2021 TJC Prelim Paper 1 Suggested Solutions

1 The curve with equation $y = p + \dfrac{1}{x+q}$, where p and q are constants, is transformed by a scaling of factor 4 parallel to the x-axis, followed by a translation of 3 units in the negative y-direction, followed by a reflection about the y-axis.

The resulting curve has asymptotes $x = 2$ and $y = 1$. Find the values of p and q. [5]

[Solution]

[Solution]

$y = p + \dfrac{1}{x+q}$ 1^{st} : x replaced by $\dfrac{x}{4}$ to get $y = p + \dfrac{1}{\frac{x}{4}+q} = p + \dfrac{4}{x+4q}$ 2^{nd}: y replaced by $y+3$ to get $y+3 = p + \dfrac{4}{x+4q} \quad \Rightarrow \quad y = p - 3 + \dfrac{4}{x+4q}$ 3^{rd} : x replaced by $-x$ to get $y = p - 3 + \dfrac{4}{-x+4q}$ Horizontal asymptote: $y = p - 3 = 1 \quad \Rightarrow \quad p = 4$ Vertical asymptote: $-x + 4q = 0$ $\Rightarrow \quad x = 4q = 2 \quad \Rightarrow \quad q = \dfrac{1}{2}$	

2 **(i)** Sketch the graphs of $y = \mathrm{e}^{-x}$ and $y = \ln(x+k)$, where $k > \mathrm{e}$. Indicate clearly the coordinates of intersection(s) with axes and the equations of any asymptotes. [2]

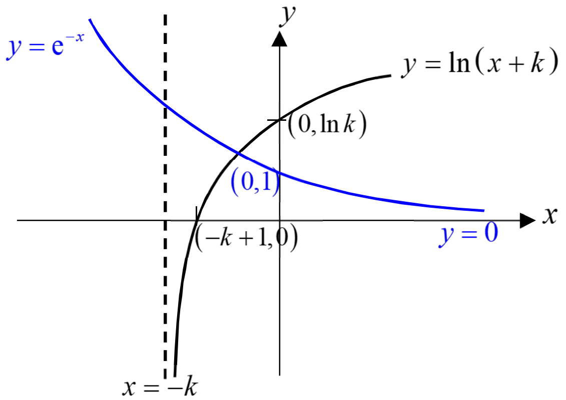
(ii) Given that $\mathrm{e}^{-a} = \ln(a+k)$ and $\mathrm{e}^{-b} = -\ln(b+k)$, solve the following inequalities in terms of *a*, *b* and *k*.

(a) $\mathrm{e}^{-x} \le \ln(x+k)$ [1]

(b) $\mathrm{e}^{-x} \le |\ln(x+k)|$ [2]

[Solution]

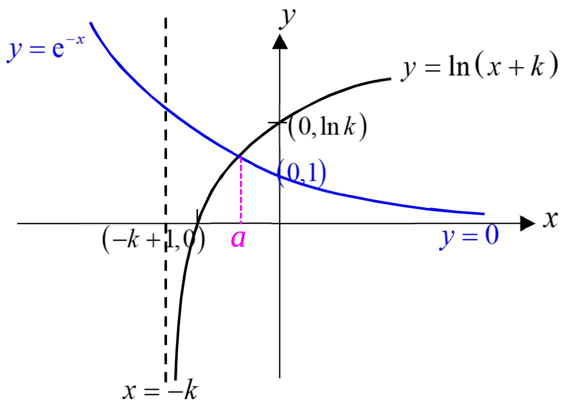
(i) Note that for $k > \mathrm{e}$, $\ln k > 1$.



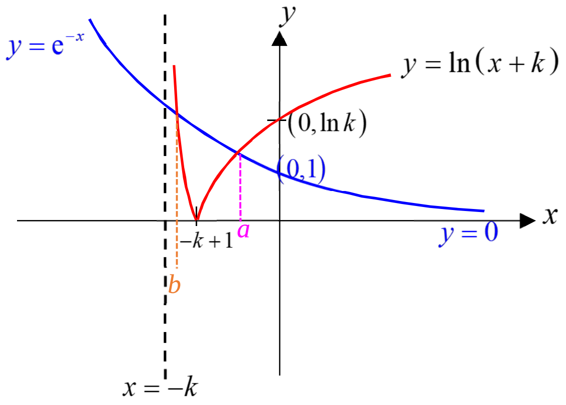
(ii) Given: $\mathrm{e}^{-a} = \ln(a+k)$

(a) $\mathrm{e}^{-x} \le \ln(x+k)$

From diagram,



solution is $x \ge a$.

Given: $e^{-a} = \ln(a+k)$ and $e^{-b} = -\ln(b+k)$ (b) $e^{-x} \le \left\|\ln(x+k)\right\|$ From diagram,  Solution is $x \ge a$ or $-k < x \le b$.	

3 **(a)** Find $\int \frac{1}{4x^2+9}\,dx$ [2]

(b) Using the substitution $x = \tan\theta$, find the exact value of $\int_1^2 \frac{1}{x^2\sqrt{1+x^2}}\,dx$. [5]

[Solution]

(a)

$$\int \frac{1}{4x^2+9}\,\mathrm{d}x = \int \frac{1}{4(x^2+(\frac{3}{2})^2)}\,\mathrm{d}x$$

$$= \frac{1}{4}\left(\frac{2}{3}\right)\tan^{-1}\frac{2x}{3} + C = \frac{1}{6}\tan^{-1}\frac{2x}{3} + C$$

Alternatively,

$$\int \frac{1}{4x^2+9}\,\mathrm{d}x = \frac{1}{2}\int \frac{2}{(2x)^2+3^2}\,\mathrm{d}x$$

$$= \frac{1}{2}\left(\frac{1}{3}\tan^{-1}\frac{2x}{3}\right) + c = \frac{1}{6}\tan^{-1}\frac{2x}{3} + c$$

(b) $x = \tan\theta$, $\frac{dx}{d\theta} = \sec^2\theta$.

