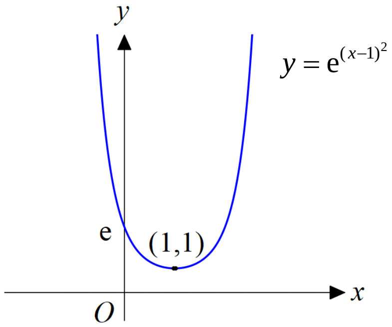
RAFFLES INSTITUTION
H2 Mathematics (9758)
2021 Year 6

2021 Year 6 H2 Math Preliminary Paper 2: Solutions

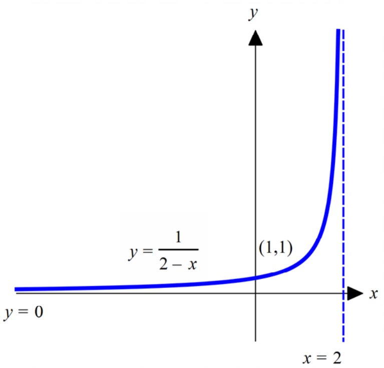
<table>
<tr><th colspan="2">Qn 1: Solutions</th></tr>
<tr><td>(i)
[1]</td><td>


</td></tr>
<tr><td>(ii)
[2]</td><td>For f^{-1} to exist, f must be a one-one function.
Least value of $k = 1$.</td></tr>
<tr><td>(iii)
[4]</td><td>
Let $y = \dfrac{1}{2-x}$

$2 - x = \dfrac{1}{y}$

$x = 2 - \dfrac{1}{y}$

$g^{-1}(x) = 2 - \dfrac{1}{x}$



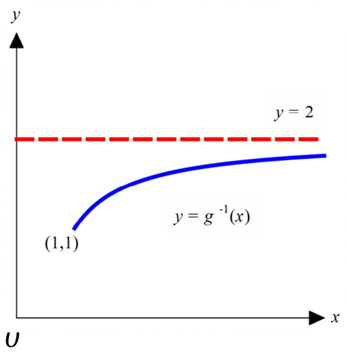
For $g^{-1}f^{-1}$ to exist, $R_{f^{-1}} \subseteq D_{g^{-1}}$.

$R_{f^{-1}} = D_f = [1, \infty)$

$D_{g^{-1}} = R_g = [1, \infty)$

Since $R_{f^{-1}} = D_{g^{-1}}$, $\therefore$ $g^{-1}f^{-1}$ exists.
</td></tr>
<tr><td>(iv)
[1]</td><td>
$[1, \infty) \xrightarrow{f^{-1}} [1, \infty) \xrightarrow{g^{-1}} [1, 2)$

$R_{g^{-1}f^{-1}} = [1, 2)$



Note: $D_{f^{-1}} = R_f = [1, \infty)$,
$R_{f^{-1}} = D_f = [1, \infty)$

OR
Since $R_{f^{-1}} = D_{g^{-1}}$, $R_{g^{-1}f^{-1}} = R_{g^{-1}} = D_g = [1, 2)$
</td></tr>
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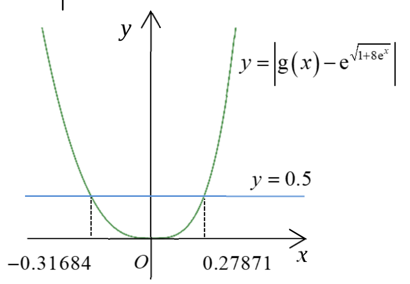
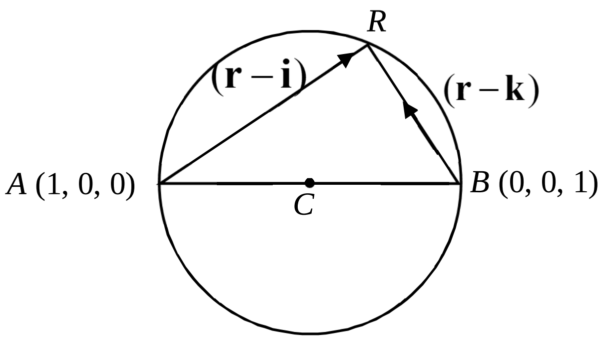
Qn 2: Solution	
(a) **[5]**	Let the 3 numbers be $\frac{x}{r}$, x and xr, where x is the middle term and r is the common ratio. $\frac{x}{r}(x)(xr)=5832$ $x^3=5832$ $x=18$ If the first number is reduced by 24, it is now $\frac{x}{r}-24$. Since the 3 numbers now form an AP, $\left(\frac{x}{r}-24\right)-x=x-xr$ Substitute $x=18$ into the above equation, $\left(\frac{18}{r}-24\right)-18=18-18r$ $3r^2-10r+3=0$ $(3r-1)(r-3)=0$ $\therefore r=\frac{1}{3}$ or $r=3$ (rejected $\because$ it is a decreasing GP, i.e. $0<r<1$) Thus, the original 3 numbers are 54, 18, 6.
(b) **(i)** **[1]**	Let A be the original area of the triangle in Fig. 1. $A=\frac{1}{2}(1)(1)\sin 60^\circ$ $=\frac{\sqrt{3}}{4}$ $T_1=\frac{1}{4}A=\frac{\sqrt{3}}{16}$ (shown)
(ii) **[3]**	Area of triangle removed in stage $1=T_1$ Area of triangles removed in stage $2=\frac{3}{4}T_1$ Area of triangles removed in stage $3=\left(\frac{3}{4}\right)^2 T_1$ $\vdots$ Area of triangles removed in stage $n=\left(\frac{3}{4}\right)^{n-1} T_1$

