

2021 TJC Prelim Paper 2

Section A: Pure Mathematics [40 marks]

- 1 The functions f and g are defined by

$$f : x \mapsto 2 - (x - 2)^2 \quad \text{for } x \leq 0, x \in \mathbb{R},$$

and

$$g : x \mapsto \begin{cases} 4 & \text{for } x \geq 3, x \in \mathbb{R} \\ x + 1 & \text{for } x < 3, x \in \mathbb{R} \end{cases}$$

- (i) Define f^{-1} in a similar form. [3]
(ii) Explain why g does not have an inverse. [2]
(iii) Find the range of values of α such that $fg(\alpha)$ exists. [2]
- 2 It is given that $f(r) = \frac{1}{r!}$ where r is a positive integer.

(i) Show that $f(r) - f(r+1) = \frac{r}{(r+1)!}$. [1]

- (ii) The sum of the first n terms of the series $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \dots$ is given by S_n . Find S_n and explain why the series converges. [4]

- (iii) By considering (ii) and using the standard series from the List of Formulae (MF26), find the exact value of $\sum_{r=2}^{\infty} \frac{r+2}{r!}$. [4]

- 3 (a) In the triangle ABC , $AC = BC = 1$ and angle $ACB = \left(\frac{1}{2}\pi + \theta\right)$ radians. Given that θ is sufficiently small angle, show that

$$AB \approx \sqrt{2} + \frac{\sqrt{2}}{2}\theta - \frac{\sqrt{2}}{8}\theta^2 \quad [5]$$

- (b) Let $f(x) = \ln(2-x)$.

- (i) Write down the first three non-zero terms of the Maclaurin series for $f(x)$, where $-2 \leq x < 2$. [3]

Denote the answer to part (i) by $g(x)$.

- (ii) Find the range of values of x for which the percentage error between the value of $g(x)$ and the value of $f(x)$ is within 10%. [2]

- 4 Relative to the origin O , the position vectors A and B are \mathbf{a} and \mathbf{b} respectively, where \mathbf{a} and \mathbf{b} are non-parallel vectors. The point C is the mid-point of OA and the point D is on the line segment AB produced such that $AD = 3AB$. The line CD meets OB at a point E .

- (i) Find the position vector of E in terms of \mathbf{b} . [4]
 (ii) The point F divides the line segment OD in the ratio 1:2. Show that A , E and F are collinear, and find $AE:AF$. [4]

Given that the vector \mathbf{a} is a unit vector perpendicular to $\mathbf{a} + 2\mathbf{b}$ and angle AOB is $\frac{5\pi}{6}$.

- (iii) Show that $|\mathbf{b}| = \frac{1}{\sqrt{3}}$. [2]

- (iv) Find the exact area of triangle ACE . [4]

Section B: Statistics [60 marks]

- 5 For the events A , B and C it is given that, $P(A) = p$, $P(B) = 0.4$, $P(A \cap C) = 0.2$ and $P(A \cap B \cap C) = P(A' \cap B' \cap C') = 0$.

It is also given that events A and B are independent, find the range of values of $P(A \cap B)$. [5]

- 6 Correlation and Regression [8]

- 7 A manufacturer claims that the mean lifetime of the Alpha light bulbs he produces is at least 1200 hours. A random sample of 80 Alpha light bulbs is taken and the mean lifetime is found to be 1198.6 hours.

Assuming that the lifetime, T hours, of the Alpha light bulbs is normally distributed with variance 35 hours^2 , test the manufacturer's claim at the 2.5% level of significance. You should state your hypotheses and define any symbols that you use. [4]

It is given that the mean lifetime of the Alpha light bulbs is 1200 hours. The manufacturer produces a new type of light bulbs known as Beta light bulbs. He claims a Beta light bulb has twice the mean lifetime of an Alpha light bulb. A random sample of 50 Beta light bulbs is taken, where the mean and standard deviation of the lifetime of this sample are k hours and 9.8 hours respectively. A test at the 5% significance level indicates that the manufacturer's claim is not valid for this sample of Beta light bulbs. Find the range of values of k , giving your answer correct to 2 decimal places. [5]

- 8 In a Junior College, it is known that the probability that a student is left-handed is 0.125. There are 25 students in a class. The number of students in a randomly chosen class who are left-handed is denoted by L .

- (i) State, in context of this question, one assumption needed to model L by a binomial distribution. Hence find the probability that there are at most 3 students who are left-handed in a randomly chosen class. [2]
- (ii) Find the probability that the number of students who are right-handed in a randomly chosen class is more than 85% of the class. [2]
- (iii) A group of students is randomly chosen from a class. Find the probability that the fifth student chosen is the third student who is left-handed. [3]
- (iv) There are n classes in the college. Find the minimum number of classes in the college such that the probability of having not more than 4 classes with at most 3 students who are left-handed in each class is less than 0.005. [3]

- 9 A random variable X has probability distribution given by

$$P(X = x) = \begin{cases} kx(6-x), & x = 1, 2, 3, 4, 5 \\ 0, & \text{otherwise.} \end{cases}$$

- (i) Show that $k = \frac{1}{35}$. [1]
- (ii) Find the mean and variance of X . [3]

40 independent observations of X are taken.

- (iii) Find the probability that the sum of the 40 observations of X exceeds 110. [3]
- (iv) Find the probability that the maximum value observed is 5. [3]

- 10 A deck of twenty cards comprises four sets of coloured cards, namely yellow, red, blue and green cards. Each set is made up of five different numbered cards, from “1” to “5”.

- (i) Four cards are randomly dealt to every player, and there are a total of five players. In how many different ways can the cards be dealt so that a particular player has only red cards? [2]
- (ii) Ten cards consisting of a set of yellow cards and a set of blue cards are arranged in a circle. Find the probability that all cards numbered “1” are next to each other and separated from the cards that are numbered “2”. [3]
- (iii) Three cards are drawn at random from the deck without replacement.
 - (a) Find the probability that the three cards have the same number. [2]
 - (b) Find the probability that at least two cards are of the same colour. [2]
 - (c) Find the probability that three cards are of green colour given that at least two cards are of the same colour. [3]

- 11** A group of ornithologists in a nature reserve is conducting a research on three endemic species of birds. They will tag and measure the beak lengths of the birds they have caught. After a large number of birds were tagged and measured, they found the beak lengths of the three species of birds have independent normal distributions. The table below gives the mean and standard deviation of the beak length of each species.

Species	Mean	Standard Deviation
A	40 mm	5 mm
B	60 mm	5 mm
C	40 mm	σ mm

Let A , B and C be the random variables, in mm, denoting the beak lengths of birds of species A, B and C respectively.

- (i) The probability of a randomly selected bird of species A having a beak length at least l mm is at least 0.9. Find the range of values of l , correct to 1 decimal place. [2]
- (ii) Find the value of k such that $P(A \leq k) = P(B \geq k)$. [2]
- (iii) One ornithologist randomly selected 3 birds of species A and 1 bird of species B. Find the probability that the difference between the total beak lengths of the birds of species A and twice the beak length of the bird of species B is at least 2 mm. [4]

It is found that 73.9% of birds of species C have beak lengths greater than 33.6 mm.

- (iv) Find the value of σ . [2]
- (v) Another ornithologist randomly selected a sample of 4 birds of species A and n birds of species C. After doing the measurements, he reported that the probability of exactly 3 birds of species A and 5 birds of species C from the sample, each has a beak length more than 33.6 mm is at least 0.0015. Find the maximum number of birds of species C in the sample. [4]