When $x = 1$, $\theta = \frac{\pi}{4}$ and $x = 2$, $\theta = \tan^{-1}2$

$$\int_1^2 \frac{1}{x^2\sqrt{1+x^2}}\,dx = \int_{\frac{\pi}{4}}^{\tan^{-1}2} \frac{1}{\tan^2\theta\sqrt{1+\tan^2\theta}}\sec^2\theta\,d\theta$$

$$= \int_{\frac{\pi}{4}}^{\tan^{-1}2} \frac{1}{\tan^2\theta(\sec\theta)}\sec^2\theta\,d\theta$$

$$= \int_{\frac{\pi}{4}}^{\tan^{-1}2} \frac{\sec\theta}{\tan^2\theta}\,d\theta$$

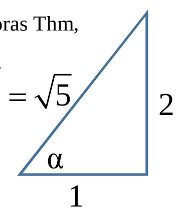
$$= \int_{\frac{\pi}{4}}^{\tan^{-1}2} \frac{\cos\theta}{\sin^2\theta}\,d\theta = \int_{\frac{\pi}{4}}^{\tan^{-1}2} \cot\theta\,\mathrm{cosec}\,\theta\,d\theta$$

$$= -\left[\mathrm{cosec}\,\theta\right]_{\frac{\pi}{4}}^{\alpha} \quad \text{where } \tan\alpha = 2$$

As $\tan\alpha = 2$, we get the right-angled triangle

By Pythagoras Thm, we have

$\sqrt{2^2+1} = \sqrt{5}$



$$= -\left[\mathrm{cosec}\,\alpha - \mathrm{cosec}\,\tfrac{\pi}{4}\right]$$

$$= -\left[\frac{\sqrt{5}}{2} - \sqrt{2}\right] = \sqrt{2} - \frac{\sqrt{5}}{2}$$

4 **(i)** Expand $\dfrac{1+ax}{(a+x)^2}$ in ascending powers of x up to and including the term in x^3, where a is a positive constant.
Given that there is no term in x, show that $a=\sqrt{2}$. [5]

(ii) The coefficient of x^n in the expansion of $\dfrac{1+\sqrt{2}x}{\left(\sqrt{2}+x\right)^2}$ where $n\in\mathbb{Z}^+$, is denoted by A_n. It is found that $A_n=(-1)^{n+1}\left(\dfrac{n-1}{2\left(\sqrt{2}\right)^n}\right)$. Find the value of n such that $|A_n|$ has the largest value. [2]

[Solution]

(i) $\dfrac{1+ax}{(a+x)^2}=(1+ax)(a+x)^{-2}$ $=a^{-2}(1+ax)\left(1+\dfrac{x}{a}\right)^{-2}$ $=a^{-2}(1+ax)\left(1-2\left(\dfrac{x}{a}\right)+\dfrac{(-2)(-3)}{2!}\left(\dfrac{x}{a}\right)^2+\dfrac{(-2)(-3)(-4)}{3!}\left(\dfrac{x}{a}\right)^3+\cdots\right)$ $=a^{-2}(1+ax)\left(1-\dfrac{2}{a}x+\dfrac{3}{a^2}x^2-\dfrac{4}{a^3}x^3+\cdots\right)$ $=a^{-2}\left(1-\dfrac{2}{a}x+\dfrac{3}{a^2}x^2-\dfrac{4}{a^3}x^3+ax-2x^2+\dfrac{3}{a}x^3+\cdots\right)$ $=\dfrac{1}{a^2}+\left(-\dfrac{2}{a^3}+\dfrac{1}{a}\right)x+\left(\dfrac{3}{a^4}-\dfrac{2}{a^2}\right)x^2+\left(-\dfrac{4}{a^5}+\dfrac{3}{a^3}\right)x^3+\cdots$ $=\dfrac{1}{a^2}+\left(\dfrac{-2+a^2}{a^3}\right)x+\left(\dfrac{3-2a^2}{a^4}\right)x^2+\left(\dfrac{-4+3a^2}{a^5}\right)x^3+\cdots$ Since there is no term in x, we have $\dfrac{-2+a^2}{a^3}=0$ $a=\sqrt{2}$ (since a is positive)	

We have $A_n = (-1)^{n+1}\left(\dfrac{n-1}{2\left(\sqrt{2}\right)^n}\right)$ $\therefore \lvert A_n \rvert = \left\lvert \dfrac{n-1}{2\left(\sqrt{2}\right)^n} \right\rvert$ Using GC, $\lvert A_1 \rvert < \lvert A_2 \rvert < \lvert A_3 \rvert = 0.3536 < \lvert A_4 \rvert = 0.375 > \lvert A_5 \rvert = 0.3536 > \lvert A_6 \rvert > \ldots$ The value of n such that $\lvert A_n \rvert$ has the largest value is 4.	

5 A curve C has parametric equations

$$x = \theta + \cos\theta, \qquad y = (1-k)\theta + \frac{1}{2}\sin 2\theta,$$

where $0 \le \theta \le \pi$.

