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**9758/02**

**13<sup>th</sup> September 2021**

### 3 Hours

Candidates answer on the Question Paper.

Additional Materials: List of Formulae (MF26)

## READ THESE INSTRUCTIONS

Write your name and class on all the work you hand in.  
Write in dark blue or black pen.  
You may use an HB pencil for any diagrams or graphs.  
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.

Write your answers in the spaces provided in the question paper.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

The use of an approved graphing calculator is expected, where appropriate. Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.

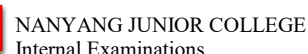
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.  
The total number of marks for this paper is 100.

For examiner's use only	
Question number	Mark
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
Total	

This document consists of **20** printed pages.



**[Turn Over**

## Section A: Pure Mathematics [40 marks]

- 1 The plane  $\Pi_1$  passes through  $(3, -1, 2)$  and is perpendicular to the line  $\mathbf{r} = 2\mathbf{i} + 4\mathbf{j} - 3\mathbf{k} + \lambda(5\mathbf{i} + 2\mathbf{j} - \mathbf{k})$ . The plane  $\Pi_2$  contains the points  $(2, 3, 2)$ ,  $(4, 1, -1)$  and  $(0, -1, 2)$ .

(i) Show that the acute angle,  $\theta$ , between the planes  $\Pi_1$  and  $\Pi_2$  is such that  $\cos \theta = \frac{\sqrt{30}}{15}$ . [3]

- (ii) Show that the line of intersection,  $L$ , of the planes  $\Pi_1$  and  $\Pi_2$  has vector equation

$$\mathbf{r} = \begin{pmatrix} 0 \\ 9 \\ 7 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}, \mu \in \mathbb{R}. \quad [3]$$

The plane  $\Pi_3$  has the equation  $4(k-2)x + (k+1)y - 4k^2z = 8$ , where  $k$  is a constant.

- (iii) The three planes  $\Pi_1$ ,  $\Pi_2$  and  $\Pi_3$  have no points in common. By considering the relationship between the line  $L$  and the plane  $\Pi_3$ , find the possible values of  $k$ . [2]

- (iv) For the positive value of  $k$  found in (iii), find the distance between  $L$  and  $\Pi_3$ . [2]

- 2 Many industries use rectangular tanks to handle their water, wastewater and chemical storage and processing needs. Because of their shape, rectangular tanks can offer tremendous cost savings for shipping compared to cylindrical tanks.

In fabricating one such industrial-strength storage tank, a customer requires water to flow into the rectangular tank with a horizontal base area  $A$ , at a constant rate of  $n$  units of volume per unit time. The water should flow out of the tank through a hole in the bottom, at a rate that is proportional to the square root of the depth of water in the tank. It is also required that when the depth of the water in the tank is  $h$ , the level of water in the tank remains constant.

- (i) Obtain a differential equation for the depth  $x$  at time  $t$ . [3]

- (ii) It is known that the tank is filled to a depth of  $4h$  initially. By using the substitution  $x = hu^2$ , show that  $u$  satisfies the differential equation  $\frac{2Ah}{n} \frac{du}{dt} = -\frac{u-1}{u}$ . [2]

- (iii) By solving this differential equation, find, in terms of  $A, h$  and  $n$ , the time needed for the depth to reach  $\frac{16}{9}h$ . Describe how  $x$  varies with  $t$ . [6]

