2021 SAJC H2 Maths Promo Paper **Duration: 3 hrs** **Marks: 100**
Attempt all questions.

1 **(a)** Let u_n and S_n denote the n^{th} term and the sum of the first n terms of a sequence respectively. Given that where $u_1 = 1$, $u_2 = 5$, the sum of the first twelve terms is 276 and S_n is a quadratic polynomial in n, find S_n in terms of n. [3]

(b) A sequence $\{v_n\}$ is defined by $v_n = v_{n-1} - \frac{3}{5}\left(\frac{2}{5}\right)^{n-2}$, $n \geq 2$ and $v_1 = 4$. By considering $\sum_{n=2}^{N}(v_n - v_{n-1})$, find v_N. Hence, explain, with a reason, whether the sequence is convergent or divergent. [5]

2 The sum of the first n terms of a series, S_n, is given by $\frac{a^n}{2^{n-1}} - 2.$, where a is a non-zero constant and $a \neq 2$.

(i) Show that T_n, the n^{th} term of the series, is $(a-2)\left(\frac{a}{2}\right)^{n-1}$. Hence show that the given series is a geometric series. [4]

(ii) Find the range of values of a for the sum to infinity to exist. [2]

(iii) Given that $a = 1$, find the least value of n for S_n to be within ± 0.2 of the value of the sum to infinity. [3]

3 The function f is defined by

$$\mathrm{f}(x) = \ln\left(\frac{\mathrm{e}^x + 1}{\mathrm{e}^x - 1}\right), \quad x > 0.$$

(i) Show that $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$. [3]

(ii) Find $\mathrm{f}^2(x)$ and hence evaluate $\mathrm{f}^{2021}(3)$, leaving your answer in exact form. [3]

4 **(i)** Given that $\mathbf{p} \times \mathbf{q} = \mathbf{0}$, what can be deduced about the vectors $\mathbf{p}$ and $\mathbf{q}$? [1]

(ii) Find the unit vector $\mathbf{r}$ such that $\mathbf{r} \times (-6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}) = \mathbf{0}$. [2]

(iii) Find the sine of the acute angle between $-6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ and the x-axis, leaving your answer in exact form. [3]

5 **(a)** **(i)** Show that $\frac{\mathrm{d}}{\mathrm{d}x}\left(3^{2x}\right)=9^{x}\ln 9$. [1]

(ii) Hence, differentiate $\tan^{-1}\left(3^{2x}\right)$, $x>0$ with respect to x. [2]

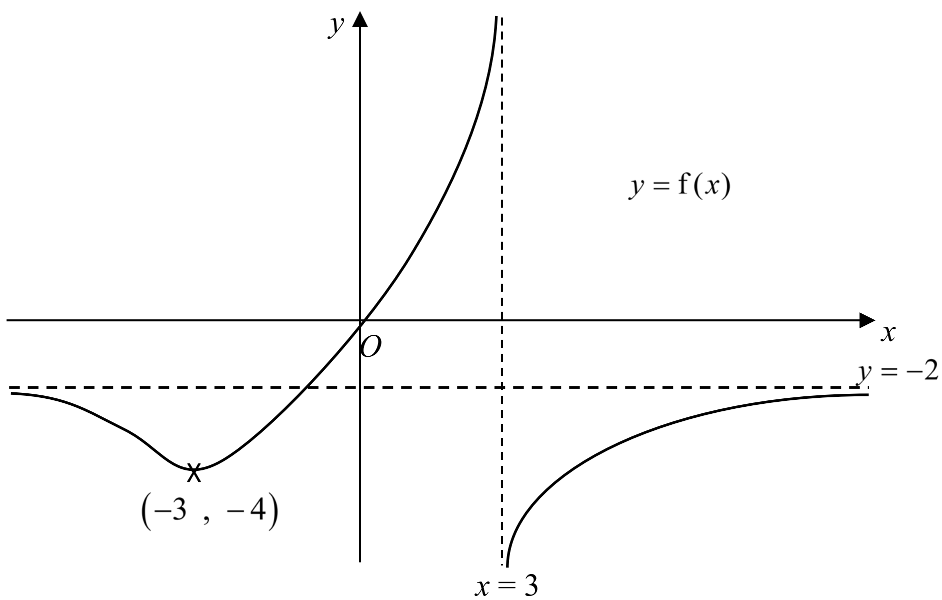
(b) Given that $y^{2y}=\mathrm{e}^{3x+2\mathrm{e}}$, where $y>\mathrm{e}^{-1}$, show that $y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\frac{2}{3}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^3$. [5]

6 **(a)** The graph of $y=\mathrm{h}(x)$, where $\mathrm{h}(x)=\mathrm{e}^{3x}$, was transformed to a new graph with equation $y=\mathrm{g}(x)$.

(i) Given that $\mathrm{g}(x)=\frac{1}{\mathrm{h}'(x)}$, express $\mathrm{g}(x)$ in the form of $k\mathrm{e}^{mx}$, where k and m are exact real constants. [1]

(ii) Describe a sequence of two transformations which would transform the graph of $y=\mathrm{h}(x)$ to the graph of $y=\mathrm{g}(x)$. [2]

(b) The diagram below shows the curve $y=\mathrm{f}(x)$. The curve has a minimum point $(-3\ ,\ -4)$ and passes through the origin. The lines $x=3$ and $y=-2$ are the vertical and horizontal asymptotes to the curve respectively.



