RAFFLES INSTITUTION
2021 YEAR 6 PRELIMINARY EXAMINATION

CANDIDATE NAME

CLASS 21

MATHEMATICS 9758/01

Paper 1 **3 hours**

Candidates answer on the Question Paper
Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name and class on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

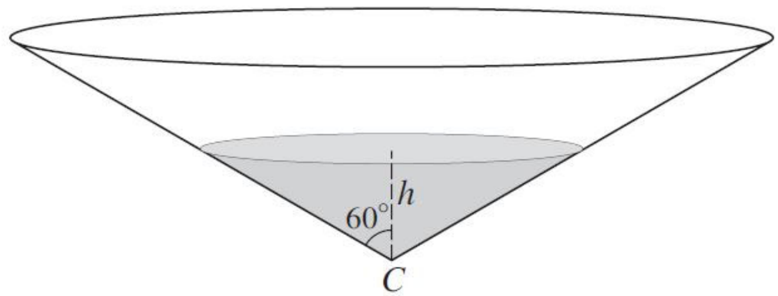
Answer **all** the questions.
Write your answers in the spaces provided in the Question Paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

FOR EXAMINER'S USE					
Q1	Q2	Q3	Q4	Q5	Q6
4	4	8	8	9	9
Q7	Q8	Q9	Q10	Q11	Total
11	11	12	12	12	100

1 Given the polynomial x^4+ax^2+bx+c has a factor $(x-2)$ and gives remainders 12 and 26 when divided by $(x-3)$ and $(x-4)$ respectively, find the values of a, b and c. [4]

2



A tank containing water is in the form of a cone with vertex C. The axis is vertical and the semi-vertical angle is 60°, as shown in the diagram. At time t = 0, the tank is filled with 94π cm^3 of water. At this instant, a tap at C is turned on and water begins to flow out at a constant rate of 2π cm^{3}s^{-1}. Denoting h cm as the depth of water at time t s, find the rate of decrease of h when t = 15, leaving your answer in exact form. [4]

[The volume V of a cone of vertical height h and base radius r is given by $V=\frac{1}{3}\pi r^2 h$.]

3 **(a)** Find $\int x\tan^{-1}x\,\mathrm{d}x$. [3]

(b) **(i)** Using the substitution $u=\frac{1}{x}$, or otherwise, find $\int \frac{\sin\left(\frac{1}{x}\right)}{x^2}\,\mathrm{d}x$. [2]

(ii) Given that n is a positive integer, evaluate the integral $\pi\int_{\frac{1}{(n+1)\pi}}^{\frac{1}{n\pi}} \frac{\sin\left(\frac{1}{x}\right)}{x^2}\,\mathrm{d}x$,

giving your answer in the form $a\pi$, where the possible values of a are to be determined. [3]

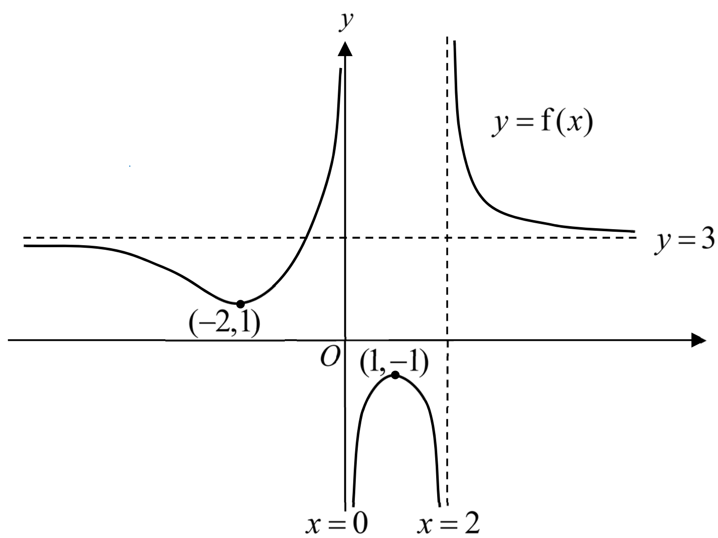
4 **(a)** Show that $(2r+1)^3-(2r-1)^3=12kr^2+k$, where k is a constant to be determined.

Use this result to find $\sum_{r=1}^{n} r^2$, giving your answer in the form $pn(qn+1)(2qn+1)$ where p and q are constants to be determined. [5]

(b) Raabe's test states that a series of positive terms of the form $\sum_{r=1}^{\infty} a_r$ converges when $\lim_{n\to\infty}\left[n\left(\frac{a_n}{a_{n+1}}-1\right)\right]>1$, and diverges when $\lim_{n\to\infty}\left[n\left(\frac{a_n}{a_{n+1}}-1\right)\right]<1$.

When $\lim_{n\to\infty}\left[n\left(\frac{a_n}{a_{n+1}}-1\right)\right]=1$, the test is inconclusive. Using the test, explain why the series $\sum_{r=1}^{\infty}\frac{1}{r^3}$ converges. [3]

5 **(a)** The graph of $y=\mathrm{f}(x)$ is shown below.



On separate diagrams, sketch the following graphs, indicating clearly the key features.

