ST ANDREW'S JUNIOR COLLEGE

PRELIMINARY EXAMINATION

MATHEMATICS	**HIGHER 2**	**9758/02**
Tuesday	**14 September 2021**	**3 hrs**

Candidates answer on the Question Paper.
Additional Materials: List of Formulae (MF26)

NAME:_____(_____) **C.G.:** _____

TUTOR'S NAME: _____

SCIENTIFIC / GRAPHIC CALCULATOR MODEL: _____

READ THESE INSTRUCTIONS FIRST

Write your name, civics group, index number and calculator models on the cover page.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions. Total marks : **100**

Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing calculator is expected, where appropriate.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

Question	1	2	3	4	5	6	7	8	9	10	11	TOTAL
Marks												
	9	11	10	10	7	7	8	9	5	12	12	100

This document consists of **7** printed pages including this page.

[Turn Over]

Section A: Pure Mathematics [40 marks]

1 A function f is said to self-inverse if $f(x) = f^{-1}(x)$ for all x in the domain of f.
The functions f and g are defined by

$$f : x \mapsto \frac{ax + 7a^2}{x - a}, \ x \in \mathbb{R}, x \neq a, \ 0 < a < 1$$
$$g : x \mapsto (x - 2)(x + 1), \ x \in \mathbb{R}, x > 3.$$

(i)   Explain why f^{-1} exists and show that f is self-inverse. **[4]**

(ii) Hence, evaluate $f^{241}\left(\dfrac{1}{a}\right)$ in terms of a, showing your working clearly. **[2]**

(iii) Show that fg exists. Find the exact range of fg in terms of a. **[3]**

2 **(a)** By using the substitution $x = \dfrac{1}{2}\sin u$, find the exact area of the region enclosed by

the curve $y = \sqrt{1 - 4x^2}$ and the x-axis. **[4]**

(b) The region A is bounded by the curve C with equation $y = e^x \sin x$, the x-axis and

the lines $x = -\pi$ and $x = \dfrac{\pi}{2}$.

(i) Find the exact area of A. **[5]**

(ii) A is now rotated about the x-axis through 2π radians. Find the volume of solid formed. **[2]**

3   The line L has equation $\mathbf{r} = -\mathbf{i} + 2\mathbf{j} - 4\mathbf{k} + \lambda(6\mathbf{i} + 3\mathbf{j} - 2\mathbf{k})$, where $\lambda \in \mathbb{R}$.
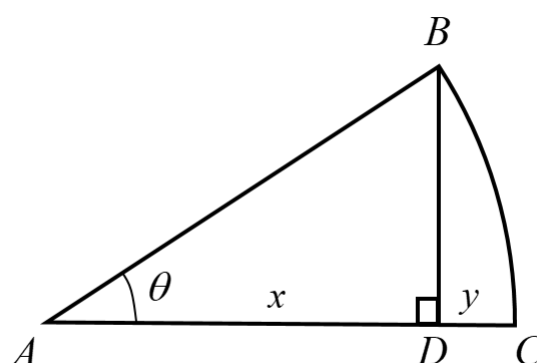
   (i)   Find the acute angle between L and the x-axis.   **[2]**

   The point P has position vector $6\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$.

   (ii)   Find the points on L which are at a distance of $\sqrt{59}$ units from P.
   Hence or otherwise find the point on L which is closest to P.   **[5]**

   (iii)   Find a cartesian equation of the plane that includes the line L and the point P.   **[3]**

4

ABC is a sector of a circle with radius AB and AC and $\angle BAC = \theta$. D is a point on AC such that $AD = x$ m, $CD = y$ m and $\angle ADB = \dfrac{\pi}{2}$.

   (i)   Show that $BD = \sqrt{y^2 + 2xy}$.   **[1]**

   (ii)   Hence show that $\sin\theta \approx (2\alpha)^{\frac{1}{2}}\left(1 - \dfrac{3}{4}\alpha\right)$, where $\alpha = \dfrac{y}{x}$, and y is small compared
   to x.   **[4]**

   You are now given that $\theta = 10^0$ and $x = 100$.
   (iii)   Use part **(ii)** to find estimates for α and y correct to 2 decimal places.   **[3]**
   (iv)   Find the actual value of y.   **[1]**
   (v)   Comparing the values of y obtained in parts **(iii)** and **(iv)**, comment on the accuracy
   of your approximations and explain your answer.   **[1]**

Section B: Statistics [60 marks]

5 The events X, Y and Z are such that $P(X) = x$, $P(Y) = y$, and $P(Z) = \dfrac{1}{4}$, where x and y are non-zero. It is given that the events X and Y are independent, events Y and Z are independent, and that events X and Z are independent.

    (i) Find an expression for $P(X \cap Z')$.   **[2]**

    (ii) Find an expression for $P(Y' \mid X')$.   **[2]**

It is now given that $P\big((X \cup Y \cup Z)'\big) = \dfrac{2}{5}$, $P(X \cap Y \cap Z) = 0$, and $y = 5x$

    (iii) By drawing a suitable Venn Diagram, find the value of x.   **[3]**

6 Shane plays a game by tossing three biased coins. If a coin shows a head, Shane scores 1 point; if a coin shows a tail, he scores -1 point.
The random variable X is the total score obtained when three such coins are tossed.
It is given that the probability of obtaining a head when a biased coin is tossed is p, where $0 < p < 1$.

