2021 NYJC JC2 Prelim Exam 9758/2 Solutions

Qn	
1(i)	$\mathbf{n}_2 = \begin{pmatrix} 2-4 \\ 3-1 \\ 2-(-1) \end{pmatrix} \times \begin{pmatrix} 2-0 \\ 3-(-1) \\ 2-2 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} -12 \\ 6 \\ -12 \end{pmatrix} = -6\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ $\cos\theta = \dfrac{\left\lvert \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix} \right\rvert}{\left\lvert \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \right\rvert \left\lvert \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix} \right\rvert} = \dfrac{10-2-2}{3(\sqrt{30})} = \dfrac{2}{\sqrt{30}} = \dfrac{2\sqrt{30}}{30} = \dfrac{\sqrt{30}}{15}$
1(ii)	$\Pi_1 : \mathbf{r} \cdot \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix} = 11 \quad \Rightarrow \quad 5x+2y-z=11$ $\Pi_2 : \mathbf{r} \cdot \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = 5 \quad \Rightarrow \quad 2x-y+2z=5$ Line of intersection between the 2 planes $2y-z=11$ $-y+2z=5$ When x=0, $\Rightarrow z=7, y=9$ $\mathbf{d} = \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ -12 \\ -9 \end{pmatrix} = -3\begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}$ $\mathbf{r} = \begin{pmatrix} 0 \\ 9 \\ 7 \end{pmatrix} + \mu\begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}, \mu \in \mathbb{R}$
1(iii)	Since 3 planes have no points in common, the line of intersection in (i) is parallel and away from the plane $4(k-2)x+(k+1)y-4k^2z=8$ $\begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 4(k-2) \\ k+1 \\ -4k^2 \end{pmatrix} = 0$ $-4k+8+4k+4-12k^2=0$ $3k^2-3=0$ $k=\pm 1$

1(iv)

Method 1

When $k=1$,

$\Pi_3 : -4x+2y-4z=8$

$$\text{Distance} = \frac{\left|\begin{pmatrix}0\\9\\7\end{pmatrix}\cdot\begin{pmatrix}-4\\2\\-4\end{pmatrix}-8\right|}{\left|\begin{pmatrix}-4\\2\\-4\end{pmatrix}\right|} = \frac{18}{6} = 3$$

Method 2

Take any point on the plane Π_3, for example $\begin{pmatrix}0\\4\\0\end{pmatrix}$.

Let $\overrightarrow{OA} = \begin{pmatrix}0\\9\\7\end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix}0\\4\\0\end{pmatrix}$.

Then distance $= \left|\overrightarrow{AB}\cdot\mathbf{n}\right| = \dfrac{\left|\begin{pmatrix}0\\-5\\-7\end{pmatrix}\cdot\begin{pmatrix}-4\\2\\-4\end{pmatrix}\right|}{6} = 3$

Method 3

Let F be the foot of perpendicular of $(0,9,7)$ onto $\Pi_3 : -4x+2y-4z=8$.

$\overrightarrow{OF} = \begin{pmatrix}0\\9\\7\end{pmatrix} + k\begin{pmatrix}-4\\2\\-4\end{pmatrix}$ for some $k \in \mathbb{R}$

.

Since F lies on Π_3, $\left[\begin{pmatrix}0\\9\\7\end{pmatrix} + k\begin{pmatrix}-4\\2\\-4\end{pmatrix}\right]\cdot\begin{pmatrix}-4\\2\\-4\end{pmatrix} = 8$