(i) Show that C has stationary points when $\cos^2\theta = \frac{k}{2}$. Hence find the range of values of k such that C has stationary points. [4]

(ii) In the case where $k = 1$, find in non-trigonometric form the exact coordinates of the turning points and sketch C. [4]

[Solution]

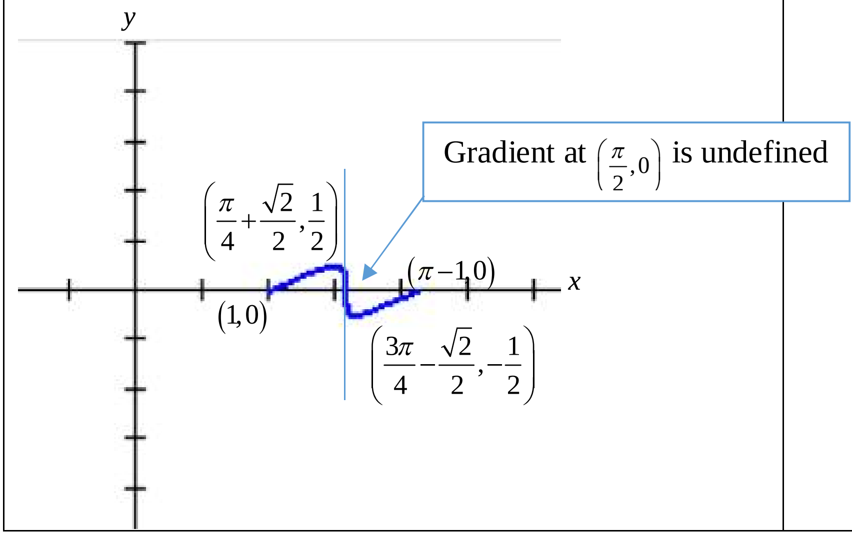
$x = \theta + \cos\theta, \qquad y = (1-k)\theta + \frac{1}{2}\sin 2\theta,$ $\frac{dx}{d\theta} = 1 - \sin\theta,$ $\frac{dy}{d\theta} = (1-k) + \frac{1}{2}2\cos 2\theta = 1 - k + 2\cos^2\theta - 1 = 2\cos^2\theta - k$ $\frac{dy}{dx} = \frac{2\cos^2\theta - k}{1-\sin\theta}$ **(i)** C has stationary points $\Rightarrow \frac{dy}{dx} = 0$ $\Rightarrow 2\cos^2\theta - k = 0$ and $1 - \sin\theta \ne 0$ $\therefore \cos^2\theta = \frac{k}{2}$ For $0 \le \theta \le \pi \Rightarrow -1 \le \cos\theta \le 1, \quad \cos\theta \ne 0$ $\Rightarrow 0 < \cos^2\theta \le 1$ $\Rightarrow 0 < \frac{k}{2} \le 1$ $\therefore 0 < k \le 2$	

(ii) $\cos^2\theta = \dfrac{1}{2} \Rightarrow \cos\theta = \pm\dfrac{1}{\sqrt{2}} \Rightarrow \theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}$

$$\theta = \frac{\pi}{4} \Rightarrow x = \frac{\pi}{4} + \cos\frac{\pi}{4} = \frac{\pi}{4} + \frac{\sqrt{2}}{2},\ y = \frac{1}{2}\sin\frac{\pi}{2} = \frac{1}{2}$$

$$\theta = \frac{3\pi}{4} \Rightarrow x = \frac{3\pi}{4} + \cos\frac{3\pi}{4} = \frac{3\pi}{4} - \frac{\sqrt{2}}{2},\ y = \frac{1}{2}\sin\frac{3\pi}{2} = -\frac{1}{2}$$

Therefore, the coordinates of turning points are $\left(\dfrac{\pi}{4} + \dfrac{\sqrt{2}}{2}, \dfrac{1}{2}\right)$ and $\left(\dfrac{3\pi}{4} - \dfrac{\sqrt{2}}{2}, -\dfrac{1}{2}\right)$.



6 **(a)** The sum of the first 5 terms of an arithmetic series is $\frac{1}{3}$ times the sum of its next 5 terms. Given that the common difference of this series is 3, find the first term of the series. [4]

(b) A teacher makes an initial payment of $\$k$ to a retirement fund on 1 January 2021, and thereafter, she contributes \$12 000 per year to it on the first day of each year. The retirement fund guarantees a compound annual interest rate of 4% on the last day of the year.

(i) Find the minimum value of k, correct to the nearest dollars such that the total value of her retirement fund at the end of the 10th year when interest is applied exceeds \$500 000. [3]

(ii) It is given that $k = 105\,000$. Find the day in which the total amount in the retirement fund will first exceed \$1 million. [3]

(a) $S_5 = \frac{1}{3}\left(S_{10} - S_5\right) \Rightarrow \quad 4S_5 = S_{10}$ $\Rightarrow \quad 4\left[\frac{5}{2}\left(2a + 4(3)\right)\right] = \frac{10}{2}\left(2a + 9(3)\right)$ $\Rightarrow \quad 20a + 120 = 10a + 135$ $\Rightarrow \quad a = \frac{3}{2}$	

(b)(i)

	Year	First Day of Year	Last day of Year
1Jan 2021	1	k	$1.04k$
	2	$1.04k+12000$	$1.04(1.04k+12000)$
	3	$1.04(1.04k+12000)$ $+12000$	$1.04(1.04(1.04k+12000)+12000)$ $=1.04^3k+12000(1.04^2+1.04)$
	n		$1.04^n k+12000(1.04^{n-1}+\cdots+1.04^2+1.04)$

For $n=10$,

$$1.04^{10}k+12000(1.04^9+\cdots+1.04^2+1.04)>500000$$

$$1.04^{10}k+12000\frac{1.04(1.04^9-1)}{0.04}>500000$$

$k>248558.11$

Minimum k = 248559 (correct to nearest dollars)

(b)(ii)

For $k=105\,000$, let T be total amount in the retirement fund.

$$T=1.04^n(105000)+12000\frac{1.04(1.04^{n-1}-1)}{0.04}>1000000$$

From GC, when $n=30$ (31 Dec 2051), $T=1001575.99$

The day in which the total amount in the retirement fund will first exceed \$1 million is

<u>31 Dec 2050.</u>

7 It is given that the equation of a curve C is $4x^2-8x^2y-32=5y^3$.

(i) Show that there is no point on C where the tangent is parallel to the y-axis. [4]

(ii) Find the equation of the tangent which is parallel to the x-axis. [3]

(iii) The point $P(x,y)$ moves along C in a way such that the x-coordinate of P is increasing at a constant rate of 2 units per second. Find the exact rate of increase of the y-coordinate at the instant when $x=1.5$. [3]

[Solution]

(i) $4x^2-8x^2y-32=5y^3$

Differentiating w.r.t. x,

$$8x-8x^2\frac{\mathrm{d}y}{\mathrm{d}x}-16xy=15y^2\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{8x-16xy}{8x^2+15y^2}$$

For tangent to be parallel to y-axis, $\frac{\mathrm{d}y}{\mathrm{d}x}$ is undefined.