(i) $y=\mathrm{f}(1-x)$, [3]

(ii) $y=\mathrm{f}'(x)$. [3]

(b) State a sequence of transformations which transform the graph of $y=\ln\left(1-\frac{x}{2}\right)$ onto the graph of $y=\ln\left(\frac{2}{1-x}\right)$. [3]

6 **Do not use a calculator in answering this question.**

(a) Show that $z = 2\mathrm{i}$ is a root of the equation $z^3 + 2z + 4\mathrm{i} = 0$. [2]
Hence find the other roots. [3]

(b) Let $w_1 = -\frac{\sqrt{6}}{2} + \frac{\sqrt{2}}{2}\mathrm{i}$ and $w_2 = 1 + \mathrm{i}$.

Find the smallest positive integer n such that $\arg\left(\frac{w_2}{w_1}\right)^n = -\frac{\pi}{2}$. [4]

7 A curve C has parametric equations

$$x = \sin t\,,\quad y = \frac{1}{3}\cos t\,,\ \text{for } -\pi \le t \le \frac{\pi}{4}.$$

(i) Find the equation of the normal to C at the point P with parameter $t = p$. [3]

(ii) The normal to C at the point when $t = -\frac{\pi}{4}$ cuts the curve again at point A. Find the coordinates of point A, correct to 2 decimal places. [4]

(iii) Sketch the graph of C, giving the coordinates of the end points in exact form. [2]

(iv) Find the area of the region bounded by C, the x-axis and the lines $x = 0$ and $x = \frac{1}{\sqrt{2}}$. [2]

8 **(i)** The curve G has equation $y = \frac{1}{1+x^2}$. Sketch the graph of G, stating the equation(s) of any asymptote(s) and the coordinates of any turning point(s). [2]

(ii) The line l intersects G at $x = 0$ and is tangential to G at the point (c, d), where $c > 0$. Find c and d, and determine the equation of l. [4]

Let R denote the region bounded by G, the x-axis and the lines $x = 0$ and $x = 1$.

(iii) By comparing the area of R and the area of the trapezoidal region between l and the x-axis for $0 \le x \le 1$, show that $\pi > 3$. [2]

(iv) By considering the volume of revolution of a suitable region rotated through 2π radians about the y-axis, show that $\ln 2 > \frac{2}{3}$. [3]

9 The equations of a plane p_1 and a line l are shown below:

$$p_1: \quad \mathbf{r}.\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 10,$$

$$l: \quad \frac{x+1}{3} = z+4, \ y = 1.$$

Referred to the origin O, the position vector of the point A is $2\mathbf{i}+\mathbf{j}-3\mathbf{k}$.

(i) Find the coordinates of the foot of perpendicular, N, from A to p_1. [4]

(ii) Find the position vector of the point B which is the reflection of A in p_1. [2]

(iii) Hence, or otherwise, find an equation of the line l', the reflection of l in p_1. [4]

(iv) Another plane, p_2, contains B and is parallel to p_1. Determine the exact distance between p_1 and p_2. [2]

10 Bob purchases a house and takes a loan of \$$A$ from a bank. The sum of money owed to the bank t months after taking the loan is denoted by \$$x$. Both x and t are taken to be continuous variables. The sum of money owed to the bank increases, due to interest, at a rate proportional to the sum owed and decreases at a constant rate r as Bob repays the bank.
When $x = a$, interest and repayment balance. Write down a differential equation relating x and t, and solve it to give x in terms of t, r, a and A. [8]

State the condition under which the sum of money owed to the bank is repaid in a finite time T months, justifying your answer. Show that $T = \frac{a}{r}\ln\left(\frac{a}{a-A}\right)$. [4]

11 $\left[\text{It is given that the volume of a sphere of radius } R \text{ is } \frac{4}{3}\pi R^3.\right]$

Craft drinks have been gaining popularity in the beverage industry in recent years. These drinks are usually freshly made and served cold, with much attention given to the ingredients that make up the drinks and the entire process of preparation.

Ice is a very important ingredient in the making of a craft drink as it affects two crucial components: the temperature and the dilution of the drink. Hence, great emphasis is placed on the shapes of the ice, as different shapes will offer different surface areas and thus have a direct impact on the taste of the drink.

An ice manufacturer, who specialises in producing cylindrical shaped ice suitable for craft drinks served in tall glasses, wants to find out information about the surface area of the cylindrical shaped ice he produces.

(i) A piece of cylindrical shaped ice has radius r, height h and a fixed volume V. Show that its surface area, S, is given by $2\pi r^2 + \frac{2V}{r}$. [2]

(ii) Use differentiation to find, in terms of V, the minimum value of S, proving that it is a minimum. You are to give your answer in the form $k\left(m\pi V^m\right)^{\frac{1}{k}}$, where k and m are positive integers to be found. Find also the ratio $r : h$ that gives this minimum value of S. [7]

There has been a growing trend to use one large piece of ice for craft drinks to create a better drinking experience for the customers. Spherical shaped ice is considered ideal as it can keep the drink at a constant cold temperature with minimal dilution.

(iii) For the minimum value of S found in part **(ii)**, show that the volume of the largest spherical shaped ice that can be carved out is $\frac{m}{k}V$, where k and m are the same integers found in part **(ii)**. [2]

(iv) State, giving a reason, whether the manufacturer should proceed to carve out spherical shaped ice from the existing cylindrical shaped ice produced. [1]