	$-10+36k=8$ $k=\frac{18}{36}$ $=\frac{1}{2}$ $\overrightarrow{OF}=\begin{pmatrix}0\\9\\7\end{pmatrix}+\frac{1}{2}\begin{pmatrix}-4\\2\\-4\end{pmatrix}=\begin{pmatrix}-2\\10\\5\end{pmatrix}$ $\therefore \text{distance}=\left\|\begin{pmatrix}-2\\10\\5\end{pmatrix}-\begin{pmatrix}0\\9\\7\end{pmatrix}\right\|=3$ (or just $\left\|\frac{1}{2}\begin{pmatrix}-4\\2\\4\end{pmatrix}\right\|$)
2(i)	Let the volume of water in the tank be V. $\frac{\mathrm{d}V}{\mathrm{d}t}=n-k\sqrt{x}$ $\frac{\mathrm{d}(Ax)}{\mathrm{d}t}=n-k\sqrt{x}$ $A\frac{\mathrm{d}x}{\mathrm{d}t}=n-k\sqrt{x}$ When $x=h,\ \frac{\mathrm{d}x}{\mathrm{d}t}=0$, therefore $A\frac{\mathrm{d}x}{\mathrm{d}t}=n-k\sqrt{h}=0\Rightarrow k=\frac{n}{\sqrt{h}}$. $A\frac{\mathrm{d}x}{\mathrm{d}t}=n-k\sqrt{x}\Rightarrow\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{n}{A}\left(1-\sqrt{\frac{x}{h}}\right)$
2(ii)	Using $x=hu^2$, differentiating wrt t, $\frac{\mathrm{d}x}{\mathrm{d}t}=2hu\frac{\mathrm{d}u}{\mathrm{d}t}$ $A\frac{\mathrm{d}x}{\mathrm{d}t}=n-k\sqrt{x}\Rightarrow 2huA\frac{\mathrm{d}u}{\mathrm{d}t}=n-\frac{n}{\sqrt{h}}\sqrt{hu^2}$ $2huA\frac{\mathrm{d}u}{\mathrm{d}t}=n-nu=-n(u-1)$ $\frac{2Ah}{n}\frac{\mathrm{d}u}{\mathrm{d}t}=-\frac{u-1}{u}$
2(iii)	$\frac{2Ah}{n}\frac{\mathrm{d}u}{\mathrm{d}t}=-\frac{u-1}{u}$ Separating the variables,

$$\int \frac{u}{u-1}\,\mathrm{d}u = -\frac{n}{2Ah}\int \mathrm{d}t$$

$$\int 1+\frac{1}{u-1}\,\mathrm{d}u = -\frac{n}{2Ah}t + C$$

$$u + \ln(u-1) = -\frac{n}{2Ah}t + C$$

When $x = 4h,\ \ u = 2$

When $x = \frac{16}{9}h,\ \ u = \frac{4}{3}$

When $x = h,\ \ u = 1$

When $t = 0,\ x = 4h,\ \ u = 2$, we have $C = 2$

When $x = \frac{16}{9}h,\ \ u = \frac{4}{3}$, we have $\frac{4}{3} + \ln\left(\frac{1}{3}\right) = -\frac{n}{2Ah}t + 2$

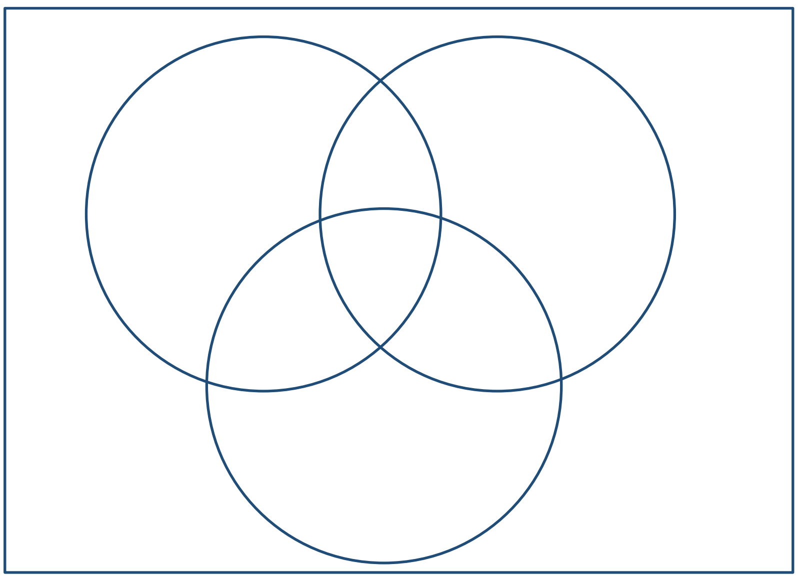
Therefore, $t = \frac{2Ah}{n}\left(\frac{2}{3} + \ln 3\right)$

As t increases, x decreases and approaches a depth of h.