	Total area of triangles removed after 10 stages, T_{10} $=T_1+\frac{3}{4}T_1+\left(\frac{3}{4}\right)^2 T_1+\cdots+\left(\frac{3}{4}\right)^9 T_1$ $=\dfrac{T_1\left(1-\left(\frac{3}{4}\right)^{10}\right)}{1-\frac{3}{4}}$ $=0.409$ (3 s.f.)
(iii) **[1]**	$\lim_{n\to\infty} T_n=\dfrac{T_1}{1-\frac{3}{4}}=\dfrac{\sqrt{3}}{4}$ **OR** $\lim_{n\to\infty} T_n=A=\dfrac{\sqrt{3}}{4}$

Qn 3: Solution	
(i) **[1]**	$\ln y=\sqrt{1+8e^x}$ Differentiate w.r.t x, $\frac{1}{y}\frac{dy}{dx}=\frac{1}{2\sqrt{1+8e^x}}\left(8e^x\right)$ $\frac{1}{y}\frac{dy}{dx}=\frac{4e^x}{\ln y}$ $(\ln y)\frac{dy}{dx}=4ye^x$ (shown) **OR** $\ln y=\sqrt{1+8e^x}$ $(\ln y)^2=1+8e^x$ Differentiate w.r.t x, $2(\ln y)\left(\frac{1}{y}\right)\frac{dy}{dx}=8e^x$ $(\ln y)\frac{dy}{dx}=4ye^x$ (shown)
(ii) **[4]**	$(\ln y)\frac{dy}{dx}=4ye^x$ Differentiate w.r.t x, $(\ln y)\frac{d^2y}{dx^2}+\frac{1}{y}\left(\frac{dy}{dx}\right)^2=4ye^x+4e^x\frac{dy}{dx}$ When $x=0$, $e^x=1$. $\ln y=3\Rightarrow y=e^3$ $3\frac{dy}{dx}=4e^3\Rightarrow\frac{dy}{dx}=\frac{4}{3}e^3$ $3\frac{d^2y}{dx^2}+\frac{1}{e^3}\left(\frac{4}{3}e^3\right)^2=4e^3+4\left(\frac{4}{3}e^3\right)$ $3\frac{d^2y}{dx^2}+\frac{16}{9}e^3=\frac{28}{3}e^3$ $\frac{d^2y}{dx^2}=\frac{1}{3}\left(\frac{28}{3}e^3-\frac{16}{9}e^3\right)=\frac{68}{27}e^3$ (shown)
(iii) **[2]**	$\ln y=\sqrt{1+8e^x}\Leftrightarrow y=e^{\sqrt{1+8e^x}}$ By Maclaurin Theorem, $e^{\sqrt{1+8e^x}}=e^3+\frac{4}{3}e^3x+\frac{68}{27}e^3\frac{x^2}{2!}+\ldots$ $=e^3\left(1+\frac{4}{3}x+\frac{34}{27}x^2\right)+\ldots$

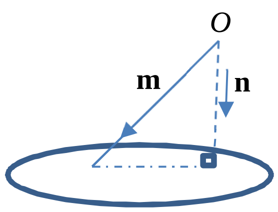
(iv) **[3]**	Let $\mathrm{g}(x)=\mathrm{e}^3\left(1+\frac{4}{3}x+\frac{34}{27}x^2\right)$. $\left\lvert \mathrm{g}(x)-\mathrm{e}^{\sqrt{1+8\mathrm{e}^x}}\right\rvert<0.5$  $x\in(-0.317, 0.279)$ (3 s.f.)