$\Rightarrow \quad 8x^2+15y^2=0$

$\Rightarrow \quad x=y=0$

Check (0, 0) is not a point on C.

So there is no point on C where the tangent is parallel to the y-axis.

(ii) For tangent to be parallel to x-axis, $\frac{\mathrm{d}y}{\mathrm{d}x}=0$.

$$8x-16xy=0 \quad \Rightarrow \quad x(1-2y)=0 \quad \Rightarrow \quad x=0 \quad \text{or} \quad y=\frac{1}{2}$$

When $x=0$, $5y^3=-32 \quad \Rightarrow \quad y=\left(-\frac{32}{5}\right)^{\frac{1}{3}}$

Check: When $y=\frac{1}{2}$,

$$4x^2-8x^2\left(\frac{1}{2}\right)-32=5\left(\frac{1}{2}\right)^3 \quad \Rightarrow \quad -32=\frac{5}{8}$$

which is invalid

The equation of the tangent parallel to the x-axis is $y=\left(-\frac{32}{5}\right)^{\frac{1}{3}}$.

(iii) Given $\frac{\mathrm{d}x}{\mathrm{d}t}=2$.

When $x=1.5$, $y=-1$.

$$\frac{\mathrm{d}y}{\mathrm{d}t}=\frac{\mathrm{d}y}{\mathrm{d}x}\cdot\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{8x-16xy}{8x^2+15y^2}\cdot\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{8(1.5)-16(1.5)(-1)}{8(1.5)^2+15(-1)^2}\cdot(2)=\frac{24}{11}$$

The y-coordinate is increasing at a rate of $\frac{24}{11}$ units/s

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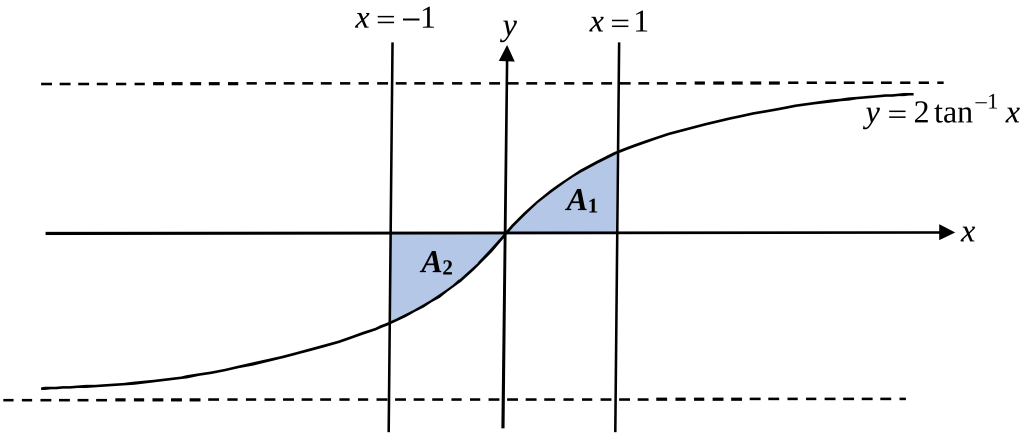

The diagram shows the curve with equation $y = 2\tan^{-1} x$ and the lines with equations $x = -1$ and $x = 1$.

(i) Write down the equations of the two horizontal asymptotes of the curve $y = 2\tan^{-1} x$. [1]

Region A_1 and A_2 are the shaded regions shown in the diagram. A_1 is bounded by the curve $y = 2\tan^{-1} x$, the x-axis and the lines $x = 1$ while A_2 is bounded by the curve $y = 2\tan^{-1} x$, the x-axis and the lines $x = -1$.

Region B is bounded by the curve $y = 2\tan^{-1} x$, the y-axis and the line $y = \dfrac{\pi}{2}$.

(ii) Find the exact area of region B and show that total area of regions A_1 and A_2 is larger than area of region B by $a\pi - b\ln 2$, where a and b are constants to be determined. [4]

(iii) Find the exact volume generated if region B is rotated completely about the y-axis. [3]

(iv) By considering a suitable translation of the graph, or otherwise, find the volume generated when region B is rotated about the line $x = 1$ through 4 right-angles, giving your answer correct to 2 decimal places. [3]

[Solution]

(i) The horizontal asymptotes are $y = \pm\pi$

(ii)

Area B $= \int_0^{\frac{\pi}{2}} \tan\frac{y}{2}\,\mathrm{d}y$

$= \left[2\ln\left|\sec\left(\frac{y}{2}\right)\right|\right]_0^{\frac{\pi}{2}}$

$= 2\ln\sec\frac{\pi}{4} - \ln\sec 0$

$= 2\ln\sqrt{2}$

$= \ln 2$

Area of regions A_1 and A_2 = $2\left(\frac{\pi}{2} - \ln 2\right) = \pi - 2\ln 2$