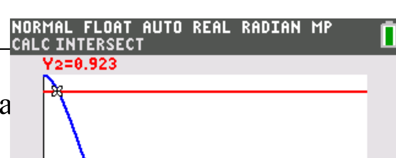
3(a)	Sum of first k terms = $\frac{k}{2}\left[2a+(k-1)d\right]$ First term of last k terms is given by $u_{n-k+1}=a+(n-k)d$ Sum of last k terms = $\frac{k}{2}\left[a+(n-k)d+a+(n-1)d\right]$ $=\frac{k}{2}\left[2a+(n-k)d+(n-1)d\right]$ Difference between the sums = $\frac{k}{2}\left[2a+(n-k)d+(n-1)d\right]-\frac{k}{2}\left[2a+(k-1)d\right]$ $=\frac{kd}{2}(n-k)+\frac{kd}{2}(n-1)-\frac{kd}{2}(k-1)$ $=\frac{kd}{2}\left(n-k+n-1-k+1\right)=\frac{kd}{2}(2n-2k)$ $=kd(n-k)$ *Alternatively:* Sum of first k terms = $\frac{k}{2}\left[2a+(k-1)d\right]$ Sum of the last k terms is $S_n-S_{n-k}=\frac{n}{2}\left[2a+(n-1)d\right]-\frac{n-k}{2}\left[2a+(n-k-1)d\right]$ $=\frac{n}{2}\left[2a+(n-1)d\right]-\frac{n}{2}\left[2a+(n-k-1)d\right]+\frac{k}{2}\left[2a+(n-k-1)d\right]$ $=\frac{n}{2}\left[kd\right]+\frac{k}{2}\left[2a+(n-k-1)d\right]$ Difference between the sums = Sum of the last k terms - Sum of the first k terms $=\frac{n}{2}\left[kd\right]+\frac{k}{2}\left[2a+(n-k-1)d\right]-\frac{k}{2}\left[2a+(k-1)d\right]$ $=\frac{n}{2}\left[kd\right]+\frac{k}{2}\left[2a+(n-k-1)d-\left(2a+(k-1)d\right)\right]$ $=\frac{n}{2}\left[kd\right]+\frac{k}{2}\left[d(n-k-1-k+1)\right]=\frac{n}{2}\left[kd\right]+\frac{k}{2}\left[d\left(n-2k\right)\right]$ $=kd\left[n-k\right]$ (shown)
3(b)	$u_r=\left(\frac{1}{3}\right)^{3r-2}+\left(\frac{1}{3}\right)^{3r-1}=\left(\frac{1}{3}\right)^{3r}\left[\left(\frac{1}{3}\right)^{-2}+\left(\frac{1}{3}\right)^{-1}\right]=12\left(\frac{1}{3}\right)^{3r}$ $\sum_{r=1}^{n}u_r=\sum_{r=1}^{n}12\left(\frac{1}{3}\right)^{3r}=12\sum_{r=1}^{n}\left(\frac{1}{3}\right)^{3r}$

