2021 TJC Prelim Paper 1

1 The curve with equation $y = p + \dfrac{1}{x+q}$, where p and q are constants, is transformed by a scaling of factor 4 parallel to the x-axis, followed by a translation of 3 units in the negative y-direction, followed by a reflection about the y-axis.
The resulting curve has asymptotes $x = 2$ and $y = 1$. Find the values of p and q. [5]

2 **(i)** Sketch the graphs of $y = \mathrm{e}^{-x}$ and $y = \ln(x+k)$, where $k > \mathrm{e}$. Indicate clearly the coordinates of intersection(s) with axes and the equations of any asymptotes. [2]

(ii) Given that $\mathrm{e}^{-a} = \ln(a+k)$ and $\mathrm{e}^{-b} = -\ln(b+k)$, solve the following inequalities in terms of a, b and k.

(a) $\mathrm{e}^{-x} \le \ln(x+k)$ [1]

(b) $\mathrm{e}^{-x} \le \left|\ln(x+k)\right|$ [2]

3 **(a)** Find $\displaystyle\int \frac{1}{4x^2+9}\,dx$ [2]

(b) Using the substitution $x = \tan\theta$, find the exact value of $\displaystyle\int_1^2 \frac{1}{x^2\sqrt{1+x^2}}\,dx$. [5]

4 **(i)** Expand $\dfrac{1+ax}{(a+x)^2}$ in ascending powers of x up to and including the term in x^3, where a is a positive constant.
Given that there is no term in x, show that $a = \sqrt{2}$. [5]

(ii) The coefficient of x^n in the expansion of $\dfrac{1+\sqrt{2}x}{\left(\sqrt{2}+x\right)^2}$ where $n \in \mathbb{Z}^+$, is denoted by A_n. It is found that $A_n = (-1)^{n+1}\left(\dfrac{n-1}{2\left(\sqrt{2}\right)^n}\right)$. Find the value of n such that $|A_n|$ has the largest value. [2]

5 A curve C has parametric equations

$$x = \theta + \cos\theta, \qquad y = (1-k)\theta + \frac{1}{2}\sin 2\theta,$$

where $0 \le \theta \le \pi$.

(i) Show that C has stationary points when $\cos^2\theta = \frac{k}{2}$. Hence find the range of values of k such that C has stationary points. [4]

(ii) In the case where $k = 1$, find in non-trigonometric form the exact coordinates of the turning points and sketch C. [4]

6 **(a)** The sum of the first 5 terms of an arithmetic series is $\frac{1}{3}$ times the sum of its next 5 terms. Given that the common difference of this series is 3, find the first term of the series. [4]

(b) A teacher makes an initial payment of $*k* to a retirement fund on 1 January 2021, and thereafter, she contributes $12 000 per year to it on the first day of each year. The retirement fund guarantees a compound annual interest rate of 4% on the last day of the year.

(i) Find the minimum value of k, correct to the nearest dollars such that the total value of her retirement fund at the end of the 10th year when interest is applied exceeds $500 000. [3]

(ii) It is given that $k = 105\,000$. Find the day in which the total amount in the retirement fund will first exceed $1 million. [3]

7 It is given that the equation of a curve C is $4x^2 - 8x^2y - 32 = 5y^3$.

(i) Show that there is no point on C where the tangent is parallel to the y-axis. [4]

(ii) Find the equation of the tangent which is parallel to the x-axis. [3]

(iii) The point $P(x, y)$ moves along C in a way such that the x-coordinate of P is increasing at a constant rate of 2 units per second. Find the exact rate of increase of the y-coordinate at the instant when $x = 1.5$. [3]

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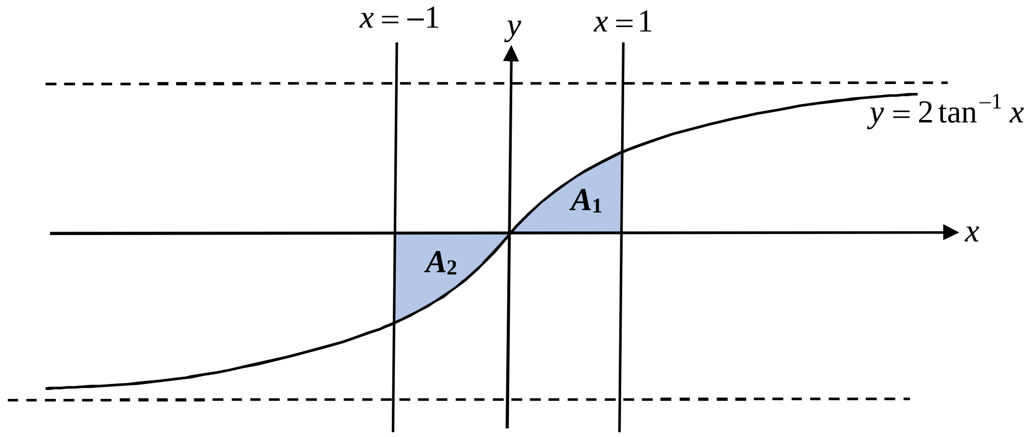

The diagram shows the curve with equation $y = 2\tan^{-1}x$ and the lines with equations $x=-1$ and $x=1$.