Qn 4: Solution	
(a) (i) [4]	$\overrightarrow{AC}\cdot\overrightarrow{BC}=(\overrightarrow{OC}-\overrightarrow{OA})\cdot(\overrightarrow{OC}-\overrightarrow{OB})$ $=(\mathbf{c}-\mathbf{a})\cdot(\mathbf{c}-\mathbf{b})$ $=(\mathbf{c}-\mathbf{a})\cdot(\mathbf{c}+\mathbf{a})$ (since $\mathbf{b}=-\mathbf{a}$) $=\mathbf{c}\cdot\mathbf{c}-\mathbf{a}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a}-\mathbf{a}\cdot\mathbf{a}$ $=\mathbf{c}\cdot\mathbf{c}-\mathbf{a}\cdot\mathbf{a}$ $=\lvert\mathbf{c}\rvert^2-\lvert\mathbf{a}\rvert^2$ (since $\mathbf{c}\cdot\mathbf{c}=\lvert\mathbf{c}\rvert^2, \mathbf{a}\cdot\mathbf{a}=\lvert\mathbf{a}\rvert^2$) $=0$ $\left(\text{since } \lvert\mathbf{c}\rvert=\lvert\mathbf{a}\rvert=\text{radius}\right)$ **OR** $\overrightarrow{AC}\cdot\overrightarrow{BC}=(\overrightarrow{OC}-\overrightarrow{OA})\cdot(\overrightarrow{OC}-\overrightarrow{OB})$ $=(\mathbf{c}-\mathbf{a})\cdot(\mathbf{c}-\mathbf{b})$ $=\mathbf{c}\cdot\mathbf{c}-\mathbf{a}\cdot\mathbf{c}-\mathbf{c}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{b}$ $=\mathbf{c}\cdot\mathbf{c}-\mathbf{a}\cdot\mathbf{c}-\mathbf{c}\cdot(-\mathbf{a})+\mathbf{a}\cdot(-\mathbf{a})$ (since $\mathbf{b}=-\mathbf{a}$) $=\mathbf{c}\cdot\mathbf{c}-\mathbf{a}\cdot\mathbf{c}+\mathbf{a}\cdot\mathbf{c}-\mathbf{a}\cdot\mathbf{a}$ (since $\mathbf{a}\cdot\mathbf{c}=\mathbf{c}\cdot\mathbf{a}$) $=\mathbf{c}\cdot\mathbf{c}-\mathbf{a}\cdot\mathbf{a}$ $=\lvert\mathbf{c}\rvert^2-\lvert\mathbf{a}\rvert^2$ (since $\mathbf{c}\cdot\mathbf{c}=\lvert\mathbf{c}\rvert^2, \mathbf{a}\cdot\mathbf{a}=\lvert\mathbf{a}\rvert^2$) $=0$ $\left(\text{since } \lvert\mathbf{c}\rvert=\lvert\mathbf{a}\rvert=\text{radius}\right)$ Since $\overrightarrow{AC}\cdot\overrightarrow{BC}=0$, $\therefore \angle ACB=90^\circ$.
(ii) [2]	Let points (1, 0, 0) and (0, 0, 1) be A and B respectively Let R be the point with position vector $\mathbf{r}$ Since $(\mathbf{r}-\mathbf{i})\cdot(\mathbf{r}-\mathbf{k})=0$, ABR is a right-angled triangle. Therefore R lies on a sphere with AB as the diameter of the sphere.  Length of line segment joining $A(1, 0, 0)$ and $B(0, 0, 1)$ is $\sqrt{2}$ Midpoint of (1, 0, 0) and (0, 0, 1) is $\left(\frac{1}{2},0,\frac{1}{2}\right)$ which is C, the centre of the sphere. Set of vectors $\mathbf{r}$ consists of position vectors of points on a sphere with diameter $\sqrt{2}$ (OR radius $\frac{\sqrt{2}}{2}$) and centre $\left(\frac{1}{2},0,\frac{1}{2}\right)$.

(b) (i) [2]

Set of vectors **r** consists of position vectors of points on a plane that contains the point M with position vector **m** and is perpendicular to the vector **n**.

If **n** is a unit vector, then $|\mathbf{m}\cdot\mathbf{n}|$ represents the shortest (perpendicular) distance from origin to the plane.



(ii) [4]

Method 1

Let X, Y and Z be points with position vectors $x\mathbf{i}, y\mathbf{j}$ and $z\mathbf{k}$ respectively, then

$$\overrightarrow{OX}=\begin{pmatrix}x\\0\\0\end{pmatrix},\overrightarrow{OY}=\begin{pmatrix}0\\y\\0\end{pmatrix},\overrightarrow{OZ}=\begin{pmatrix}0\\0\\z\end{pmatrix}$$

A normal to the plane π

$$=\overrightarrow{XY}\times\overrightarrow{XZ}=\begin{pmatrix}-x\\y\\0\end{pmatrix}\times\begin{pmatrix}-x\\0\\z\end{pmatrix}=\begin{pmatrix}yz\\xz\\xy\end{pmatrix}$$

$$\therefore d=\left|\overrightarrow{OX}\cdot\frac{\begin{pmatrix}yz\\xz\\xy\end{pmatrix}}{\sqrt{y^2z^2+x^2z^2+x^2y^2}}\right|=\left|\frac{xyz}{\sqrt{y^2z^2+x^2z^2+x^2y^2}}\right|$$

$$d^2=\frac{x^2y^2z^2}{y^2z^2+x^2z^2+x^2y^2}$$

$$\frac{y^2z^2+x^2z^2+x^2y^2}{x^2y^2z^2}=\frac{1}{d^2}$$

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{d^2} \qquad \text{(shown)}$$