	$$=12\left[\left(\frac{1}{3}\right)^3+\left(\frac{1}{3}\right)^6+\ldots+\left(\frac{1}{3}\right)^{3n}\right]$$ $$=12\left[\frac{\left(\frac{1}{3}\right)^3\left(1-\left(\frac{1}{3}\right)^{3n}\right)}{1-\left(\frac{1}{3}\right)^3}\right]=\frac{6}{13}\left(1-\frac{1}{27^n}\right)$$ As $n\to\infty,\frac{1}{27^n}\to 0,$ therefore, $\sum_{r=1}^{\infty}u_r=\frac{6}{13}$ Alternatively: $$u_r=\left(\frac{1}{3}\right)^{3r-2}+\left(\frac{1}{3}\right)^{3r-1}=\left(\frac{1}{3}\right)^{3r-1}\left[\left(\frac{1}{3}\right)^{-1}+1\right]=4\left(\frac{1}{3}\right)^{3r-1}$$ $$\sum_{r=1}^{n}u_r=\sum_{r=1}^{n}4\left(\frac{1}{3}\right)^{3r-1}=4\sum_{r=1}^{n}\left(\frac{1}{3}\right)^{3r-1}$$ $$=4\left[\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^5+\ldots+\left(\frac{1}{3}\right)^{3n-1}\right]$$ $$=4\left[\frac{\left(\frac{1}{3}\right)^2\left(1-\left(\frac{1}{3}\right)^{3n}\right)}{1-\left(\frac{1}{3}\right)^3}\right]=\frac{6}{13}\left(1-\frac{1}{27^n}\right)$$ As $n\to\infty,\frac{1}{27^n}\to 0,$ therefore, $\sum_{r=1}^{\infty}u_r=\frac{6}{13}$
3(c) (i)	$$z+z^2+\ldots+z^n=\frac{z(1-z^n)}{1-z}$$
3(c) (ii)	$$\frac{z(1-z^n)}{1-z}=\frac{\mathrm{e}^{\mathrm{i}\theta}(1-\mathrm{e}^{\mathrm{i}n\theta})}{1-\mathrm{e}^{\mathrm{i}\theta}}$$ $$=\frac{\mathrm{e}^{\mathrm{i}\theta}.\mathrm{e}^{\mathrm{i}\frac{n\theta}{2}}\left(\mathrm{e}^{-\mathrm{i}\frac{n\theta}{2}}-\mathrm{e}^{\mathrm{i}\frac{n\theta}{2}}\right)}{\mathrm{e}^{\mathrm{i}\frac{\theta}{2}}\left(\mathrm{e}^{-\mathrm{i}\frac{\theta}{2}}-\mathrm{e}^{\mathrm{i}\frac{\theta}{2}}\right)}$$ $$=\frac{\mathrm{e}^{\mathrm{i}\frac{(n+1)\theta}{2}}(-2\mathrm{i}\sin\frac{n\theta}{2})}{-2\mathrm{i}\sin\frac{\theta}{2}}=\frac{\sin\frac{n\theta}{2}\mathrm{e}^{\mathrm{i}\frac{(n+1)\theta}{2}}}{\sin\frac{\theta}{2}}$$ For ‘show’ question, it is necessary to show the working clearly to reach the answer given to get full credits.
3(c) (iii)	$$z+z^2+\ldots+z^n=\frac{\mathrm{e}^{\mathrm{i}\theta}(1-\mathrm{e}^{\mathrm{i}n\theta})}{1-\mathrm{e}^{\mathrm{i}\theta}}$$ $$\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{\mathrm{i}2\theta}+\ldots+\mathrm{e}^{\mathrm{i}n\theta}=\frac{\mathrm{e}^{\mathrm{i}\theta}(1-\mathrm{e}^{\mathrm{i}n\theta})}{1-\mathrm{e}^{\mathrm{i}\theta}}$$

