RAFFLES INSTITUTION
2021 YEAR 6 PRELIMINARY EXAMINATION

CANDIDATE NAME

CLASS 21

MATHEMATICS — 9758/02

Paper 2 — **3 hours**

Candidates answer on the Question Paper
Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name and class on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the Question Paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

FOR EXAMINER'S USE

SECTION A: PURE MATHEMATICS				
Q1	Q2	Q3	Q4	Total
8	10	10	12	40

SECTION B: PROBABILITY AND STATISTICS						
Q5	Q6	Q7	Q8	Q9	Q10	Total
6	8	9	12	12	13	60

TOTAL
100

Section A: Pure Mathematics [40 marks]

1 Functions f and g are defined by

$$\mathrm{f}: x \mapsto \mathrm{e}^{(x-1)^2}, \quad x \in \mathbb{R},$$

$$\mathrm{g}: x \mapsto \frac{1}{2-x}, \quad x \in \mathbb{R}, \quad 1 \le x < 2.$$

(i) Sketch the graph of $y = \mathrm{f}(x)$. [1]

(ii) If the domain of f is restricted to $x \ge k$, state with a reason the least value of *k* for which the function f^{-1} exists. [2]

In the rest of the question, the domain of f is $x \ge k$, using the value of *k* found in part **(ii)**.

(iii) Find $\mathrm{g}^{-1}(x)$ and show that the composite function $\mathrm{g}^{-1}\mathrm{f}^{-1}$ exists. [4]

(iv) Find the range of $\mathrm{g}^{-1}\mathrm{f}^{-1}$. [1]

2 **(a)** Three consecutive terms of a decreasing geometric progression has a product of 5832. If the first number is reduced by 24, these 3 numbers in the same order will form an arithmetic progression. Find the three terms of the geometric progression. [5]

(b) The fractal called Sierpiński Triangle is depicted below. Fig. 1 shows an equilateral triangle of side 1. In stage 1, the triangle in Fig. 1 is divided into four smaller *identical* equilateral triangles and the middle triangle is removed to give the triangle shown in Fig. 2. In stage 2, the remaining three equilateral triangles in Fig. 2 are each divided into four smaller *identical* equilateral triangles and the middle triangles are removed to give the triangle shown in Fig. 3 and the process continues.

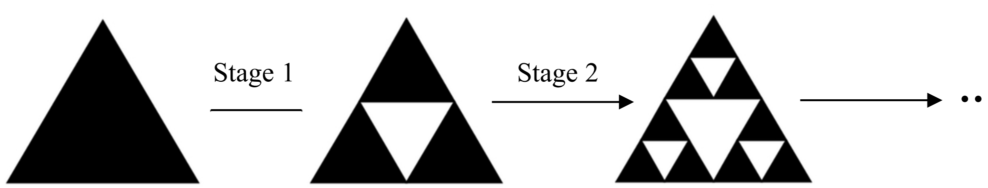


Fig. 1 **Fig. 2** **Fig. 3**

Let T_n be the total area of triangles removed after *n* stages of the process.

(i) Show that $T_1 = \dfrac{\sqrt{3}}{16}$. [1]

(ii) Find T_{10}. [3]

(iii) State the exact value of $\lim_{n \to \infty} T_n$. [1]

3 It is given that $\ln y=\sqrt{1+8\mathrm{e}^{x}}$.

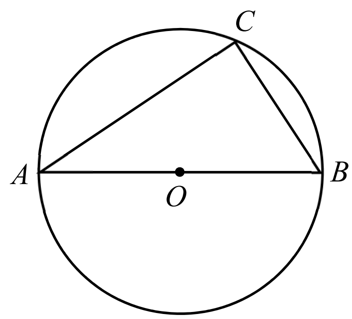
(i) Show that $(\ln y)\dfrac{\mathrm{d}y}{\mathrm{d}x}=4y\mathrm{e}^{x}$. [1]

(ii) Show that the value of $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ when $x=0$ is $\dfrac{68}{27}\mathrm{e}^{3}$. [4]

(iii) Hence find the Maclaurin series for $\mathrm{e}^{\sqrt{1+8\mathrm{e}^{x}}}$ up to and including the term in x^2. [2]

(iv) Denoting the answer found in part **(iii)** as $\mathrm{g}(x)$, find the set of values of x for which $\mathrm{g}(x)$ is within ±0.5 of the value of $\mathrm{e}^{\sqrt{1+8\mathrm{e}^{x}}}$. [3]

4 **(a)** **(i)**



Referred to the origin O, points A, B and C have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively. The three points lie on a circle with centre O and diameter AB (see diagram).
Using a suitable scalar product, show that the angle ACB is $90°$. [4]

(ii) The variable vector $\mathbf{r}$ satisfies the equation $(\mathbf{r}-\mathbf{i})\cdot(\mathbf{r}-\mathbf{k})=0$. Describe the set of vectors $\mathbf{r}$ geometrically. [2]

(b) **(i)** The variable vector $\mathbf{r}$ satisfies the equation $\mathbf{r}\cdot\mathbf{n}=\mathbf{m}\cdot\mathbf{n}$, where $\mathbf{m}$ and $\mathbf{n}$ are constant vectors. Describe the set of vectors $\mathbf{r}$ geometrically.
Give the geometrical meaning of $|\mathbf{m}\cdot\mathbf{n}|$ if $\mathbf{n}$ is a unit vector. [2]