Method 2

Let the normal vector to the plane be $\begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix}$ where $\sqrt{n_x^2 + n_y^2 + n_z^2} = 1$ (unit vector)

$$d = \left| \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} \right| \Rightarrow d = |xn_X| \Rightarrow |n_x| = \frac{d}{|x|}$$

Similarly, $d = \left| \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix} \cdot \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} \right| \Rightarrow |n_y| = \frac{d}{|y|}$

and, $d = \left| \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} \cdot \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} \right| \Rightarrow |n_z| = \frac{d}{|z|}$

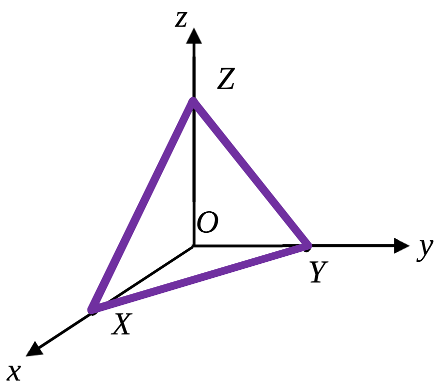
$$\sqrt{n_x^2 + n_y^2 + n_z^2} = 1$$

Therefore $\Rightarrow \left(\frac{d}{|x|}\right)^2 + \left(\frac{d}{|y|}\right)^2 + \left(\frac{d}{|z|}\right)^2 = 1$

$$\Rightarrow \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{1}{d^2} \quad \text{(Shown)}$$

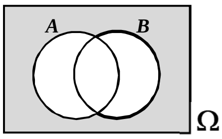
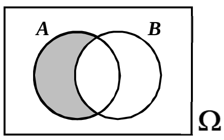
Method 3 (Geometry)

Let X, Y and Z be points with position vectors $x\mathbf{i}, y\mathbf{j}$ and $z\mathbf{k}$ respectively, then

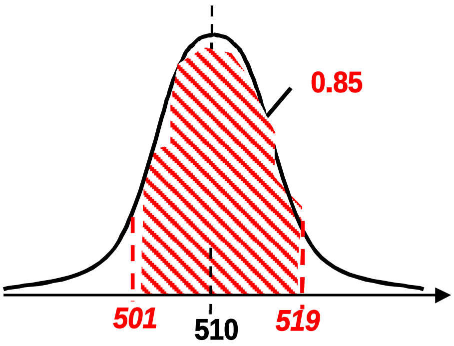


Using the base OXY, volume of the tetrahedral $OXYZ$

	$$=\frac{1}{3}(\textit{Area} \text{ of triangle } OXY)(OZ)$$ $$=\frac{1}{3}\left(\frac{1}{2}[OX][OY]\right)(OZ)$$ $$=\frac{1}{6}\lvert xyz\rvert$$ Using the base *XYZ*, volume of the tetrahedral *OXYZ* $$=\frac{1}{3}(\textit{Area} \text{ of triangle } XYZ)(d)$$ $$=\frac{1}{3}\left(\frac{1}{2}\left\lvert\overrightarrow{XY}\times\overrightarrow{XZ}\right\rvert\right)(d)$$ $$=\frac{1}{3}\left(\frac{1}{2}\left\lvert\begin{pmatrix}-x\\y\\0\end{pmatrix}\times\begin{pmatrix}-x\\0\\z\end{pmatrix}\right\rvert\right)(d), \quad \overrightarrow{XY}=\begin{pmatrix}-x\\y\\0\end{pmatrix} \text{ and } \overrightarrow{XZ}=\begin{pmatrix}-x\\0\\z\end{pmatrix}$$ $$=\frac{1}{3}\left(\frac{1}{2}\left\lvert\begin{pmatrix}yz\\xz\\xy\end{pmatrix}\right\rvert\right)(d)$$ $$=\frac{d}{6}\sqrt{y^2z^2+x^2z^2+x^2y^2}$$ Hence $$\frac{d}{6}\sqrt{y^2z^2+x^2z^2+x^2y^2}=\frac{1}{6}\lvert xyz\rvert$$ $$\Rightarrow d^2\left(y^2z^2+x^2z^2+x^2y^2\right)=x^2y^2z^2$$ $$\Rightarrow \frac{y^2z^2+x^2z^2+x^2y^2}{x^2y^2z^2}=\frac{1}{d^2}$$ $$\Rightarrow \frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{d^2}$$