(i) Sketch, including the coordinates of the point(s) of intersections with the axes, turning point(s) and equation(s) of asymptote(s), if any, the following:

(a) $y=-\frac{1}{2}\mathrm{f}(x-3)$ [4]

(b) $y=\frac{1}{\mathrm{f}(x)}$. [3]

(ii) State the coordinates of the point(s) where the curve $y=\mathrm{f}'(x)$ cuts the axes. [1]

7 A curve C has equation $y = \mathrm{f}(x)$, where $\mathrm{f}(x) = \dfrac{2x^2 + 6x + k}{x+3}$, k is a non-zero real constant. It is given that the gradient of the curve C is always positive.

(i) Find the range of values of k. [3]

(ii) Sketch the curve C for the range of values of k in **(i)**, showing clearly, if any, the equation(s) of the asymptote(s) and axial intercept(s). [3]

(iii) By adding a suitable curve to the graph of $y = \mathrm{f}(x)$ in **(ii)**, deduce the number of distinct real roots of the equation $\dfrac{1}{4}\left[\dfrac{\left(2x^2 + 6x + k\right)^2}{(x+3)^2}\right] - x^2 = 1$. [2]

(iv) Given that $k = -1$ and drawing two suitable graphs, solve the inequality $\mathrm{f}(x) > |x - 3|$. [4]

8 A curve C has parametric equations

$$x = 2\sin\theta + 2 \ , \ y = 3\cos\theta - 1 \ ,$$

where $\theta \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$.

(i) The point $P\left(\sqrt{2} + 2 \ , \ \dfrac{3}{2}\sqrt{2} - 1\right)$ lies on the curve C. **Without converting the parametric equations into cartesian form**, show that the equation of the normal to the curve C at P is $6y = 4x - 14 + 5\sqrt{2}$. [5]

(ii) Find the area of the quadrilateral bounded by the y-axis, x-axis, the normal at the point P and the horizontal line passing through P, correct to 3 decimal places. [3]

(iii) What can be said about the tangents to C as $\theta \to \pm\dfrac{\pi}{2}$? [1]

(iv) Draw the curve C, showing clearly the features of the curve at the points where $\theta = -\dfrac{\pi}{2}$ and $\dfrac{\pi}{2}$. [2]

9 When referred to the origin O, the points A and B have position vectors $5\mathbf{j}+15\mathbf{k}$ and $\mathbf{i}-\mathbf{j}+3\mathbf{k}$ respectively.

The line l_1 has equation $\mathbf{r}=\begin{pmatrix}-2\\1\\3\end{pmatrix}+\lambda\begin{pmatrix}-1\\2\\3\end{pmatrix}$, $\lambda\in\mathbb{R}$.

(i) Find a vector equation of line l_2 passing through the points A and B. [2]

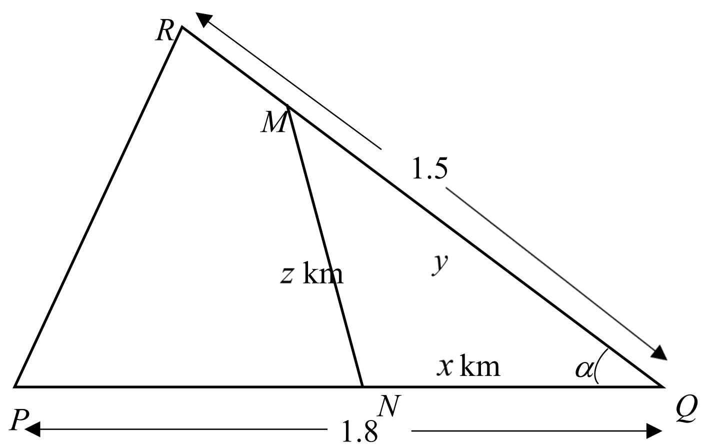
(ii) Find the coordinates of the point C, where l_1 and l_2 intersect. [3]

(iii) Find the position vector of the point F, the foot of the perpendicular from A to the line l_1. [4]

(iv) Find the vector equation of the line of reflection of l_2 in the line l_1. [3]

(v) Find a cartesian equation of the plane that contains the lines l_1 and l_2. [3]

10 A property developer wants to develop a triangular plot of land PQR as shown in the diagram below.
One section, NQM, is to be used for residential development and the other section, $PNMR$, is to be used for commercial development where M is on RQ and N is on PQ.
It is given that $NQ = x$ km, $QM = y$ km, $MN = z$ km, $RQ = 1.5$ km, $PQ = 1.8$ km, and a fixed angle $\angle NQM = \alpha$ radians where $\alpha\in\left(0,\frac{\pi}{2}\right)$.



(i) To achieve the requirements set out by the government on the use of the plot, the developer plans the use such that the residential development and commercial development takes up the same area each in the plot PQR.
Show that $z^2 = x^2+\dfrac{1.8225}{x^2}-2.7\cos\alpha$. [4]

The developer wants to build a fence on the boundary MN. In order to minimize the construction costs, he decides that the boundary MN should be of minimum length.

(ii) Using differentiation, find the value of x which will minimise the length MN, giving your answers correct to 3 decimal places. [7]

(iii) Given that $\angle NQM = \alpha = \dfrac{\pi}{3}$, sketch the graph showing the relationship of the square of the length MN as the length of NQ varies. [3]