NANYANG JUNIOR COLLEGE

JC2 PRELIMINARY EXAMINATION

Higher 2

CANDIDATE NAME

CT CLASS | 2 | 0 | | |

Centre Number/ Index Number | | | | | / | | | |

MATHEMATICS **9758/01**

Paper 1 **31st August 2021**

3 Hours

Candidates answer on the Question Paper.

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS

Write your name and class on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing calculator is expected, where appropriate.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

For examiner's use only	
Question number	**Mark**
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
Total	

1 The points P and Q have position vectors $\mathbf{p}$ and $\mathbf{q}$ respectively, which are non-zero and non-parallel. The points P and Q are fixed and R, with position vector $\mathbf{r}$, varies.

(a) Given that $(2\mathbf{r}-\mathbf{p})\times\mathbf{p}=\mathbf{0}$, describe geometrically the set of all possible position vectors of $\mathbf{r}$. [2]

(b) If $\mathbf{r}=\mathbf{q}+\lambda\mathbf{p}$, where $\lambda<0$, show that the area of triangle PQR is $k\,|\mathbf{p}\times\mathbf{q}|$ where k is a constant to be determined in terms of λ . [3]

2 **(i)** On the same axes, sketch the curves with equations $y=\left|\dfrac{ax-3a+2}{3-x}\right|$ and $y=\dfrac{a}{3}x$, where $a>1$, giving the equations of the asymptotes and the coordinates of the points where the curves meet the axes. [3]

(ii) Hence, solve the inequality $\left|\dfrac{ax-3a+2}{3-x}\right|>\dfrac{a}{3}x$, giving your answer in terms of a. [3]

3 **(i)** By first expressing $\dfrac{1}{r!(r+2)}$ in the form $\dfrac{A}{(r+2)!}+\dfrac{B}{(r+1)!}$ where A and B are constants, find $\displaystyle\sum_{r=1}^{n}\frac{1}{r!(r+2)}$. [3]

(ii) Hence, explain why the series $\displaystyle\sum_{r=1}^{\infty}\frac{1+\left(\frac{1}{3}\right)^{r}r!(r+2)}{r!(r+2)}$ converges and find the exact sum to infinity of this series. [4]

4 A curve C has cartesian equation

$$y=\frac{x}{1-x},\ 0\le x<1.$$

(i) The distance between a general point (x,y) on C and the fixed point $(1,0)$ is denoted by s. Show that $s^2=(x-1)^2+y^2$. [1]

(ii) Use differentiation to determine the coordinates of the point on C which has the minimum distance from the point $(1,0)$, giving both coordinates correct to 4 decimal places. [5]

[You need not prove that this distance is a minimum]

(iii) Find this minimum distance. [1]

5 **(a)** The curve $y = \mathrm{f}(x)$ has a horizontal asymptote $y = k$ and cuts the axes at $(a, 0)$ and $(0, b)$, where a, b and k are non-zero constants. It is given that $\mathrm{f}^{-1}(x)$ exists. State, if possible, the coordinates of the points where the following curves cut the axes and the equations of their asymptotes.

(i) $y = \mathrm{f}(2x - 3)$

(ii) $y = \mathrm{f}^{-1}(x)$ [2]

(b) The function g is given by $\mathrm{g} : x \mapsto \ln\left(\dfrac{\mathrm{e}^x + 5}{\mathrm{e}^x - 1}\right)$, for $x \in \mathbb{R}, x > 0$.

(i) Find $\mathrm{g}^{-1}(x)$ and state its domain. [3]

The function h is defined by

$$\mathrm{h} : x \mapsto 1 + \sqrt{9 - (x-2)^2}, \text{ for } x \in \mathbb{R}, -1 \leq x \leq 5.$$

(ii) Find the exact solutions of $\mathrm{g}^{2021}\mathrm{h}(x) = \ln 2$, giving your answer in its simplest form. [3]

6 **(a)** Find the exact value of $\displaystyle\int_{-3}^{-1} \frac{|x+2|}{x^2 + 4x + 5}\,\mathrm{d}x$. [4]

(b) **(i)** Write down $\displaystyle\int \frac{\sin(\ln x)}{x}\,\mathrm{d}x$, where $x > 0$. [1]

(ii) Hence find $\int x \sin(\ln x)\,\mathrm{d}x$. [4]

7 It is given that $y = \ln(1 + \mathrm{e}^x)$.

(i) Show that $(1 + \mathrm{e}^x)\dfrac{\mathrm{d}y}{\mathrm{d}x} - \mathrm{e}^x = 0$. [1]

(ii) By further differentiation of the result in **(i)**, find the Maclaurin series for y, up to and including the term in x^2. Hence find the series for $\dfrac{\mathrm{e}^x}{1 + \mathrm{e}^x}$ up to and including x. [4]

(iii) Using appropriate expansion from the List of Formulae (MF26), verify the correctness of the series $\dfrac{\mathrm{e}^x}{1 + \mathrm{e}^x}$ found in **(ii)**. [3]

8 A curve C has parametric equations

$$x = a\sin^5 t,\ \ y = a\cos^5 t$$

where a is a positive constant.