Qn 5: Solution	
(i) **[1]**	$\mathrm{P}(A\cup B)$ $=1-\mathrm{P}(A'\cap B')$ $=1-0.15$ $=0.85$  $A'\cap B'$
(ii) **[3]**	$\mathrm{P}(B\mid A)=\dfrac{\mathrm{P}(B\cap A)}{\mathrm{P}(A)}$ $0.4=\dfrac{\mathrm{P}(B\cap A)}{0.3}$ $\mathrm{P}(B\cap A)=0.12$ $\mathrm{P}(A\cup B)=\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A\cap B)$ $0.85=0.3+\mathrm{P}(B)-0.12$ $\mathrm{P}(B)=0.67$
(iii) **[2]**	$\mathrm{P}(A\mid B')$ $=\dfrac{\mathrm{P}(A\cap B')}{\mathrm{P}(B')}$ $=\dfrac{\mathrm{P}(A)-\mathrm{P}(A\cap B)}{1-\mathrm{P}(B)}$ $=\dfrac{0.3-0.12}{1-0.67}$ $=\dfrac{6}{11}$  $A\cap B'$

Qn 6: Solution	
(i) **[2]**	Null hypothesis, $H_0 : \mu = 980$ Alternative hypothesis, $H_1 : \mu < 980$ where μ is the population mean weekly salary.
(ii) **[5]**	Let X be the weekly earning of an I-ber driver (in \$). Using GC, $\bar{x} = 939.5$ Perform a 1-tailed test at 5% significance level. Under H_0, $\bar{X} \sim \mathrm{N}\left(980, \frac{88^2}{10}\right)$ $p\text{-value} = 0.0728 > 0.05$, hence we do not reject H_0, and conclude that, based on the test carried out by the recruitment manager, there is insufficient evidence for the managing director to conclude at 5% level of significance that the mean weekly earnings of a driver is less than \$980. Assumption: Assume that the weekly earnings of the I-ber drivers are normally distributed.
(iii) **[1]**	To reject H_0, smallest level of significance = 7.3% (1 d.p)

Qn 7: Solution	
(i) **[1]**	Let X and Y be the amount, in ml, of hand sanitiser in a small and large bottle respectively. Then $X \sim \mathrm{N}\left(108, 5^2\right)$ and $Y \sim \mathrm{N}\left(510, \sigma^2\right)$ $\mathrm{P}(X < 100) = 0.054799 = 0.0548$ (3 s.f.)
(ii) **[2]**	Required probability $= {}^{99}C_4 (0.054799)^4 (1-0.054799)^{95} (0.054799)$ $= 0.00880$ (3 s.f.)
(iii) **[3]**	$\mathrm{P}\left(\lvert Y-510\rvert < 9\right) = 0.85$ $\mathrm{P}\left(\frac{\lvert Y-510\rvert}{\sigma} < \frac{9}{\sigma}\right) = 0.85$ $\mathrm{P}\left(\lvert Z\rvert < \frac{9}{\sigma}\right) = 0.85$ From G.C, $\frac{9}{\sigma} = 1.4395$ $\sigma = 6.25$ (3 s.f.) **OR**  $\mathrm{P}(501 < Y < 519) = 0.85$ $\mathrm{P}\left(\frac{501-510}{\sigma} < \frac{Y-510}{\sigma} < \frac{519-510}{\sigma}\right) = 0.85$ $\mathrm{P}\left(-\frac{9}{\sigma} < Z < \frac{9}{\sigma}\right) = 0.85$ From G.C, $\frac{9}{\sigma} = 1.4395$ $\sigma = 6.25$ (3 s.f.)
(iv) **[3]**	$Y - 5X \sim \mathrm{N}\left(510 - 5(108), 6^2 + 5^2\left(5^2\right)\right)$ i.e $Y - 5X \sim \mathrm{N}(-30, 661)$ $\mathrm{P}(Y - 5X < 0) = 0.878$ (3 s.f.)

Qn 8: Solution	
(i) **[1]**	The number of different arrangements $=\dfrac{8!}{2!2!}=10080$ (shown)
(ii) **[3]**	The consonants are M, M, N, T, Y and the vowels are I, I, U. Number of ways to arrange the consonants $=\dfrac{5!}{2!}=60$ $\uparrow\ C_1\ \uparrow\ C_2\ \uparrow\ C_3\ \uparrow\ C_4\ \uparrow\ C_5\ \uparrow$ Number of ways to choose 3 slots to insert the vowels $={}^6C_3=20$ Number of ways to arrange the vowels $=\dfrac{3!}{2!}=3$ $\therefore$ The number of different arrangements $=60\times 20\times 3=3600$ **<u>Method 2</u>** (complementary approach) : Case 1 : 3 vowels are together (as a unit) $\boxed{\text{I I U}}$ M M N T Y Number of arrangements = $\dfrac{6!}{2!}\times 3$ = 1080. Case 2 : Group 2 'I's together (as a unit) M M N T Y $\boxed{\text{I I}}$ U Number of arrangements = $\dfrac{7!}{2!}$ = 2520. Case 3 : Group one I and one U together (as a unit) M M N T Y $\boxed{\text{I U}}$ I Number of arrangements = $\dfrac{7!}{2!}\times 2=5040$. Hence, the required number of different arrangements = 10080 – 2520 – 5040 + 1080 = 3600.