- 3 (a) An arithmetic progression has  $n$  terms and a common difference of  $d$ . Prove that the difference between the sum of the last  $k$  terms and the sum of the first  $k$  terms is  $(n-k)kd$ . [4]
- (b) The  $r$ th term,  $u_r$ , of a series is given by  $u_r = \left(\frac{1}{3}\right)^{3r-2} + \left(\frac{1}{3}\right)^{3r-1}$ . Express  $\sum_{r=1}^n u_r$  in the form  $A\left(1 - \frac{B}{27^n}\right)$ , where  $A$  and  $B$  are constants to be found. Find the sum to infinity of the series. [4]
- (c) (i) Write down the sum of the geometric series  $z + z^2 + \dots + z^n$ . [1]
- (ii) By putting  $z = e^{i\theta}$  in your result, show that this sum can be written as  $\frac{\sin \frac{n\theta}{2} e^{i\left(\frac{n+1}{2}\right)\theta}}{\sin \frac{\theta}{2}}$ . [2]
- (iii) Hence, by using the identity  $e^{i\theta} = \cos \theta + i \sin \theta$ , or otherwise, show that
- $$\sin \theta + \sin 2\theta + \dots + \sin n\theta = \frac{\sin \frac{n\theta}{2} \sin \frac{(n+1)\theta}{2}}{\sin \frac{\theta}{2}}. \quad [3]$$
- 4 Air is pumped into a spherical elastic ball through a tiny hole at a constant rate of  $10 \text{ cm}^3$  per second. Assuming that the ball maintains a spherical shape throughout the process, find
- (i) the rate of increase of the radius of the ball at the instant when the radius is 5 cm, [2]
- (ii) the rate of increase of the surface area of the ball at the instant when the radius of the ball is increasing at the rate of  $\frac{5}{72\pi} \text{ cm per second}$ . [3]
- [It is given that the surface area and volume of a sphere of radius  $r$  are  $4\pi r^2$  and  $\frac{4}{3}\pi r^3$  respectively.]

### Section B: Probability and Statistics [60 marks]

- 5 A school teacher is interested in the amount of time spent for revision per day for his students. He decided to select 5 students from each of his four classes.
- (i) Explain why this method may not be appropriate. [1]
- (ii) Suggest how the method can be improved to get an appropriate sample and why it should be done this way. [2]
- (iii) If it is further known that there are 20 students in each of his four classes, find the number of possible samples that he can have if he selects 5 students from each class. [1]

- 6 The events  $A$ ,  $B$  and  $C$  are such that  $P(A)=0.4$ ,  $P(B)=0.8$ ,  $P(C)=0.7$ . The events  $A$  and  $B$  are independent; and the events  $B$  and  $C$  are independent.
- (i) If  $A$ ,  $B$  and  $C$  are independent events, find the value of  $P(A \cap B' \cap C)$ . [2]
- (ii) If  $P(A \cap B \cap C) = 0.25$ , find the least and greatest value of  $P(A \cap B' \cap C)$ . [4]
- 7 This question is about the arrangement of the nine letters in the word PINEAPPLE.
- (i) Find the number of ways of arranging all nine letters of the word such that the letters are **not** in alphabetical order. [2]
- (ii) Find the number of different ways of arranging all nine letters of the word PINEAPPLE such that no vowel (A, E, I) is next to another vowel. [2]

It is now given that 4 letters are chosen to form another arrangement.

- (iii) Find the probability that it consists of at least 2 identical letters. [4]
- 8 A factory produces surgical masks. On average, a proportion  $p$  of mask fail to meet the requirement of surgical standard. A mask that fails to meet the requirement is considered faulty. The masks are packed in boxes for sale to retail outlets. It should be assumed that the number of faulty surgical masks in a box follows a binomial distribution.
- For quality control purposes a random sample of 10 masks from a box is tested.
- If there are no faulty masks, the box is accepted for sale.
  - If there are more than 2 faulty masks found in this sample of 10, the box is rejected.
  - If there are 1 or 2 faulty masks found in this sample, a further sample of 5 masks is randomly selected. The box will be accepted if this sample of 5 masks has no faulty masks and will be rejected otherwise.
- (i) Given that the probability of a box being accepted is 0.923, find  $p$ . [3]
- (ii) Find the expected number of masks to be sampled for quality control. [2]
- The inspection was conducted in batches. In one batch, 60 boxes of masks are chosen to be inspected.
- (iii) Find the probability that the 60<sup>th</sup> box is the 5<sup>th</sup> box to be rejected if at least 5 boxes of masks are rejected. [3]

- 9** A bag contains 2 blue counters, 1 white counter and  $n$  red counters, where  $n > 2$ . In a game, John removes counters randomly from the bag, one at a time, until he has taken out 2 red counters. The total number of blue counters John removes from the bag is denoted by  $C$ .

