ST ANDREW'S JUNIOR COLLEGE

PRELIMINARY EXAMINATION

MATHEMATICS HIGHER 2 9758/01

Monday 30 August 2021 3 hrs

Candidates answer on the Question Paper.
Additional Materials: List of Formulae (MF26)

NAME:____(_____) C.G.: __________

TUTOR'S NAME: ____

SCIENTIFIC / GRAPHIC CALCULATOR MODEL: ______________________

READ THESE INSTRUCTIONS FIRST

Write your name, civics group, index number and calculator models on the cover page.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions. Total marks : **100**

Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing calculator is expected, where appropriate.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

Question	1	2	3	4	5	6	7	8	9	10	TOTAL
Marks											
	9	7	11	8	9	10	13	8	13	12	100

This document consists of **6** printed pages including this page.

1 **(i)** The curve $y=\dfrac{ax^2-2ax+4}{x-b}$ has an oblique asymptote $y=x$ and a vertical asymptote $x=2$. State the value of b and find the value of a. **[2]**

(ii) By using an algebraic method, show that y cannot take values between -2 and 6. **[4]**

Using the information found in part **(i)** and part **(ii)**,

(iii) sketch the graph of $y=\dfrac{ax^2-2ax+4}{x-b}$, stating clearly any axial intercept(s), asymptotes and the coordinates of the turning point(s). **[3]**

2 **(a)** A curve has equation $y=\mathrm{f}(x)$, where

$$\mathrm{f}(x)=\begin{cases}1-3x & -1\le x\le 1,\\ -2-(x-1)^2 & 1<x\le 2,\\ 2 & \text{otherwise.}\end{cases}$$

Sketch the graph of $y=\mathrm{f}(x)$ for $-1\le x\le 4$, stating clearly the coordinates of any axial intercepts. State the range of f. **[4]**

(b) A curve undergoes the transformations *A, B* and *C* in succession. The transformations *A, B* and *C* are given as follows:

A: Translate 1 unit in the positive *x*-direction;

B: Scale parallel to *x*-axis by a scale factor of 5;

C: Reflect in the *y*-axis.

The equation of the resulting curve is $y=\mathrm{e}^{x-2}-x$. Determine the equation of the original curve. **[3]**

3 **(a)** A sequence of numbers u_1, u_2, u_3, … has a sum S_n where $S_n=\sum_{r=1}^{n}u_r$. It is given that $S_n=A(2^n)+Bn^2+C$, where *A, B* and *C* are non-zero constants.

(i) Find an expression for u_n in terms of *A*, *B* and *n*. **[2]**

(ii) It is also given that $u_1=7$ and $S_2=25$. Find *A*, *B* and *C*. **[4]**

(b) Show that $r(r+1)(r+2)-(r-2)(r-1)r=kr^2$, where k is a constant to be determined.

Use this result to deduce that $\sum_{r=1}^{n}r^2=\dfrac{n(n+1)(2n+1)}{6}$. **[5]**

4 Sketch the graphs of $y = 1 + \dfrac{a-2}{x-a}$, and $y = -\dfrac{1}{a}x + \dfrac{2}{a}$ on a single diagram, where a is a positive constant and $1 < a < 2$, showing all asymptotes and axial intercepts clearly. **[4]**

(i) Using the graphs, solve, in terms of a, $1 + \dfrac{a-2}{x-a} > -\dfrac{1}{a}x + \dfrac{2}{a}$. **[1]**

(ii) Hence, solve $1 + \dfrac{ax-2x}{1-ax} > -\dfrac{1}{ax} + \dfrac{2}{a}$. **[3]**

5 **(a)** The sum of the first n terms of a sequence is denoted by S_n. Given that $S_n = \mathrm{e}^n - 1$, prove that the sequence is a geometric progression. **[3]**

(b) The area of island S is $2880\ \text{km}^2$ at the end of 2019. Due to the rise in sea level, the area of the island decreases gradually every year. At the end of 2020, the area decreases by $64\ \text{km}^2$. Two companies, A and B, are engaged to study the trend in the decrease in the area of Island S.

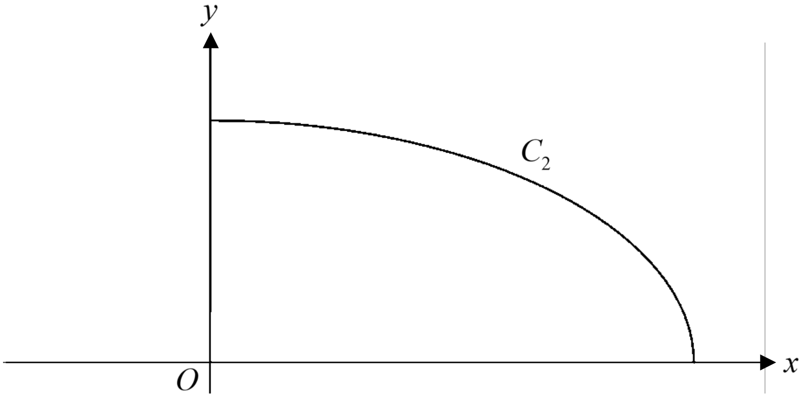
(i) According to Company A, the decrease in area in each subsequent year is $3\ \text{km}^2$ less than that of the previous year. The decrease in area continues annually up to and including the year when the decrease is first less than $5\ \text{km}^2$ occurs. Find the area of Island S when the decrease is first less than $5\ \text{km}^2$. **[4]**

(ii) According to Company B, the decrease in area in each subsequent year is $\dfrac{5}{6}$ that of the previous year. Determine the theoretical area of Island S in the long run. **[2]**

6 A curve C_1 has equation $\dfrac{4-x^2y^2}{x^2+y^2}=\dfrac{1}{2}$, where $y \geq 0$.

(i) Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x}=-\dfrac{x+2xy^2}{y+2x^2y}$. **[3]**

(ii) The point P on C_1 has x-coordinate 2. The tangent to C_1 at P cut the x-axis at point Q. Find the exact coordinates of Q. **[4]**



The diagram above shows a second curve C_2 with parametric equations

$$x=k^2\sin\theta, \qquad y=k\cos\theta,$$

where $0 \leq \theta \leq \dfrac{\pi}{2}$ and k is a positive constant.