	<u>Method 3</u> (complementary approach) : Case 1 : 3 vowels are together (as a unit) $\boxed{\text{I I U}}$ M M N T Y Number of arrangements = $\dfrac{6!}{2!}\times 3$ = 1080. Case 2 : Group 2 'I's together (as a unit), and U is another unit not adjacent to I-I $\boxed{\text{I I}}$ $\boxed{\text{U}}$ M M N T Y ↑ ↑ ↑ ↑ ↑ ↑ Number of ways to arrange the 5 consonants = $\dfrac{5!}{2!}$ = 60. Numbers of ways to insert 2 units = ${}^{6}C_{2}\times 2 = 30$. Number of different arrangements = 60 × 30 = 1800. Case 3 : Group one I and U together (as a unit), and the other I is another unit not adjacent to I-U unit $\boxed{\text{I U}}$ $\boxed{\text{I}}$ M M N T Y ↑ ↑ ↑ ↑ ↑ ↑ Number of ways to arrange the 5 consonants = $\dfrac{5!}{2!}$ = 60. Numbers of ways to insert 2 units = ${}^{6}C_{2}\times 2\times 2 = 60$. Number of different arrangements = 60 × 60 = 3600. Hence, the required number of different arrangements = 10080 – 1080 – 1800 – 3600 = 3600.
(iii) **(a)** **[1]**	Events A and B are not mutually exclusive. This is because both events can occur at the same time, for example arrangements such as MMUNIITY, IINTUMMY and YUMMIINT.
(iii) **(b)** **[3]**	$\mathrm{P}(A)=\mathrm{P}(B)=\dfrac{7!}{2!}\div 10080=\dfrac{1}{4}$, so $\mathrm{P}(A)\times\mathrm{P}(B)=\dfrac{1}{16}$ $\mathrm{P}(A\cap B)=\dfrac{6!}{10080}=\dfrac{1}{14}$ $\mathrm{P}(A\cap B)\neq\mathrm{P}(A)\times\mathrm{P}(B)$ $\Rightarrow$ Events A and B are not independent.

(iv)
[4]

Required probability

$$= \mathrm{P}(A' \cap B')$$

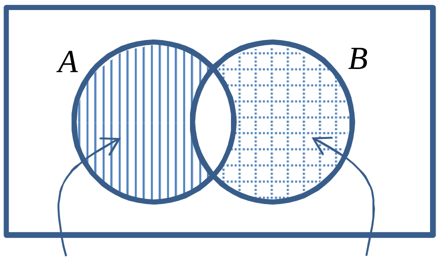
$$= 1 - \mathrm{P}(A \cup B)$$

$$= 1 - \left[\mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)\right]$$

$$= 1 - \left[\frac{1}{4} + \frac{1}{4} - \frac{1}{14}\right]$$

$$= \frac{4}{7}$$

<u>Method 2</u> (Venn Diagram) :



$\mathrm{P}(A) - \mathrm{P}(A \cap B) = \frac{1}{4} - \frac{1}{14} = \frac{5}{28}$ $\mathrm{P}(B) - \mathrm{P}(A \cap B) = \frac{1}{4} - \frac{1}{14} = \frac{5}{28}$

Required probability = $1 - \frac{5}{28} - \frac{5}{28} - \frac{1}{14} = \frac{4}{7}$.

<u>Method 3</u> (complementary approach) :

Case 1 : Group 2 ‘M’s together (as a unit), and the two ‘I’s are to be separated from each other.

I   I

$\uparrow$ $\boxed{\text{MM}}$ $\uparrow$ U $\uparrow$ N $\uparrow$ T $\uparrow$ Y $\uparrow$

Number of ways = $5! \times {}^6C_2$ = 1800.

Case 2 : Group 2 ‘I’s together (as a unit), and the two ‘M’s are to be separated from each other.

M   M

$\uparrow$ $\boxed{\text{I I}}$ $\uparrow$ U $\uparrow$ N $\uparrow$ T $\uparrow$ Y $\uparrow$

Number of ways = $5! \times {}^6C_2 = 1800$.

Case 3 : Group 2 'I's together (as a unit), and group 2 'M's together (as a unit)

$\boxed{\text{I I}}$ $\boxed{\text{M M}}$ U N T Y

Number of ways = 6! = 720.

Hence, the required probability = $1 - \frac{1800}{10080} - \frac{1800}{10080} - \frac{720}{10080} = \frac{4}{7}$.

<u>Method 4</u> (complementary approach) :

Case 1 : Group 2 'M's together (as a unit)

$\boxed{\text{MM}}$ U N T Y I I

Number of ways = $\frac{7!}{2!} = 2520$.

Case 2 : Group 2 'I's together (as a unit)

$\boxed{\text{I I}}$ U N T Y M M

Number of ways = $\frac{7!}{2!} = 2520$.

Case 3 : Group 2 'I's together (as a unit), and group 2 'M's together (as a unit)

$\boxed{\text{I I}}$ $\boxed{\text{M M}}$ U N T Y

Number of ways = 6! = 720.