The tangent to C at a general point with parameter t cuts the coordinate axes at the points A and B.

(i) Denoting the origin by O, show that $OA^{\frac{2}{3}} + OB^{\frac{2}{3}} = a^{\frac{2}{3}}$. [5]

(ii) As t varies, the midpoint of the line segment AB traces out a curve. Find the cartesian equation of this curve in its simplest form. [3]

(iii) Find, in terms of a, the area enclosed by the curve C, giving your answer in the form ka^2 where k is to be determined correct to 2 decimal places. [3]

9 The complex numbers z and w where $w \neq 0$ satisfy the relation

$$2z = |w| + 1.$$

(i) It is given that a is a real number and that a and z satisfy the equation $2z^3 - 5z^2 + 2z + (a+3)\mathrm{i} = 0$. Explain, with justification, why $a = -3$ and that the only possible value of z is 2. [6]

It is given that $\arg w = \dfrac{\pi}{3}$.

(ii) Express the complex number w in the form $p + q\mathrm{i}$ where p and q are in non-trigonometric form. [2]

(iii) Find the least integer n such that $\left|w^n\right| > 20212021$. [2]

(iv) Find the least positive integer k such that w^k is a positive real number. [2]

10 One day, Eddie came home from a birthday party and brought back a helium filled balloon. After playing with it, he accidentally released the balloon at the point (1, 2, 3) and it floated vertically upwards at a speed of 1 unit per second. Shortly after t seconds, a sudden gust of wind caused the balloon to move in the direction of $\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$.

You may assume that $z = 0$ refers to the horizontal ground.

(i) Find the angle in which the balloon has changed in direction after the gust of wind blew it away. [3]

(ii) Find the Cartesian equation of the plane that the balloon is moving along. [3]

(iii) Given that the balloon eventually stayed at the point (2, 6, 12) on the ceiling, find the time t when the gust of wind blew the balloon away. [3]

Eddie decides to shoot the balloon down with his catapult.

(iv) Assume he was holding his catapult at (3, 2, 1) initially and he walked along the path parallel to $2\mathbf{i} + \mathbf{j}$. Find the position vector of the point where he should place his catapult so that the distance between his catapult and the balloon is at its minimum. Hence find this distance. [4]

11 In economics, a supply and demand chart is made up of two curves: the supply curve and demand curve. The supply curve is a function that shows how the price of a product, P, is related to the quantity, q, supplied during a given period of time. The demand curve is a function that shows how the price of the same product is related to the quantity demanded during a given period of time. Due to the nature of the curves, they will intersect at a point which is known as the *equilibrium point*.

For a particular product, the demand curve is given by the equation $D(q) = 75 - 1.22^q$ where $0 \le q \le 21$ and the supply curve is given by the equation $S(q) = 2(1.22)^q - 1$ using the same domain.

(i) Sketch both curves on the same diagram. Your sketch should indicate the axial intercepts of both curves. [3]

(ii) Find the coordinates of the equilibrium point. [1]

Let p_e be the price of the product at equilibrium point. The area between the demand curve, the line $P = p_e$ and the line $q = 0$ is defined as the consumer surplus. In a similar fashion, the area between the supply curve, the line $P = p_e$ and the line $q = 0$ is defined as the producer surplus.

(iii) Without using a graphing calculator, determine the consumer surplus and producer surplus, leaving both answer to 2 decimal places. [4]

A global increase in production lead to a shift of the supply curve to the right. The equation of the new curve is given by $S(q - a)$ where a is a positive constant. The price of the product at the new equilibrium point is denoted by p_e^*.

(iv) Show that $\dfrac{p_e^* + 1}{p_e + 1} = \dfrac{3}{(1.22)^a + 2}$. [4]

(v) For the case when $a = 4$, find the increase in the consumer surplus, leaving your answer to the nearest whole number. [2]