(i) Write down the equations of the two horizontal asymptotes of the curve $y = 2\tan^{-1}x$. [1]

Region A_1 and A_2 are the shaded regions shown in the diagram. A_1 is bounded by the curve $y=2\tan^{-1}x$, the x-axis and the lines $x=1$ while A_2 is bounded by the curve $y=2\tan^{-1}x$, the x-axis and the lines $x=-1$.

Region B is bounded by the curve $y = 2\tan^{-1}x$, the y-axis and the line $y = \frac{\pi}{2}$.

(ii) Find the exact area of region B and show that total area of regions A_1 and A_2 is larger than area of region B by $a\pi - b\ln 2$, where a and b are constants to be determined. [4]

(iii) Find the exact volume generated if region B is rotated completely about the y-axis. [3]

(iv) By considering a suitable translation of the graph, or otherwise, find the volume generated when region B is rotated about the line $x = 1$ through 4 right-angles, giving your answer correct to 2 decimal places. [3]

9 **(i)** Verify that one of the roots of the equation $z^3-(1+2\mathrm{i})z^2+(a-1+\mathrm{i})z-a(1+\mathrm{i})=0$ where a is real, is $1+\mathrm{i}$. [2]

(ii) Show that the other 2 roots z_1 and z_2 can be expressed as $z_1=\dfrac{\sqrt{-1-4a}+\mathrm{i}}{2}$ and $z_2=\dfrac{-\sqrt{-1-4a}+\mathrm{i}}{2}$. [3]

(iii) Find the range of a such that z_1 and z_2 are purely imaginary. [2]

(iv) Given that $\arg(z_1)=\dfrac{\pi}{3}$, find a. [3]

Hence find $|z_2|$. [2]

10 On a mangrove swamp, scientists are investigating the population of mudskippers and crabs. Initially, there are 800 mudskippers and 800 crabs on the mangrove swamp. At time t years, the number of mudskippers and crabs are M and C respectively.

(a) For the mudskippers, the scientists discovers that every year, the growth rate is 0.4% of the population size and the death rate is 5 mudskippers per year.

(i) Write down a differential equation relating M and t. [1]

(ii) Solve the differential equation in part **(i)** and determine what happens to the mudskipper population in the future. [4]

(b) For the crabs, the scientists propose that C and t are related by the differential equation $\dfrac{\mathrm{d}C}{\mathrm{d}t} = 8C - 0.005C^2$.

(i) Find the number of crabs in the mangrove swamp when the rate of change of C is a maximum. [2]

(ii) Find C in terms of t. Sketch a graph of C against t. [5]

11 Coordinates axes $Oxyz$ are set up with the origin O at the base of an airport control tower. The x-axis is due East, the y-axis due North and the z-axis vertical. The units of distances are kilometres. An airplane A takes off from the point X. For the first 4 minutes, the position vector of A at time t minutes after take-off, is given by

$$\mathbf{r} = (2+t)\mathbf{i} + (1+2t)\mathbf{j} + 3t\mathbf{k}, \quad 0 \le t \le 4.$$

(i) State the coordinates of X. [1]

(ii) Find the acute angle the flight path makes with the horizontal. [2]

(iii) The airplane enters a cloud at a height of 5 km. Find the coordinates of the point where it enters the cloud. [2]

A second airplane B takes off from the point $(-2,-1,0)$ at the same time as the first airplane A and is traveling at a constant speed in a straight line for the first 4 minutes. Two minutes after take-off, B is at the point $(1,5,\alpha)$.

(iv) Find in terms of α, the position vector of B after t minutes where $0 \le t \le 4$. Explain if it is possible for the two airplanes to collide in the first 4 minutes. [4]

(v) At $t = 4$, a third airplane C was spotted to be equidistant from the first two airplanes. At the same instant, two buildings on the ground D and E are such that A and B are equidistant from both D and E, i.e. $AD = BD$ and $AE = BE$. Find the Cartesian equation of the plane in terms of α in which C, D and E lie. [4]