(i) Show that  $P(C=1) = \frac{4(n-1)}{(n+2)(n+1)}$ . Find  $P(C=c)$  for all other possible values of  $c$ . [4]

(ii) Show that  $E(C) = \frac{4}{n+1}$  and  $\text{Var}(C) = \frac{g(n)}{(n+2)(n+1)^2}$  where  $g(n)$  is a quadratic polynomial to be determined. [3]

(iii) John plays the game twice and the number of blue counters obtained from these 2 games are  $C_1$  and  $C_2$  respectively. It is known that  $P(|C_1 - C_2| > 0) < \frac{1}{5}$ . Find the least value of  $n$ . [3]

- 10** An office worker, Natalie, has diabetes and has to monitor her blood glucose levels, which vary throughout the day. The results from a sample of 75 readings,  $x$  (in mmol/L), taken at random times over a week, are summarised by  $\sum x = 511.5$  and  $\sum x^2 = 4027.89$ .

(i) Calculate unbiased estimates of the population mean and variance for the blood glucose levels. [2]

(ii) Test at 5% significance level whether Natalie's mean blood glucose level,  $\mu$  (in mmol/L), is greater than 6.0. You should state your hypotheses and give your conclusion in context. [4]

(iii) State, giving a reason, whether the conclusion of the test in part (ii) would be valid if the 75 readings were all taken at weekends. [1]

Following a change in her diet, Natalie claims that her mean blood glucose level  $\mu$  is now less than 6.0. She takes another random sample of 75 readings and notes that the total blood glucose levels is now 420. Using this sample, Natalie concludes that there is no reason to reject her claim at 6% level of significance.

(iv) Find the range of possible values of the variance used in calculating the test statistic. [4]

(v) Explain why there is no need for Natalie to know anything about the population distribution of the glucose blood levels when carrying out the tests in (ii) and (iv). [1]

**11 In this question you should state clearly all the distributions that you use, together with the values of the appropriate parameters.**

NY Pasta Brava runs a pasta delivery service in the district of Serangoon. Its pasta are first prepared at a central kitchen before being sent out for delivery through its food delivery partner. To ensure freshness of each order, NY Pasta Brava only starts preparing the next customer's order after the previous customer's order has been sent out for delivery.

The time taken for NY Pasta Brava to prepare each customer's order follows a normal distribution with mean  $\mu$  minutes and standard deviation  $\sigma$  minutes. The time taken for its food delivery partner to send each order also follows a normal distribution with mean 24 minutes and standard deviation 6 minutes.

- (i) It is given that the preparation time for an order is equally likely to be faster than 13 minutes and slower than 18 minutes. The boss also recorded that on average, 49.5% of orders were prepared between 14.5 and 16.5 minutes. State the value of  $\mu$  and show that  $\sigma = 1.50$ . [2]

- (ii) Sketch the distribution for the preparation times between 10.5 minutes and 20.5 minutes. [2]

To improve the efficiency of the pasta preparation process, the chef of NY Pasta Brava purchases new pasta machines and reorganises the kitchen with designated workstations to prevent bottlenecks such that the preparation time for an order is reduced by 15%.

- (iii) To maintain customer satisfaction, NY Pasta Brava aims to keep the average time for its orders to reach its customers below 38 minutes for each day. Given that there were 85 randomly chosen orders in total on that day, find the probability that NY Pasta Brava succeeds in maintaining customer satisfaction on that day. [3]

At Merlion Pasta Bar, the time taken to prepare each customer's order has mean 13.5 minutes and standard deviation 7 minutes.

- (iv) Explain why the time taken to prepare each customer's order at Merlion Pasta Bar is unlikely to follow a normal distribution with this mean and standard deviation. [1]
- (v) For  $n$  randomly chosen orders, where  $n$  is large, find the least value of  $n$  such that with the improved pasta preparation process at NY Pasta Brava, the mean preparation time of NY Pasta Brava for each order is faster than that of Merlion Pasta Bar by more than 80% of the time. [4]