Promo Practice Paper 2 [NYJC 2021] 100marks

1 **(a)** Differentiate $\sin^{-1}x + x\sqrt{1-x^2}$ with respect to x, expressing your answer in its simplest form. Hence, find $\int \sqrt{1-x^2}\,\mathrm{d}x$. [4]

(b) Find $\displaystyle\int \frac{x^2}{\sqrt{4x^3+1}}\,\mathrm{d}x$. [2]

2 **(i)** By using an algebraic method, solve the inequality $\dfrac{x+3}{x+4} \le \dfrac{5}{1-2x}$. [4]

(ii) Hence, solve the inequality $\dfrac{x^2+3}{x^2+4} \le \dfrac{5}{1-2x^2}$. [2]

3 Referred to the origin O, points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. Point P lies on OA such that $OP = 2PA$ and point Q lies on AB such that $5AQ = 4QB$. Show that the equation of the line l passing through P and Q can be written as

$$\mathbf{r} = \frac{2}{3}\mathbf{a} + \lambda(4\mathbf{b} - \mathbf{a}), \text{ where } \lambda \in \mathbb{R}. \qquad [3]$$

Point X lies on l such that AX is perpendicular to l. If $|\mathbf{a}| = \sqrt{3}$, $|\mathbf{b}| = \dfrac{1}{2}$ and $\mathbf{a}$ is perpendicular to $\mathbf{b}$, find the position vector of X in terms of $\mathbf{a}$ and $\mathbf{b}$. [4]

4 The function f is defined by

$$\mathrm{f}(x) = \frac{1}{x^2+1}, \quad x \in \mathbb{R}, \quad x \ge k.$$

(i) State the minimum value of k for which the function f^{-1} exists. [1]

For the rest of the question, use the value of k found in part **(i)**.

(ii) Sketch the graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$ on the same diagram, showing clearly the relationship between them. [3]

The function g is defined by

$$\mathrm{g}(x) = \frac{x^2+1}{x}, \quad x \in \mathbb{R}, \quad x > 0.$$

(iii) By finding fg(x) or otherwise, solve $\mathrm{g}(x) = \mathrm{f}^{-1}\left(\dfrac{1}{5}\right)$. [4]

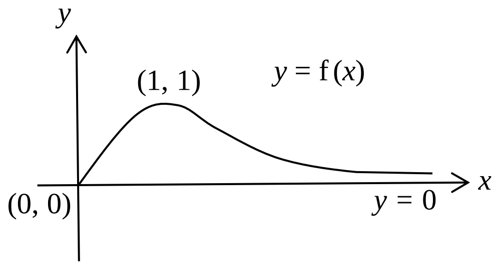
5 **(a)** The curve C_1 and C_2 have equations $y = \dfrac{x}{x^2+1}$ and $y = \sqrt{\dfrac{5}{4} - x^2}$ respectively.

(i) Sketch C_1 and C_2 on the same diagram, stating the exact coordinates of any points of intersection with the axes and stationary points, and the equation(s) of any asymptote(s). [4]

(ii) State the coordinates of the point of intersection of C_1 and C_2. [1]

(iii) Hence solve the inequality $\dfrac{x}{x^2+1} \geq \sqrt{\dfrac{5}{4} - x^2}$. [1]

(b) The diagram below shows a sketch of the graph of $y = \mathrm{f}(x)$. The graph meets the origin (0, 0), has a turning point at (1, 1) and the equation of the asymptote is $y = 0$.



On separate diagrams, draw sketches of the graphs of

(i) $y = \mathrm{f}\left(|x|\right)$, [2]

(ii) $y = \mathrm{f}'(x)$, [2]

stating the coordinates of the turning point(s), point(s) of intersection with the x-axis and equation(s) of asymptote(s) when it is possible to do so.

6 It is given that $y = \sqrt{1 + \ln(1 + \sin 2x)}$.

(i) Show that $y\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\cos 2x}{1 + \sin 2x}$. [1]

(ii) Show that $y\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = \dfrac{k}{(1 + \sin 2x)}$, where k is a constant to be determined. [3]

(iii) Hence show that the Maclaurin series of y is $1 + x - \dfrac{3}{2}x^2 + \dfrac{13}{6}x^3 + \ldots$. [3]

(iv) Expand $\left(1 + x - \dfrac{3}{2}x^2 + \dfrac{13}{6}x^3\right)^2$ in powers of x up to and including the term in x^3, simplifying the coefficients. By using the standard series expansions of $\sin x$ and $\ln(1+x)$ from the List of Formulae (MF26), explain briefly how the result can be used as a check on the correctness of the first four terms in the series for y. [3]

7 A curve C has parametric equations

$$x = 3t^2, \quad y = a\left(t^3 + 1\right),$$

where a is a positive constant.

(i) Sketch C, giving the coordinates of any point(s) where the curve meets the axes. [2]

The tangent to C at point $A(3, 2a)$ makes an angle of $\frac{\pi}{3}$ with the positive x-axis.

(ii) Show that $a = 2\sqrt{3}$, and find the equation of the tangent to C at A in the form $y = mx + c$, where m and c are constants to be determined. [5]

(iii) The tangent and the normal to C at A meet the x-axis at T and N respectively. Find the exact area of triangle ATN. [3]

8 **RI Promo 9758/2020/Q7a and ASRJC Promo 9758/2020/Q8a**

(a) One root of the equation $zz^* + 2\mathrm{i}z = a + 6\mathrm{i}$, where a is real, is $z = 3 - 7\mathrm{i}$. Find the value of a and the other root. [4]

Do not use a calculator in answering the following question.