Area A_1 + Area A_2 – Area B

$= \pi - 2\ln 2 - \ln 2$

$= \pi - 3\ln 2.$

So $a = 1$ and $b = 3$

Method 2

Area of regions A_1 and A_2

$= 2\int_0^1 2\tan^{-1} x\, dx$ by symmetry

$= \left[4x\tan^{-1} x\right]_0^1 - \int_0^1 \frac{4x}{1+x^2}\, dx$

$= \pi - \left[2\ln(1+x^2)\right]_0^1$

$= \pi - 2\ln 2$

Area B = $\frac{\pi}{2} - \frac{1}{2}(\pi - 2\ln 2) = \ln 2$

Area A_1 + Area A_2 – Area B

$= \pi - 2\ln 2 - \ln 2$

$= \pi - 3\ln 2.$

So $a = 1$ and $b = 3$

(iii) Volume for B rotated about y-axis

$= \pi\int_0^{\frac{\pi}{2}} x^2 dy$

$= \pi\int_0^{\frac{\pi}{2}} \tan^2\left(\frac{y}{2}\right) dy$

$= \pi \int_0^{\frac{\pi}{2}} (\sec^2\left(\frac{y}{2}\right) - 1) dy$

$= \pi \left[2\tan\frac{y}{2} - y\right]_0^{\frac{\pi}{2}}$

$= \pi(2 - \frac{\pi}{2})$

(iv) Translating the curve $y = 2\tan^{-1} x$ by -1 unit along x-axis, we have $y = 2\tan^{-1}(x+1)$,

i.e. $x = \tan\dfrac{y}{2} - 1$

Volume of region A$_1$ rotated about the line $x = 1$ is $\pi\int_0^1 \left(\tan\dfrac{y}{2} - 1\right)^2 \mathrm{d}y$

Volume of the cylinder formed by A_1 and B when rotated about $x = 1$ is $\pi(1^2)\dfrac{\pi}{2} = \dfrac{\pi^2}{2}$

The required volume

$= \dfrac{\pi^2}{2} - \pi\int_0^{\frac{\pi}{2}} \left(\tan\dfrac{y}{2} - 1\right)^2 \mathrm{d}y = 3.01$ (correct to 2 dps)

9 **(i)** Verify that one of the roots of the equation $z^3-(1+2\mathrm{i})z^2+(a-1+\mathrm{i})z-a(1+\mathrm{i})=0$ where a is real, is $1+\mathrm{i}$. [2]

(ii) Show that the other 2 roots z_1 and z_2 can be expressed as $z_1=\dfrac{\sqrt{-1-4a}+\mathrm{i}}{2}$ and $z_2=\dfrac{-\sqrt{-1-4a}+\mathrm{i}}{2}$. [3]

(iii) Find the range of a such that z_1 and z_2 are purely imaginary. [2]

(iv) Given that $\arg(z_1)=\dfrac{\pi}{3}$, find a. [3]

Hence find $|z_2|$. [2]

[Solution]

(i) Sub $z=1+\mathrm{i}$, we have

$$LHS=(1+\mathrm{i})^3-(1+2\mathrm{i})(1+\mathrm{i})^2+(a-1+\mathrm{i})(1+\mathrm{i})-a(1+\mathrm{i})$$
$$=-2+2\mathrm{i}+4-2\mathrm{i}+a(1+\mathrm{i})-2-a(1+\mathrm{i})$$
$$=0=RHS$$

(ii) **From (i),** one of the factors of

$z^3-(1+2\mathrm{i})z^2+(a-1+\mathrm{i})z-a(1+\mathrm{i})$ is $z-1-\mathrm{i}$

Method 1

Let $(z-(1+\mathrm{i}))(z^2+Az+B)=$

$$z^3-(1+2\mathrm{i})z^2+(a-1+\mathrm{i})z-a(1+\mathrm{i})$$

Comparing coefficient of z^2 : $A-(1+\mathrm{i})=-(1+2\mathrm{i})\Rightarrow A=-\mathrm{i}$

Comparing the constant term: $-B(1+\mathrm{i})=-a(1+\mathrm{i})\Rightarrow B=a$

Method 2: Using Long Division, we have

$$z^3-(1+2\mathrm{i})z^2+(a-1+\mathrm{i})z-a(1+\mathrm{i})=(z-1-\mathrm{i})(z^2-\mathrm{i}z+a)$$

Therefore, $z^3-(1+2\mathrm{i})z^2+(a-1+\mathrm{i})z-a(1+\mathrm{i})=0$

$\Rightarrow\quad (z-1-\mathrm{i})(z^2-\mathrm{i}z+a)=0$

$\Rightarrow\quad z=1-\mathrm{i}$ or $z^2-\mathrm{i}z+a=0$

The other 2 roots are $z=\dfrac{\mathrm{i}\pm\sqrt{(-\mathrm{i})^2-4(1)(a)}}{2(1)}=\dfrac{\mathrm{i}\pm\sqrt{-1-4a}}{2}$ Therefore, we have $z_1=\dfrac{\sqrt{-1-4a}+\mathrm{i}}{2}$ and $z_2=\dfrac{-\sqrt{-1-4a}+\mathrm{i}}{2}$ **(iii)** For z_1 and z_2 to be purely imaginary, $-1-4a\leq 0$. Therefore, $a\geq -\dfrac{1}{4}$. **(iv)** Since $\arg(z_1)=\dfrac{\pi}{3}$, it means that real part of z_1 must be $\dfrac{\sqrt{-1-4a}}{2}$, i.e. $-1-4a>0$ Therefore, $\dfrac{\frac{1}{2}}{\frac{\sqrt{-1-4a}}{2}}=\tan\dfrac{\pi}{3}=\sqrt{3}$ $\sqrt{-1-4a}=\dfrac{1}{\sqrt{3}}$ Squaring both sides, we have $-1-4a=\dfrac{1}{3}\Rightarrow a=-\dfrac{1}{3}$ Therefore, $\lvert z_2\rvert=\left\lvert\dfrac{-\sqrt{-1-4\left(\frac{-1}{3}\right)}+\mathrm{i}}{2}\right\rvert$ $=\left\lvert\dfrac{-\sqrt{\frac{1}{3}}+\mathrm{i}}{2}\right\rvert=\dfrac{1}{2}\sqrt{\left(-\sqrt{\dfrac{1}{3}}\right)^2+1}=\dfrac{1}{\sqrt{3}}=\dfrac{\sqrt{3}}{3}$	