Hence, the required probability = $1 - \frac{2520}{10080} - \frac{2520}{10080} + \frac{720}{10080} = \frac{4}{7}$.

Qn 9: Solution	
(a) **[2]**	1. The probability of drawing red balls from 2^{nd} draw does not remain at a constant. If first ball is red, probability of red for 2^{nd} ball is 4/7 If first ball is black, probability of red for 2^{nd} ball is 3/6. 2. The drawing of balls are not independent of each other as it involves replacement.
(b) **(i)** **[2]**	$\mathrm{P}(R \geq 1) = \mathrm{P}(\text{at least 1 red}) = 1 - \mathrm{P}(\text{all black})$ $= 1 - \left(\frac{1}{2}\right)\left(\frac{2}{5}\right)\left(\frac{1}{4}\right)$ $= 1 - \frac{1}{20} = \frac{19}{20}$
(ii) **[3]**	Let A be the event the first ball drawn is black. Then $\mathrm{P}(A \mid R \geq 1)$ $= \frac{\mathrm{P}(A \cap \{R \geq 1\})}{\mathrm{P}(R \geq 1)}$ $= \frac{\mathrm{P}(\{B, R\} + \{B, B, R\})}{\frac{19}{20}}$ $= \frac{\frac{1}{2}\left(\frac{3}{5}\right) + \frac{1}{2}\left(\frac{2}{5}\right)\left(\frac{3}{4}\right)}{\frac{19}{20}}$ $= \frac{9}{19}$ Alternatively, $\mathrm{P}(A \mid R \geq 1)$ $= \frac{\mathrm{P}(A \cap \{R \geq 1\})}{\mathrm{P}(R \geq 1)}$ $= \frac{\mathrm{P}(B) - P(BBB)}{\frac{19}{20}}$ $= \frac{\frac{1}{2} - \frac{1}{2}\left(\frac{2}{5}\right)\left(\frac{1}{4}\right)}{\frac{19}{20}}$ $= \frac{9}{19}$

(c)
[2]

P(all 31 balls red)

$$=\left(\frac{3}{6}\right)\left(\frac{4}{7}\right)\left(\frac{5}{8}\right)\left(\frac{6}{9}\right)\left(\frac{7}{10}\right)\ldots\left(\frac{30}{33}\right)\left(\frac{31}{34}\right)\left(\frac{32}{35}\right)\left(\frac{33}{36}\right)$$

$$=\frac{3(4)(5)}{34(35)(36)}$$

$$=\frac{1}{34(7)(3)}$$

$$=\frac{1}{714}$$

(d)
[3]

P(all red balls in first n draws) $=\mathrm{P}\left(R=n\right)$

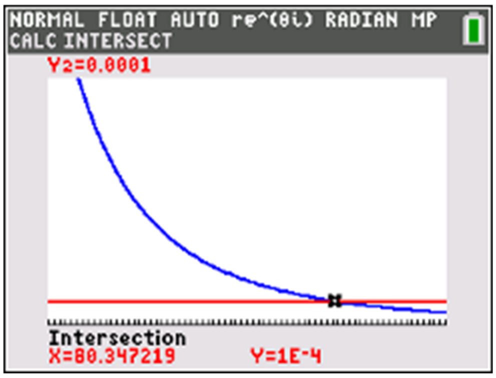
$$=\left(\frac{3}{6}\right)\left(\frac{4}{7}\right)\left(\frac{5}{8}\right)\left(\frac{6}{9}\right)\ldots\left(\frac{n}{n+3}\right)\left(\frac{n+1}{n+4}\right)\left(\frac{n+2}{n+5}\right)$$

$$=\frac{60}{(n+3)(n+4)(n+5)}$$

From GC,

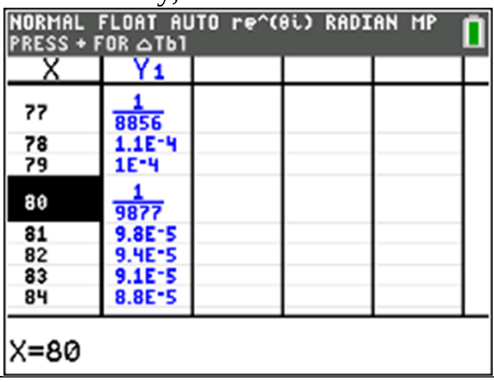
$$\frac{60}{(n+3)(n+4)(n+5)}>0.0001$$

$$\Rightarrow 0<n<80.347$$



Maximum amount Isaac would win with probability exceeding 0.0001 is $8000.

Alternatively, from GC table of values:



NORMAL FLOAT AUTO re^(θi) RADIAN MP
PRESS + FOR ΔTbl

X	Y_1			
77	$\frac{1}{8856}$			
78	1.1E-4			
79	1E-4			
80	$\frac{1}{9877}$			
81	9.8E-5			
82	9.4E-5			
83	9.1E-5			
84	8.8E-5			

X=80

Qn 10: Solution

(i) [5]

The following table * shows the probabilities of obtaining the number of heads corresponding to the number of throws.

