-\frac{1}{k^n}\right) - \frac{2}{3}\left(1-\frac{1}{k^{n-1}}\right)$$

$$= \frac{2}{3}\left(\frac{1}{k^n}\frac{1}{k^{-1}} - \frac{1}{k^n}\right) = \frac{2}{3}\left(\frac{k}{k^n} - \frac{1}{k^n}\right)$$

$$= \frac{2(k-1)}{3}\left(\frac{1}{k}\right)^n$$

$$\frac{u_n}{u_{n-1}} = \frac{\frac{2(k-1)}{3}\left(\frac{1}{k}\right)^n}{\frac{2(k-1)}{3}\left(\frac{1}{k}\right)^{n-1}} = \left(\frac{1}{k}\right)^{n-(n-1)} = \frac{1}{k} \qquad (k \neq 0)$$

which is a constant independent of *n*.

Hence the sequence is geometric with

common ratio $r = \frac{1}{k}$ and

first term, $u_1 = \frac{2(k-1)}{3}\left(\frac{1}{k}\right)^1 = \frac{2(k-1)}{3k}$

or $u_1 = S_1 = \frac{2}{3}\left(1-\frac{1}{k}\right)$

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To prove the existence of a GP, we need to show that the ratio of $\frac{u_n}{u_{n-1}}$ or $\frac{u_{n+1}}{u_n}$ is a constant independent of *n*. This means that the working should start in terms of *n* and ultimately show that ratio obtained does not contain *n*.

Showing $\frac{u_2}{u_1} = \frac{u_3}{u_2}$ is insufficient as it merely shows that the first 3 terms are geometric.

It is a good practice to simplify where possible e.g. it is more desirable to simplify $r = \frac{k+1}{k^2+k} = \frac{k+1}{k(k+1)} = \frac{1}{k}$

Many candidates missed out the last sub-part of finding u_1. When dealing with a question with many sub-parts that are not explicitly labelled, it is important to tick off a part when the solution to that part has been completed. This will minimize the likelihood of missing out on a sub-part.

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(a)(ii)

The new series $\frac{1}{u_1} + \frac{1}{u_2} + \ldots + \frac{1}{u_n}$ is geometric with

common ratio = $\frac{1}{\frac{1}{k}} = k$

Sum to infinity exists when $|k| < 1,\ k \neq 0$,

$\Rightarrow\ -1 < k < 1,\ k \neq 0$

First term = $\frac{1}{u_1} = \frac{3k}{2(k-1)}$

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When the reciprocal of the terms of a given GP are used to form another GP in the same successive order, the common ratio will be the reciprocal of the common ratio of the original GP. Hence this can be stated as a deduction directly instead incurring unnecessary working to find the common ratio.

In addition, since the common ratio is derived from the original common ratio, it must obey all conditions (both explicit and inherent) of the original ratio.

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Sum to infinity, $S_\infty = \dfrac{\frac{3k}{2(k-1)}}{1-k}$ $= \dfrac{3k}{2(k-1)} \times \dfrac{1}{1-k}$ $= -\dfrac{3k}{2(1-k)^2}$	
(b)(i) $S_n = \frac{1}{3}\left(S_{2n} - S_n\right)$ $4S_n = S_{2n}$ $4 \times \frac{n}{2}\left(2a + (n-1)d\right) = \frac{2n}{2}\left(2a + (2n-1)d\right)$ $4na + 2n^2d - 2nd = 2na + 2n^2d - nd$ $2na = nd$ $2a = d$ (shown)	Simplify $S_n = \frac{1}{3}\left(S_{2n} - S_n\right)$ to $4S_n = S_{2n}$ will make subsequent working easier.
(b)(ii) Subst $d = 2a$ into S_n: $\frac{n}{2}\left(2a + (n-1)2a\right) = 98$ $n^2a = 98$ Since a and n are positive integers, $a = 2$, $n = 7$	By guess and check