10 On a mangrove swamp, scientists are investigating the population of mudskippers and crabs. Initially, there are 800 mudskippers and 800 crabs on the mangrove swamp. At time t years, the number of mudskippers and crabs are M and C respectively.

(a) For the mudskippers, the scientists discovers that every year, the growth rate is 0.4% of the population size and the death rate is 5 mudskippers per year.

(i) Write down a differential equation relating M and t. [1]

(ii) Solve the differential equation in part **(i)** and determine what happens to the mudskipper population in the future. [4]

(b) For the crabs, the scientists propose that C and t are related by the differential equation $\frac{\mathrm{d}C}{\mathrm{d}t} = 8C - 0.005C^2$.

(i) Find the number of crabs in the mangrove swamp when the rate of change of C is a maximum. [2]

(ii) Find C in terms of t. Sketch a graph of C against t. [5]

(i)	Growth rate = $\frac{0.4}{100}M = 0.004M$ Death rate = 5 $\frac{\mathrm{d}M}{\mathrm{d}t} = \text{Growth rate} - \text{death rate}$ $= 0.004M - 5$	
(ii)	$\frac{\mathrm{d}M}{\mathrm{d}t} = 0.004M - 5 = 0.004(M - 1250)$ $\frac{1}{M-1250}\frac{\mathrm{d}M}{\mathrm{d}t} = 0.004$ Integrate wrt t: $\int \frac{1}{M-1250}\,\mathrm{d}M = \int 0.004\,\mathrm{d}t$ $\Rightarrow \ln\lvert M-1250\rvert = 0.004t + c$ $\Rightarrow \lvert M-1250\rvert = \mathrm{e}^{0.004t+c}$ $\Rightarrow M - 1250 = \pm\mathrm{e}^{0.004t+c}$ $\Rightarrow M - 1250 = A\mathrm{e}^{0.004t}, A = \pm\mathrm{e}^{c}$ At $t=0, M=800$, $\Rightarrow A = -450$ $\therefore M = 1250 - 450\mathrm{e}^{0.004t}$ As $t \to \infty$, $450\mathrm{e}^{0.004t} \to \infty$, $M = 1250 - 450\mathrm{e}^{0.004t} \to -\infty$ This means that the mudskipper population will become extinct in the future. $M = 1250 - 450\mathrm{e}^{0.004t} = 0 \quad \Rightarrow \quad \mathrm{e}^{0.004t} = \frac{25}{9} \quad \Rightarrow \quad t = 255.4$ The mudskipper will be extinct in the 256th year.	•
b(i)	<u>Method 1:</u> $\frac{\mathrm{d}C}{\mathrm{d}t} = 8C - 0.005C^2$.	

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Let R be the rate of change of C. i.e. $R=\frac{\mathrm{d}C}{\mathrm{d}t}$

To maximise R, let $\frac{\mathrm{d}R}{\mathrm{d}C}=0$. Note R is in terms of C.

$$\frac{\mathrm{d}R}{\mathrm{d}c}=\frac{\mathrm{d}\left(\frac{\mathrm{d}C}{\mathrm{d}t}\right)}{\mathrm{d}C}=8-0.01C=0 \quad \Rightarrow \quad C=800.$$

$$\frac{\mathrm{d}^2R}{\mathrm{d}C^2}=\frac{\mathrm{d}^2\left(\frac{\mathrm{d}C}{\mathrm{d}t}\right)}{\mathrm{d}C^2}=-0.01<0$$

$\Rightarrow$ The number of crabs when growth rate is maximum is 800.

<u>Method 2:</u>

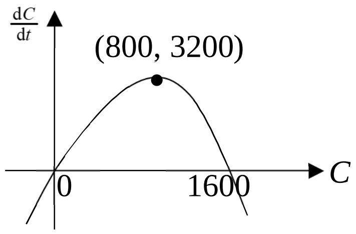
$$\frac{\mathrm{d}C}{\mathrm{d}t}=-0.005\left(C^2-1600C\right)$$

$$=-0.005(C-800)^2+3200\leq 3200$$

$\frac{\mathrm{d}C}{\mathrm{d}t}=3200$ is maximum when C = 800.

<u>Method 3:</u>

$\frac{\mathrm{d}C}{\mathrm{d}t}=C(8-0.005C)=0 \Rightarrow C$ = 0 or 1600



From the graph, $\frac{\mathrm{d}C}{\mathrm{d}t}=3200$ is maximum when C = 800.

