TEMASEK JUNIOR COLLEGE

2021 JC 1 H2 Mathematics Promotional Examination

1 **(a)** Describe a sequence of transformations which would transform a curve $y = \mathrm{f}(x)$ onto the curve $y = \mathrm{f}(1-2x)$. [3]

(b) Sketch the graph of $y = \left|2 - 3\mathrm{e}^{x}\right|$, showing the relevant features. [2]

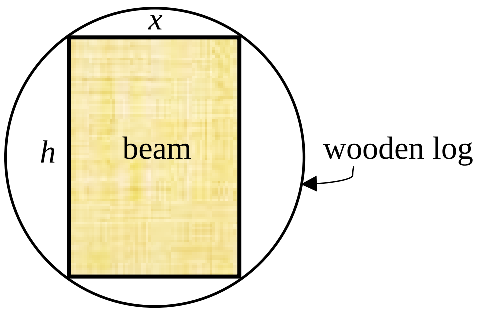
Hence find the exact range of values of x that satisfy the inequality $\left|2-3\mathrm{e}^{x}\right| > \mathrm{e}^{2x}$, for $x \geq 0$. [4]

2 The position vectors of the points A and B relative to O are $\mathbf{a}$ and $\mathbf{b}$ respectively, where $\mathbf{a}$ and $\mathbf{b}$ are non-zero and non-parallel vectors. The point C, with position vector $\mathbf{c}$, is the reflection of A in the line OB. Given that $\mathbf{b}$ is a unit vector,

(i) show that $\mathbf{c} = 2(\mathbf{a}.\mathbf{b})\mathbf{b} - \mathbf{a}$, [2]

(ii) show that $\mathbf{c} \times \mathbf{a} = 2(\mathbf{a}.\mathbf{b})(\mathbf{c} \times \mathbf{b})$. [2]

3 In the Emazon forest, trees are chopped down and the wooden logs are transported to the timber factory where they are cut into beams. In the figure shown below, it is assumed that the cross-section of each wooden log is a circle. Each beam is cut such that the cross-section is a rectangle and the 4 corners of the rectangle touch the circumference of the log to reduce wastage. It is given that the strength, S, of a beam is proportional to the product of its height h and the square of its width x.

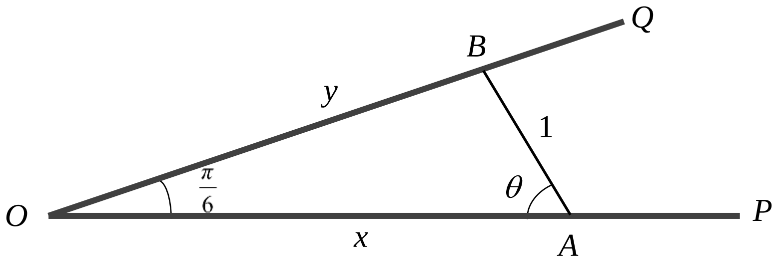


A particular wooden log of radius r has been sent for cutting.

(i) Show that $S = kx^2\sqrt{4r^2 - x^2}$, where k is a real constant. [2]

(ii) Using differentiation, find the exact width, in terms of r, of the strongest beam that can be cut from the log. [5]

4 In the figure (not drawn to scale), POQ is a rail and $\angle POQ = \frac{\pi}{6}$. AB is a rod of fixed length 1 m which is free to slide on the rail with end A on OP and end B on OQ. It is given that $OA = x$ m, $OB = y$ m and $\angle OAB = \theta$ radians.



(i) Express x and y in terms of θ. [2]

(ii) Given that S is the area of triangle OAB, find S in terms of θ. [1]

Given that B is moving towards O at a rate of 0.2 ms^{-1} at the instant when $\theta = \frac{\pi}{4}$,

(iii) find the rate of change of S at this instant. [4]

5 A curve C has equation $y = \dfrac{\alpha - x^2}{1 - x}$ where α is a real constant and $\alpha \neq 1$.

(i) Find the equations of the asymptotes of C. [2]

It is given that C has positive gradient for all $x \in \mathbb{R}$.

(ii) Find the range of values of α. [4]

(iii) Sketch C, giving the equations of the asymptotes and the coordinates of the axial intercepts. [2]

6 **(a)** Find $\int \sin px \sin qx \, dx$, where p and q are real constants. [2]

(b) Find the exact value of $\int_{-\ln 2}^{0} \left(\dfrac{e^{2x}}{e^{2x} + 1} \right) dx$, giving your answer as a single logarithm.