No. of tosses (No. shown on the ball drawn)		Number of heads recorded, X				
		0	1	2	3	4
	1	T: $\frac{1}{4}\times\binom{1}{0}\left(\frac{1}{2}\right)$	H: $\frac{1}{4}\times\binom{1}{1}\left(\frac{1}{2}\right)$	-	-	-
	2	TT: $\frac{1}{4}\times\binom{2}{0}\left(\frac{1}{2}\right)^2$	HT or TH: $\frac{1}{4}\times\binom{2}{1}\left(\frac{1}{2}\right)^2$	HH: $\frac{1}{4}\times\binom{2}{2}\left(\frac{1}{2}\right)^2$	-	-
	3	TTT: $\frac{1}{4}\times\binom{3}{0}\left(\frac{1}{2}\right)^3$	HTT or THT or HTT: $\frac{1}{4}\times\binom{3}{1}\left(\frac{1}{2}\right)^3$	HHT or HTH or THH: $\frac{1}{4}\times\binom{3}{2}\left(\frac{1}{2}\right)^3$	HHH: $\frac{1}{4}\times\binom{3}{3}\left(\frac{1}{2}\right)^3$	-
	4	TTTT: $\frac{1}{4}\times\binom{4}{0}\left(\frac{1}{2}\right)^4$	HTTT, THTT, TTHT or TTTH: $\frac{1}{4}\times\binom{4}{1}\left(\frac{1}{2}\right)^4$	HHTT, HTHT, HTTH, THHT, THTH or TTHH: $\frac{1}{4}\times\binom{4}{2}\left(\frac{1}{2}\right)^4$	HHHT, HHTH, HTHH or THHH: $\frac{1}{4}\times\binom{4}{3}\left(\frac{1}{2}\right)^4$	HHHH: $\frac{1}{4}\times\binom{4}{4}\left(\frac{1}{2}\right)^4$

$$\mathrm{P}(X=0)=\mathrm{P}\left(\text{T, TT, TTT or TTTT}\right)=\frac{1}{4}\left[\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}\right]=\frac{15}{64}\ \text{(shown)}$$

x	0	1	2	3	4
$\mathrm{P}(X=x)$	$\frac{15}{64}$	$\frac{13}{32}$	$\frac{1}{4}$	$\frac{3}{32}$	$\frac{1}{64}$

(ii) [3]

$$\mu=\mathrm{E}(X)$$

$$=\left(0\times\frac{15}{64}\right)+\left(1\times\frac{13}{32}\right)+\left(2\times\frac{1}{4}\right)+\left(3\times\frac{3}{32}\right)+\left(4\times\frac{1}{64}\right)$$

$$=\frac{5}{4}=1.25$$

	$\mathrm{P}(X>\mu)=\mathrm{P}\left(X>\frac{5}{4}\right)$ $=\mathrm{P}(X=2)+\mathrm{P}(X=3)+\mathrm{P}(X=4)$ $=\frac{1}{4}+\frac{3}{32}+\frac{1}{64}$ $=\frac{23}{64}$ or 0.359375 $\mathrm{E}(X^2)=\left(0^2\times\frac{15}{64}\right)+\left(1^2\times\frac{13}{32}\right)+\left(2^2\times\frac{1}{4}\right)+\left(3^2\times\frac{3}{32}\right)+\left(4^2\times\frac{1}{64}\right)=\frac{5}{2}$ $\sigma^2=\mathrm{Var}(X)$ $=\mathrm{E}(X^2)-[\mathrm{E}(X)]^2$ $=\frac{5}{2}-\left(\frac{5}{4}\right)^2$ $=\frac{15}{16}$ (shown)
(iii) **[2]**	Let Y be the number of games, out of ten, with at least two heads. $Y\sim \mathrm{B}\left(10,\frac{23}{64}\right)$ $\mathrm{P}(Y\geq 2)$ $=1-\mathrm{P}(Y\leq 1)$ $=1-0.076953$ $=0.923$ (3 s.f.)
(iv) **[3]**	Let $\bar{X}$ be the average number of heads recorded in 50 games. Since $n=50$ is large, by Central Limit Theorem, $\bar{X}\sim \mathrm{N}\left(1.25,\frac{15}{16(50)}\right)$ or $\bar{X}\sim \mathrm{N}\left(1.25,\frac{0.9375}{50}\right)$ approximately. $\mathrm{P}(\bar{X}<1)=0.0339$ (3 s.f.) Alternatively, Since $n=50$ is large, by Central Limit Theorem, $X_1+X_2+\cdots X_{50}\sim \mathrm{N}\left(50\times 1.25, 50\times\frac{15}{16}\right)$ approximately. $\mathrm{P}(\bar{X}<1)=\mathrm{P}(X_1+X_2+\cdots X_{50}<50)=0.0339$ (3 s.f.)