(iii) The area bounded by the curve C_2, the x-axis and the y-axis is equal to the area of triangle OPQ, where O is the origin. Find the value of k correct to 2 decimal places. **[3]**

7 **(a)** It is given that $z=1+\sqrt{3}\mathrm{i}$ is a root of the equation $3z^3+az^2+bz-8=0$, where a and b are real numbers. Find the exact values of a and b and hence solve the equation completely, giving all the roots in exact form. **[5]**

(b) **(i)** Using Euler's formula that $\mathrm{e}^{\mathrm{i}\theta}=\cos\theta+\mathrm{i}\sin\theta$, show that $(\cos\theta+\mathrm{i}\sin\theta)^4=\cos 4\theta+\mathrm{i}\sin 4\theta$. **[1]**

(ii) Using result shown in **(i)**, by comparing the real parts, show that $\cos 4\theta=\cos^4\theta-6\cos^2\theta\sin^2\theta+\sin^4\theta$.
Obtain an expression for $\sin 4\theta$ in terms of $\cos\theta$ and $\sin\theta$.

Hence show that $\tan 4\theta=\dfrac{4\tan\theta-4\tan^3\theta}{1-6\tan^2\theta+\tan^4\theta}$. **[4]**

(iii) Hence, find the possible values of $\tan\theta$ given that $\tan 4\theta=4$, leaving your answers correct to 3 significant figures. **[3]**

8 The points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$, respectively, relative to the origin O. The points O , A and B are not collinear. The point P lies on AB between A and B such that AP: PB = $(1-\lambda):\lambda$, where $0<\lambda<1,\ \lambda\in\mathbb{R}$.

(i) Write down the position vector of P in terms of $\mathbf{a}$, $\mathbf{b}$ and λ . **[1]**

It is given that OP bisects $\angle AOB$.

(ii) Show that $\lambda=\dfrac{b}{a+b}$, where $a=|\mathbf{a}|$ and $b=|\mathbf{b}|$. **[3]**

The point Q also lies on AB between A and B, and is such that $AP = BQ$.

(iii) Find an expression for $\left(OQ^2-OP^2\right)$ in terms of a and b. **[4]**

9 A scientist is investigating two models for the motion of a falling object of mass 1 kg. At time t seconds, the object has fallen a distance of x metres with velocity v m s^{-1}. The object in both models falls vertically from a building with an initial velocity of 0 m s^{-1}.

(i) In the first model, the motion of the object is modelled by the differential equation $\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2}=9.8-0.2\left(\dfrac{\mathrm{d}x}{\mathrm{d}t}\right)^2$. By using the substitution $v=\dfrac{\mathrm{d}x}{\mathrm{d}t}$, show that the differential equation can be written as $\dfrac{\mathrm{d}v}{\mathrm{d}t}=9.8-0.2v^2$. Hence, show that $v=\dfrac{7\left(1-\mathrm{e}^{-\frac{14}{5}t}\right)}{1+\mathrm{e}^{-\frac{14}{5}t}}$. **[5]**

(ii) In the second model, the scientist considers that for any falling object, it experiences a downward gravitational acceleration of 9.8 m s^{-2} and an upward acceleration R due to air resistance, where $R\le 9.8$. The upward acceleration is directly proportional to v, with a constant of proportionality $k>0$.
The rate of change of v for the falling object is modelled as the difference between the gravitational acceleration and the upward acceleration due to air resistance. Form a differential equation relating v to t and k, and solve this differential equation to obtain v as a function of t and k. **[3]**

The terminal velocity of an object is the value of v after a long time.

(iii) Given that the objects in both models achieve the same terminal velocity, find the value of k. **[2]**

(iv) Using the value of k in **(iii)**, justify which model predicts that the object reaches 80% of its terminal velocity earlier. **[3]**

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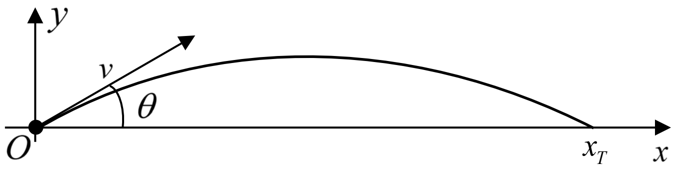

Figure 1

From a point O, a projectile is launched with a fixed velocity v ms^{-1} at a fixed angle of elevation θ from the horizontal, where v is a positive real constant and $0 < \theta < \frac{\pi}{2}$. The horizontal displacement, x metres, and the vertical displacement, y metres, of the projectile at time t seconds may be modelled by the parametric equations

$$x = (v\cos\theta)t, \qquad y = (v\sin\theta)t - \frac{1}{2}gt^2,$$

where g is a constant known as the acceleration due to gravity.

(i) Show that the time taken by the projectile to hit x_T is given by $\dfrac{2v\sin\theta}{g}$. **[2]**

(ii) A is the area enclosed by the path of the projectile as shown in **Figure 1** and the x-axis. **Without expressing *y* in terms of *x*,** show that $A = \dfrac{2v^4\sin^3\theta\cos\theta}{3g^2}$. **[4]**

(iii) Hence find the exact maximum value of A as θ varies in terms of v and g. **[6]**

End of Paper