	$\cos\theta + \mathrm{i}\sin\theta + \cos 2\theta + \mathrm{i}\sin 2\theta + \ldots + \cos n\theta + \mathrm{i}\sin n\theta = \dfrac{\sin\frac{n\theta}{2}\mathrm{e}^{\mathrm{i}\frac{(n+1)\theta}{2}}}{\sin\frac{\theta}{2}}$ $= \dfrac{\sin\frac{n\theta}{2}\left(\cos\frac{(n+1)\theta}{2} + \mathrm{i}\sin\frac{(n+1)\theta}{2}\right)}{\sin\frac{\theta}{2}}$ Comparing the imaginary parts, $\sin\theta + \sin 2\theta + \ldots + \sin n\theta = \dfrac{\sin\frac{(n+1)\theta}{2}\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}$
4	(i) $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{\mathrm{d}V}{\mathrm{d}t} = 4\pi r^2 \frac{\mathrm{d}r}{\mathrm{d}t}$. (1) Substitute $\frac{\mathrm{d}V}{\mathrm{d}t} = 10, r = 5$ into (1) gives $10 = 4\pi\left(5^2\right)\frac{\mathrm{d}r}{\mathrm{d}t} \Rightarrow \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{1}{10\pi}$. So the radius of the ball is increasing at the rate of $\frac{1}{10\pi}$ cm/s. (ii) $A = 4\pi r^2 \Rightarrow \frac{\mathrm{d}A}{\mathrm{d}t} = 8\pi r \frac{\mathrm{d}r}{\mathrm{d}t}$. (2) Substitute $\frac{\mathrm{d}V}{\mathrm{d}t} = 10, \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{5}{72\pi}$ into (1) gives $10 = 4\pi r^2\left(\frac{5}{72\pi}\right) \Rightarrow r = 6$. Substitute $\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{5}{72\pi}, r = 6$ into (2) gives $\frac{\mathrm{d}A}{\mathrm{d}t} = 8\pi(6)\left(\frac{5}{72\pi}\right) = \frac{10}{3}$. So the surface area of the ball is increasing at the rate of $\frac{10}{3}$ cm^2/s.

5(i)	The class sizes of the classes may not be the same. Thus the students will not have equal chance of being selected.
5(ii)	The teacher must obtain a random sample in which every student has equal chance of being selected. This will ensure that the sample is not biased.
5(iii)	Number of samples = $\binom{20}{5}^4 = 5.78\times10^{16}$
6(i)	Since A, B and C are independent, A, C and B' are also independent. Thus $P(A\cap C\cap B')=(0.4)(0.7)(0.2)=0.056$
6(ii)	 A B C y 0.25 0.07 $0.08-y$ 0.17 0.31 $0.14-y$ Let $y=P(A\cap C\cap B')$ From the Venn Diagram, we must have

	$0.08-y\ge 0$ and $0.14-y\ge 0$. Thus $y\le 0.08$. Using $P(A\cup B\cup C)\le 1$, we have $1.02-y\le 1$. Thus $y\ge 0.02$. Thus least value of y is 0.02 and greatest value of y is 0.08
7(i)	Number of ways $=\dfrac{9!}{3!2!}-1=30239$
7(ii)	Number of ways $=\dfrac{5!}{3!}\times {}^6C_4\times\dfrac{4!}{2!}=3600$
7(iii)	Number of ways to form 4 letters = n(4 diff letters)+n(2 same, 2 diff)+n(3 same, 1 diff)+ n(2 pairs of same letters) $={}^6C_4(4!)+{}^2C_1\,{}^5C_2(\dfrac{4!}{2!})+{}^1C_1\,{}^5C_1(\dfrac{4!}{3!})+\dfrac{4!}{2!2!}$ Probability required $=\dfrac{{}^2C_1\,{}^5C_2(\frac{4!}{2!})+{}^1C_1\,{}^5C_1(\frac{4!}{3!})+\frac{4!}{2!2!}}{{}^6C_4(4!)+{}^2C_1\,{}^5C_2(\frac{4!}{2!})+{}^1C_1\,{}^5C_1(\frac{4!}{3!})+\frac{4!}{2!2!}}$ $=0.425$ or $\dfrac{133}{313}$
8(i)	Let X be the number of rejected masks out of 10 masks. $X\sim \mathrm{B}(10,p)$ Let Y be the number of defective masks out of 5 pens. $Y\sim B(5,p)$ P(box is accepted) = 0.923 $\mathrm{P}(X=0)+(\mathrm{P}(X=1)+\mathrm{P}(X=2))\mathrm{P}(Y=0)$ =0.923