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<u>Method 1: using Partial Fraction</u>

$$\frac{\mathrm{d}C}{\mathrm{d}t}=8C-0.005C^2=0.005C(1600-C)$$

$$\Rightarrow \frac{1}{C(1600-C)}\frac{\mathrm{d}C}{\mathrm{d}t}=0.005$$

Integrate wrt t:

$$\int\frac{1}{C(1600-C)}\,\mathrm{d}C=\int 0.005\,\mathrm{d}t$$

$$\Rightarrow \frac{1}{1600}\left[\ln|C|-\ln|1600-C|\right]=0.005t+d$$

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$$\Rightarrow \ln\left|\frac{C}{1600-C}\right| = 8t + 1600d$$

$$\Rightarrow \frac{C}{1600-C} = \pm e^{8t+1600d}$$

$$\Rightarrow \frac{C}{1600-C} = Be^{8t}, B = \pm e^{1600d}$$

When $t = 0$, $C = 800$, $B = 1$

$$\therefore C = \frac{1600e^{8t}}{1+e^{8t}} \quad (\text{or } C = \frac{1600}{1+e^{-8t}})$$

Method 2: Complete the square and use MF26

$$\frac{dC}{dt} = -0.005\left[(C-800)^2 - 800^2\right]$$

$$\frac{1}{(C-800)^2 - 800^2}\frac{dC}{dt} = -0.005$$

$$\int \frac{1}{(C-800)^2 - 800^2} dC = \int -0.005 dt$$

$$\frac{1}{2(800)}\ln\left|\frac{(C-800)-800}{C-800+800}\right| = -0.005t + d$$

$$\left|\frac{(C-800)-800}{C-800+800}\right| = e^{-8t+1600d}$$

$$\frac{C-1600}{C} = \pm e^{-8t+1600d}$$

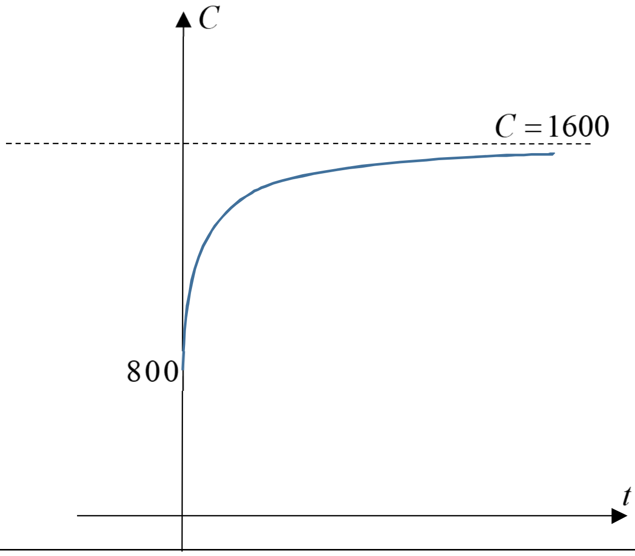
$$\frac{C-1600}{C} = Be^{-8t}, B = \pm e^{1600d}$$

When $t = 0$, $C = 800$

$B = -1$

$$1 - \frac{1600}{C} = -e^{-8t}$$

$$\frac{1600}{C} = 1 + e^{-8t} \Rightarrow C = \frac{1600}{1+e^{-8t}}$$



11 Coordinates axes *Oxyz* are set up with the origin *O* at the base of an airport control tower. The *x*-axis is due East, the *y*-axis due North and the *z*-axis vertical. The units of distances are kilometres. An airplane *A* takes off from the point *X*. For the first 4 minutes, the position vector of *A* at time *t* minutes after take-off, is given by

$$\mathbf{r} = (2+t)\mathbf{i} + (1+2t)\mathbf{j} + 3t\mathbf{k}, \quad 0 \le t \le 4.$$

(i) State the coordinates of *X*. [1]

(ii) Find the acute angle the flight path makes with the horizontal. [2]

(iii) The airplane enters a cloud at a height of 5 km. Find the coordinates of the point where it enters the cloud. [2]

A second airplane *B* takes off from the point $(-2,-1,0)$ at the same time as the first airplane *A* and is traveling at a constant speed in a straight line for the first 4 minutes. Two minutes after take-off, *B* is at the point $(1,5,\alpha)$.

(iv) Find in terms of α, the position vector of *B* after *t* minutes where $0 \le t \le 4$. Explain if it is possible for the two airplanes to collide in the first 4 minutes. [4]

(v) At *t* = 4, a third airplane *C* was spotted to be equidistant from the first two airplanes. At the same instant, two buildings on the ground *D* and *E* are such that *A* and *B* are equidistant from both *D* and *E*, i.e. *AD* = *BD* and *AE* = *BE*. Find the Cartesian equation of the plane in terms of α in which *C*, *D* and *E* lie. [4]

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[Solution]

(i) $\overrightarrow{OA} = \begin{pmatrix}2\\1\\0\end{pmatrix} + t\begin{pmatrix}1\\2\\3\end{pmatrix}, \quad 0 \le t \le 4$

When $t=0$, coordinates of *X* = (2, 1, 0)

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(ii) Let θ be the required angle.

Normal to the horizontal plane is $\begin{pmatrix}0\\0\\1\end{pmatrix}$

$$\sin\theta = \frac{\left|\begin{pmatrix}1\\2\\3\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}\right|}{\sqrt{1^2+2^2+3^2}\sqrt{1^2}} = \frac{3}{\sqrt{14}}$$

$\theta = 53.3^\circ$ (1 d.p.)

Alternative Method,

$$\cos(90^\circ - \theta) = \frac{\left|\begin{pmatrix}1\\2\\3\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}\right|}{\sqrt{1^2+2^2+3^2}\sqrt{1^2}} = \frac{3}{\sqrt{14}}$$

$90^\circ - \theta = 36.7^\circ$ (1 d.p.)

$\theta = 53.3^\circ$ (1 d.p.)