(ii) The plane π passes through the points with position vectors $x\mathbf{i}, y\mathbf{j}$ and $z\mathbf{k}$ where x, y and z are non-zero constants. It is given that d is the perpendicular distance from the origin to π. Show, by finding the normal of π, or otherwise, that $\dfrac{1}{x^2}+\dfrac{1}{y^2}+\dfrac{1}{z^2}=\dfrac{1}{d^2}$. [4]

Section B: Probability and Statistics [60 marks]

5 For events A and B it is given that $\mathrm{P}(A)=0.3$, $\mathrm{P}(B|A)=0.4$ and $\mathrm{P}(A'\cap B')=0.15$. Find

(i) $\mathrm{P}(A\cup B)$, [1]

(ii) $\mathrm{P}(B)$, [3]

(iii) $\mathrm{P}(A|B')$. [2]

6 The recruitment manager of the private car hire company, I-ber, claims that the mean weekly earnings of a full-time driver is \$980. The managing director suspects that the mean weekly earnings is less than \$980 and he instructs the recruitment manager to carry out a hypothesis test on a sample of drivers. It is given that the population standard deviation of the weekly earnings is \$88.

(i) State suitable hypotheses for the test, defining any symbols that you use. [2]

The recruitment manager takes a random sample of 10 drivers. He finds that the weekly earnings in dollars, are as follows.

942 950 905 1003 883 987 924 920 913 968

(ii) Find the mean weekly earnings of the sample of these 10 drivers. Carry out the test, at 5% level of significance, for the recruitment manager. Give your conclusion in context and state a necessary assumption for the test to be valid. [5]

(iii) Find the smallest level of significance at which the test would result in rejection of the null hypothesis, giving your answer correct to 1 decimal place. [1]

7 **In this question, you should state clearly all the distributions that you use, together with the values of the appropriate parameters.**

A company sells hand sanitiser in bottles of two sizes – small and large. The amounts, in ml, of hand sanitiser in the small and large bottles, are modelled as having independent normal distributions with means and standard deviations as shown in the table.

	Mean	Standard deviation
Small bottles	108	5
Large bottles	510	σ

(i) Find the probability that the amount of hand sanitiser in a randomly chosen small bottle is less than 100 ml. [1]

(ii) During a quality control check on a batch of small bottles of hand sanitiser, 100 small bottles are randomly chosen to be inspected by an officer one at a time. Once he finds five bottles, each with amount of hand sanitiser less than 100 ml, that batch will be rejected. Find the probability that he had to check through all 100 bottles to reject that batch. [2]

(iii) Given that the amount of hand sanitiser in 85% of the large bottles lie within 9 ml of the mean, find σ. [3]

(iv) Given instead that $\sigma = 6$, find the probability that the amount of hand sanitiser in a randomly chosen large bottle is less than five times the amount of hand sanitiser in a randomly chosen small bottle. [3]

8 This question is about arrangements of all eight letters in the word IMMUNITY.

(i) Show that the number of different arrangements of the eight letters that can be made is 10080. [1]

(ii) Find the number of different arrangements that can be made with no two vowels next to each other. [3]

One of the 10080 arrangements in part **(i)** is randomly chosen.
Let A denote the event that the two I's are next to each other and let B denote the event that the two M's are next to each other.

(iii) Determine, with a reason, whether A and B are

(a) mutually exclusive, [1]

(b) independent. [3]

(iv) Find the probability that the chosen arrangement contains no two adjacent letters that are the same. [4]

9 A bag initially contains 3 red balls and 3 black balls. Whenever a red ball is drawn from the bag, it is put back into the bag together with an extra red ball. Whenever a black ball is drawn from the bag, it is not put back into the bag and no extra balls are added.

Isaac draws n balls from the bag, one after another, where $n \in \mathbb{Z}^+$, and R denotes the number of red balls out of the n balls drawn.

(a) Give two reasons why R cannot be modelled using a Binomial distribution. [2]

(b) For $n = 3$, find

(i) $\mathrm{P}(R \geq 1)$, [2]

(ii) the probability that the first ball drawn is black given that at least 1 of the 3 balls drawn is red. [3]

(c) For $n = 31$, show that $\mathrm{P}(R = 31) = \dfrac{1}{714}$. [2]

(d) Isaac wins 100 dollars for each red ball he draws if all the balls he draws from the bag are red, and does not win any money otherwise. What is the maximum amount of money Isaac would win if the probability of all the balls he draws are red exceeds 0.0001? [3]

10 A bag contains four balls numbered 1, 2, 3 and 4. In a game, a ball is drawn at random from the bag and then a fair coin is tossed a number of times that is equal to the number shown on the ball drawn. The random variable X is the number of heads recorded.

(i) Show that $\mathrm{P}(X = 0) = \dfrac{15}{64}$. Find $\mathrm{P}(X = x)$ for all other possible values of x. [5]

(ii) Denoting the expectation and variance of X by μ and σ^2 respectively, find $\mathrm{P}(X > \mu)$ and show that $\sigma^2 = \dfrac{15}{16}$. [3]

Adam plays this game 10 times.

(iii) Find the probability that there are at least two games with at least 2 heads recorded. [2]

Bill plays this game 50 times.

(iv) Using a suitable approximation, estimate the probability that the average number of heads recorded is less than 1. [3]