(b) It is given that two complex numbers z and w satisfy the following equations

$$\mathrm{i}w + z = 5$$
$$w^2 + (4\mathrm{i} - 1)\mathrm{z} = -11 + 18\mathrm{i}$$

Find z and w. [4]

9 **CJC Promo 9758/2021/Q9**

With reference to the origin O, the points A and B have position vectors $\mathbf{a} = -\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = 2\mathbf{j} + 5\mathbf{k}$ respectively.

(i) Find a vector equation of the line l_1 that passes through point A and is parallel to the vector $\mathbf{a}$. [1]

(ii) Find the exact length of projection of $\mathbf{b}$ on l_1. Hence find d, the exact perpendicular distance from the point B to l_1. [4]

(iii) Using the value of d found in part **(ii)**, find the position vector of the point C, the foot of perpendicular from the point B to l_1. [3]

(iv) The line l_2 passes through point B and is parallel to vector $\mathbf{b}$. Find a cartesian equation of l_3 which is the reflection of l_2 in l_1. [3]

10 Mrs Tan wants to build a wooden fence using vertical planks of equal width and thickness but different lengths. In her plan, the length of the first wooden plank is 2 metres and the length of the planks forms a geometric progression. The length of the 10th plank is 1.5 metres.

(i) If the cost of wooden plank is \$18 per metre, show that the cost of the fence will never exceed \$1200. [4]

Mrs Tan realised that her plan is not feasible and now wants to build her fence using several identical panels. Each panel comprises 10 vertical planks with identical dimensions to the first 10 planks in her original plan.

(ii) Find the total length, T metres, of the planks to be used in each panel. [2]

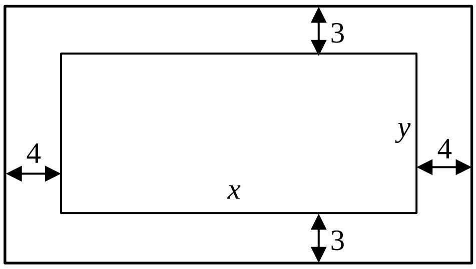
(iii) She hires a contractor to install the fence. The contractor misunderstands her instructions and uses 10 planks to construct a panel so that the lengths form an arithmetic progression with common difference d metres. If the total length of the planks to be used for one panel is still T metres and the length of the first plank is still 2 metres, find the value of d. [2]

The contractor offers to paint the fence for Mrs Tan. He buys a 5-litre can of paint to do the paint job. To save costs, he fills the can to the 5-litre mark with turpentine to form a uniform mixture when the level of paint in the can falls to the 4-litre mark. He then repeats this process whenever the level of the mixture falls to the 4-litre mark.

(iv) State, in terms of n, an expression for the volume of paint remaining in the mixture after the nth refill. [2]

(v) Find the minimum number of refills taken before the mixture is more than 80% turpentine. [2]

11 A designer wishes to create a piece of artwork with painted area of 1200 cm^2 on a rectangular piece of canvas. The painted area measures x cm by y cm and is surrounded by an unpainted border with top and bottom margins of 3 cm each, and side margins of 4 cm each on the canvas, as shown in the diagram below.



(i) By differentiation, find the dimensions of the canvas with the smallest area. [6]

(ii) What is the largest possible area of the canvas if $30 \le x \le 50$? [2]

At an exhibition, a spotlight illuminates a circular region of radius $\frac{2}{\sqrt{\pi}}$ cm on the artwork. The area of this circular region then increases at a constant rate of 20 cm^2 per minute.

(iii) Find the rate of change of the radius **after** 3 minutes. [4]

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Answers

1	(a) $2\sqrt{1-x^2}$; $\frac{1}{2}\left(\sin^{-1}x + x\sqrt{1-x^2}\right)+C$ (b) $\frac{1}{6}\sqrt{4x^3+1}+C$
2	(i) $-4 < x < \frac{1}{2}$ (ii) $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$
3	$\overrightarrow{OX} = \frac{1}{21}(17\mathbf{a} - 12\mathbf{b})$
4	$k = 0$ (iii) $x = 1$
5	(a) (ii) $\left(1, \frac{1}{2}\right)$ (iii) $1 \le x \le \frac{\sqrt{5}}{2}$
6	(i) k = -2 (iv) $1+2x-2x^2+\frac{4}{3}x^3+...$
7	(ii) $y=\sqrt{3}x+\sqrt{3}$ (iii) $32\sqrt{3}$
8	(a) $a = 72$, $3+9\text{i}$ (b) $z=6+2\text{i}$ or $3+2\text{i}$ $w=\text{i}-2$ or $-2\text{i}-2$
9	(i) $\mathbf{r} = \begin{pmatrix}-1\\1\\2\end{pmatrix}+k\begin{pmatrix}-1\\1\\2\end{pmatrix}=\lambda\begin{pmatrix}-1\\1\\2\end{pmatrix}$, $\lambda\in\mathbb{R}$ (ii) $2\sqrt{6}$; $\sqrt{5}$ (iii) $\mathbf{c} = \begin{pmatrix}-2\\2\\4\end{pmatrix}$ (iv) $-\frac{x}{4}=\frac{y}{2}=\frac{z}{3}$
10	(ii)17.4 m (iii) $d \approx -0.0579$m (iv) $5\left(\frac{4}{5}\right)^n$ (v) 8
11	(i) $x = 48$ cm and $y = 36$ cm (ii) 1748 cm^2 0.705 cm/min