[2]

(c) By considering $x - 1 = A(1 - 2x) + B$, where A and B are constants to be determined, find $\displaystyle\int \frac{x - 1}{\sqrt{6 + x - x^2}} \, dx$. [5]

7 **(a)** The function g is defined by

$$\mathrm{g}: x \mapsto x + \frac{1}{x}, \quad x > 0.$$

(i) Explain why g does not have an inverse. [2]

(ii) If the domain of g is restricted to the subset of $\mathbb{R}$ for which $x \geq p$, find the minimum value of p such that g^{-1} exists. Using this value of p, find an expression for $\mathrm{g}^{-1}(x)$, stating the domain. [5]

(b) The functions f and h are defined by

$$\mathrm{f}: x \mapsto k - (x+1)^2, \ -2 < x < 2 \text{ and } k \text{ is a constant,}$$
$$\mathrm{h}: x \mapsto \ln(x+3), \quad x > 0.$$

Find the minimum value of k such that the composite function hf exists. [2]

8 **(a)** Find $\int x\mathrm{e}^{\frac{1}{2}x}\,\mathrm{d}x$. [2]

(b) A curve has parametric equations

$$x = \mathrm{e}^t - 2t, \quad y = \mathrm{e}^{\frac{1}{2}t}, \quad \text{where } 0 \leq t \leq 2.$$

(i) Sketch the curve, giving the exact coordinates of the end-points. [2]

(ii) Find the exact x-coordinate of the point on the curve where the tangent to the curve is parallel to the y-axis. [4]

(iii) Find the exact area of the region bounded by the curve, the lines $y = 1$ and $y = \mathrm{e}$, and the y-axis. [4]

9 An ellipse C has equation $x^2 + 3y^2 = a^2$ where a is a positive constant.

(i) Sketch C. [1]

(ii) The region enclosed by C, the line $y = x$ and the positive x-axis is denoted by R.

By using the substitution $x = a\sin\theta$, find the exact area of R in the form $k\pi a^2$ where k is a constant to be determined. [7]

It is given that $a = \sqrt{10}$. The region enclosed by C and the line $x = 2$, where $x > 0$ is denoted by S.

(iii) Write down the equation of the curve obtained when C is translated 2 units in the negative x-direction. Hence or otherwise, find the volume of the solid formed when S is rotated through 2π radians about the line $x = 2$, giving your answer to 3 significant figures. [4]

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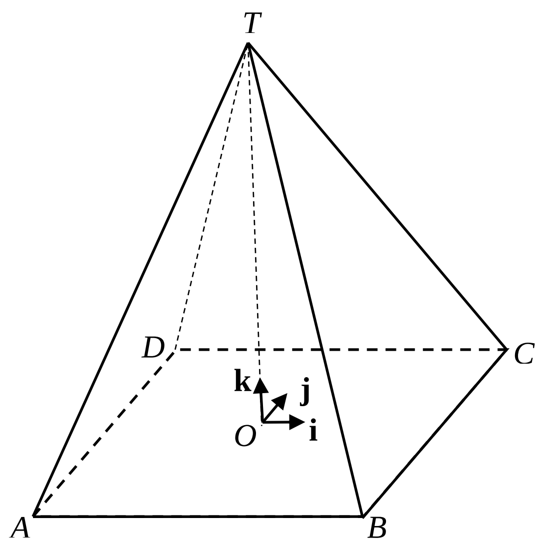

A pyramid has a horizontal rectangular base *ABCD*, where *AB* = 4 units and *AD* = 2 units. The vertex *T* is 5 units vertically above the centre of the base, *O*. Perpendicular unit vectors **i**, **j** and **k** are parallel to *AB, AD* and *OT* respectively. The point *N* is on *TB* such that *TN* : *NB* = 5 : 1.

(i) Find a vector equation of the line *AB*. [1]

(ii) *P* is a point on *AB* such that *PN* is perpendicular to *TB*. Find the position vector of *P*. [4]

(iii) Find a cartesian equation of the plane *TBC*. [3]

A line *l* has cartesian equation $\frac{x+2}{5} = y = \frac{z}{k}$, where *k* is a positive real constant.

(iv) Find the exact range of values of *k* such that *l* is inclined at an angle of less than 45° to the horizontal plane. [3]

11 **(a)** The sum, S_n, of the first *n* terms of a sequence $u_1, u_2, \ldots, u_n, \ldots$ is given by $\frac{2}{3}\left(1-\frac{1}{k^n}\right)$, where *k* is a constant and $k \neq 1$.

(i) Show that the sequence is a geometric progression, and state the values of the common ratio and the first term in terms of *k*. [4]

(ii) The sum of the first *n* terms of another geometric series is given by $\frac{1}{u_1}+\frac{1}{u_2}+\ldots+\frac{1}{u_n}$. State the range of values of *k* for the sum to infinity of this series to exist and find the sum to infinity in terms of *k*. [3]

(b) An arithmetic progression has first term *a* and common difference *d*. The sum of the first *n* terms is one-third the sum of the next *n* terms of the arithmetic progression.

(i) Show that $d = 2a$. [3]

It is also given that *a* and *d* are positive integers.

(ii) Hence find the value of *n* if the sum of the first *n* terms is 98. [2]