    (i) Tabulate the probability distribution of X.   **[3]**

    (ii) Find E(X) in terms of p.

    Hence show that $\left(\dfrac{X+3}{6}\right)$ is an unbiased estimator of p.   **[4]**

7 A class committee consists of 6 boys and 3 girls. They are to sit at random around a table for a meeting.

    (i) Find the number of possible arrangements if not all 3 girls are seated together. **[3]**

    (ii) Find the number of possible arrangements with all 3 girls separated.   **[2]**

    (iii) Find the probability that there are exactly two boys seated between any two girls.

        [3]

8 A company that sells potato chips in packets claims that the mean amount of sodium content per packet is 798 mg.

(a) It is given that the standard deviation of sodium content in the packets of potato chips is 6.5 mg. Upon receiving complaints that the potato chips were too salty, the local food agency chooses a large random sample of n packets of potato chips and found that the mean sodium content is 799.5 mg.
Given that the local food agency concludes at the 5% level of significance that the company has understated the amount of sodium content, find the set of values of n. **[4]**

(b) The company decided to try out a new healthier recipe that reduces the amount of sodium content per packet. The company collected data from a random sample of 50 packets and found that the mean is 796.3 mg and standard deviation is 6.2 mg.
    (i) Test at 5% level of significance whether the mean amount of sodium in a packet of potato chips has been reduced from the original claim, after the introduction of the new healthier recipe. **[4]**
    (ii) Explain, in the context of the question, the meaning of "at 5% level of significance" for the test in **(b)(i)**. **[1]**

9 **(i)** Sketch a scatter diagram that might be expected when x and y are related approximately as given in each of the cases (A), (B) and (C) below. In each of the case, your diagram should include 6 points, approximately equally spaced with respect to x, and with all x- and y-values positive. The letters a, b, c, d, e and f represent constants.
(A) $y = a + bx^2$, where a is positive and b is negative.
(B) $y = c + d \ln x$, where c is positive and d is negative. **[2]**

A motoring website gives the following information about the distance travelled, y km, by a certain type of car at different speeds, x km h^{-1}, on a fixed amount of fuel.

Speed, x	88	96	104	112	120	128
Distance, y	144	147	144	138	126	107

(ii) Draw a scatter diagram for these values, labelling the axes. **[1]**
(iii) Explain which of the three cases in part **(i)** is the most appropriate for modelling these values, and calculate the product moment correlation coefficient for this case. **[2]**

10   During a pandemic, an unknown infectious disease spread through a population. Researchers wanted to study the probability, p, that a person is infected. They collected some data from a random sample of 25 people.

(i)   State, in the context of this question, two assumptions needed for the number of infected people to be well modelled by a binomial distribution.   **[2]**

Assume now that these assumptions do in fact hold.

(ii)   It is given that the probability that at most 2 people are infected is 0.0982. Write an equation in p and solve for p.   **[2]**

A total of 30 random samples of 25 people were chosen from this population.

(iii) Find the probability that the total number of samples with at most 2 infected people is at least 5.   **[3]**

It is now given that $p = 0.10$.

A diagnostic test for the presence of the disease was carried out on the population. It was found that the percentage of infected people who were tested positive is 85%, while the percentage of non-infected people who were tested negative is 96%.

(iv)   Find the probability that a person who tested positive carries the disease.   **[3]**

(v)   Hence or otherwise, find the probability that a person who tested positive does not carry the disease.   **[1]**

(vi)   Discuss briefly if the diagnostic test is worthwhile.   **[1]**

11(a) Anne, a Bubble Tea (BBT) seller, intends to increase the sales of BBT using a drink vending machine which delivers BBT into a cup when cash payment is made into the machine. The volume of BBT dispensed is normally distributed with mean 210 ml and standard deviation 5 ml. The capacity of a cup is 220 ml and the nominal amount of BBT in a cup is stated as 212 ml.

(i)   Find the probability that a cup overflows when BBT is dispensed into the cup.
    [1]

(ii)   A customer bought five cups of BBT from the vending machine.
       Find the probability that at most one cup of BBT will overflow.   **[2]**

(iii) Anne received complaints from some customers that there is a high proportion of cups with less than the nominal amount of BBT. It is assumed that the standard deviation of the volume of BBT dispensed is fixed while the mean volume of BBT dispensed could be adjusted. Find the range of the mean volume of BBT dispensed such that not more than 10% of the cups will contain less than the nominal amount of BBT.   **[3]**

(iv)   Another customer bought n cups of BBT. Find the approximate probability that the total volume of BBT dispensed exceeds $211n$ ml as n becomes very large.   **[2]**

(v)   Anne wishes to gather feedback about her BBT. She decided to interview 50 customers who bought BBT from the vending machine during lunch time. Give a reason as to whether Anne would obtain a random sample of customers. **[1]**

[Turn Over]

(b) On another occasion, Anne deployed a staff to operate a BBT counter at a wedding reception. The average volume of BBT per serving is 200 ml and the standard deviation of the volume of BBT is given to be 10 ml. Find the probability that the mean volume of 60 servings of BBT prepared by the staff is less than 198 ml.

[3]

End of Paper