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(iii) $\overrightarrow{OA}=\begin{pmatrix}2\\1\\0\end{pmatrix}+t\begin{pmatrix}1\\2\\3\end{pmatrix}, \quad 0\le t\le 4$ Since z-axis is vertical, at height 5 km, $3t=5$ $\Rightarrow \quad t=\frac{5}{3}$ Position vector of the point when it enters the cloud $=\begin{pmatrix}2\\1\\0\end{pmatrix}+\frac{5}{3}\begin{pmatrix}1\\2\\3\end{pmatrix}=\begin{pmatrix}\frac{11}{3}\\\frac{13}{3}\\5\end{pmatrix}$ Coordinates of the point $=\left(\frac{11}{3},\frac{13}{3},5\right)$	
(iv) $\begin{pmatrix}1\\5\\\alpha\end{pmatrix}-\begin{pmatrix}-2\\-1\\0\end{pmatrix}=\begin{pmatrix}3\\6\\\alpha\end{pmatrix}=2\begin{pmatrix}\frac{3}{2}\\3\\\frac{\alpha}{2}\end{pmatrix}$ Position vector of the air plane B after t minutes, $\overrightarrow{OB}=\begin{pmatrix}-2\\-1\\0\end{pmatrix}+t\begin{pmatrix}\frac{3}{2}\\3\\\frac{\alpha}{2}\end{pmatrix}, \quad 0\le t\le 4$ **Alternative method,** Let $\overrightarrow{OB}=\begin{pmatrix}-2\\-1\\0\end{pmatrix}+t\boldsymbol{d}$ where $0\le t\le 4$ When $t=2$, $\begin{pmatrix}1\\5\\\alpha\end{pmatrix}=\begin{pmatrix}-2\\-1\\0\end{pmatrix}+2\boldsymbol{d} \quad \Rightarrow \boldsymbol{d}=\begin{pmatrix}\frac{3}{2}\\3\\\frac{\alpha}{2}\end{pmatrix}$ $\therefore \quad \overrightarrow{OB}=\begin{pmatrix}-2\\-1\\0\end{pmatrix}+t\begin{pmatrix}\frac{3}{2}\\3\\\frac{\alpha}{2}\end{pmatrix}$ If the two airplanes collide, $\overrightarrow{OA}=\overrightarrow{OB}$ for some $0\le t\le 4$	

$$\begin{pmatrix}2\\1\\0\end{pmatrix}+t\begin{pmatrix}1\\2\\3\end{pmatrix}=\begin{pmatrix}-2\\-1\\0\end{pmatrix}+t\begin{pmatrix}\frac{3}{2}\\3\\\frac{\alpha}{2}\end{pmatrix}$$ $$t\begin{pmatrix}\frac{1}{2}\\1\\\frac{\alpha}{2}-3\end{pmatrix}=\begin{pmatrix}4\\2\\0\end{pmatrix}$$ There is no solution to the above equation. Hence the two airplanes will not collide.	
(v) At $t = 4$, $\overrightarrow{OA}=\begin{pmatrix}2\\1\\0\end{pmatrix}+4\begin{pmatrix}1\\2\\3\end{pmatrix}=\begin{pmatrix}6\\9\\12\end{pmatrix}$, $\overrightarrow{OB}=\begin{pmatrix}-2\\-1\\0\end{pmatrix}+4\begin{pmatrix}\frac{3}{2}\\3\\\frac{\alpha}{2}\end{pmatrix}=\begin{pmatrix}4\\11\\2\alpha\end{pmatrix}$. The plane contains all the points that each has equal distance to points A and B is the plane that contains the perpendicular bisector of AB and perpendicular to $\overrightarrow{AB}$. Let M be the Mid-point of AB, $$\overrightarrow{OM}=\frac{1}{2}\left[\begin{pmatrix}6\\9\\12\end{pmatrix}+\begin{pmatrix}4\\11\\2\alpha\end{pmatrix}\right]=\begin{pmatrix}5\\10\\6+\alpha\end{pmatrix}$$ $$\overrightarrow{AB}=\begin{pmatrix}4\\11\\2\alpha\end{pmatrix}-\begin{pmatrix}6\\9\\12\end{pmatrix}=\begin{pmatrix}-2\\2\\2\alpha-12\end{pmatrix}=2\begin{pmatrix}-1\\1\\\alpha-6\end{pmatrix}$$ Normal to the plane = $\begin{pmatrix}-1\\1\\\alpha-6\end{pmatrix}$ Vector equation of the plane: $$r\bullet\begin{pmatrix}-1\\1\\\alpha-6\end{pmatrix}=\begin{pmatrix}5\\10\\6+\alpha\end{pmatrix}\bullet\begin{pmatrix}-1\\1\\\alpha-6\end{pmatrix}$$ $$=-5+10+\alpha^2-36$$ $$=\alpha^2-31$$ Cartesian equation of the plane: $$-x+y+(\alpha-6)z=\alpha^2-31$$ **Alternative method 1**: Let coordinates of point C be (x, y, z) and M be the Mid-point of AB.	

$\overrightarrow{AB} \perp \overrightarrow{CM} \Rightarrow \overrightarrow{AB} \bullet \overrightarrow{CM} = 0$ $\Rightarrow \begin{pmatrix} -1 \\ 1 \\ \alpha-6 \end{pmatrix} \bullet \begin{pmatrix} x-5 \\ y-10 \\ z-\alpha-6 \end{pmatrix} = 0$ $\Rightarrow -x+5+y-10+(\alpha-6)z-(\alpha^2-36)=0$ $\Rightarrow -x+y+(\alpha-6)z=\alpha^2-31$ **Alternative method 2**: At $t = 4$, $\overrightarrow{OA} = \begin{pmatrix} 6 \\ 9 \\ 12 \end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix} 4 \\ 11 \\ 2\alpha \end{pmatrix}$. Let coordinates of point C be (x, y, z) and $AC = BC$. $\Rightarrow \sqrt{(6-x)^2+(9-y)^2+(12-z)^2} = \sqrt{(4-x)^2+(11-y)^2+(2\alpha-z)^2}$ $\Rightarrow 124-4x+4y+(4\alpha-24)z=4\alpha^2$ $\Rightarrow -x+y+(\alpha-6)z=\alpha^2-31$	