	Using GC, $p = 0.043231 = 0.0432$ (correct to 3 s.f)
8(ii)	Let M be the number of masks sampled m : 10, 15 $P(M=m)$: $P(X=0)+P(X>2) = 0.650503$; $P(X=1)+P(X=2) = 0.349497$ Expected number of masks sampled = 10 (0.650503) + 15 (0.349497) = 11.747 =11.7 <u>Alternative method</u> Let A be the number of additional masks sampled a : 0, 5 $P(A=a)$: $P(X=0)+P(X>2) = 0.650503$; $P(X=1)+P(X=2) = 0.349497$ Expected number of masks sampled = 10 + 5 (0.349497) = 11.747 =11.7
8(iii)	Let A be the number of rejected boxes out of 60 boxes. $A \sim B(60, 0.077)$ Let C be the number of rejected boxes out of 59 boxes. $C \sim B(59, 0.077)$ P(the 60^{th} box is the 5^{th} box to be rejected | at least 5 boxes of masks are rejected) $= \frac{P(C=4)\times 0.077}{P(A\geq 5)} = 0.030335 = 0.0303$
9(i)	$P(C=1) = 2!P(BRR) + 3!P(BWRR)$ $= 2\frac{2\cdot n\cdot(n-1)}{(n+3)(n+2)(n+1)} + 6\frac{2\cdot 1\cdot n\cdot(n-1)}{(n+3)(n+2)(n+1)(n)}$ $= \frac{4n(n-1)+12(n-1)}{(n+3)(n+2)(n+1)} = \frac{4(n-1)}{(n+2)(n+1)}$ $P(C=0) = P(RR) + 2!P(WRR)$ $= \frac{n\cdot(n-1)}{(n+3)(n+2)} + 2\frac{1\cdot n\cdot(n-1)}{(n+3)(n+2)(n+1)}$ $= \frac{n(n-1)(n+1+2)}{(n+3)(n+2)(n+1)} = \frac{n(n-1)}{(n+2)(n+1)}$ $P(C=2) = 1 - P(C=0) - P(C=1)$ $= 1 - \frac{n(n-1)}{(n+2)(n+1)} - \frac{4(n-1)}{(n+2)(n+1)}$ $= \frac{n^2+3n+2-n^2+n-4n+4}{(n+2)(n+1)} = \frac{6}{(n+2)(n+1)}$

9(ii)

$$\mathrm{E}(C^2)=\frac{4(n-1)}{(n+2)(n+1)}+\frac{24}{(n+2)(n+1)}$$
$$=\frac{4(n+5)}{(n+2)(n+1)}$$

$$\mathrm{Var}(C)=\mathrm{E}(C^2)-[\mathrm{E}(C)]^2$$
$$=\frac{4(n+5)}{(n+2)(n+1)}-\left[\frac{4}{n+1}\right]^2$$
$$=\frac{4(n+1)(n+5)-16(n+2)}{(n+2)(n+1)^2}$$
$$=\frac{4n^2+8n-12}{(n+2)(n+1)^2}=\frac{4(n+3)(n-1)}{(n+2)(n+1)^2}$$

9(iii)

$$P\left(|C_1-C_2|>0\right)=1-P\left(|C_1-C_2|=0\right)$$
$$=1-P(C_1=C_2)$$
$$=1-\left[P(C=0)\right]^2+\left[P(C=1)\right]^2+\left[P(C=2)\right]^2$$
$$=1-\frac{n^2(n-1)^2}{(n+2)^2(n+1)^2}-\frac{16(n-1)^2}{(n+2)^2(n+1)^2}-\frac{36}{(n+2)^2(n+1)^2}$$

Using GC,

n	$P\left(\lvert C_1-C_2\rvert>0\right)$
33	0.20093
34	0.19605

Thus least $n=34$.

10(i)

Unbiased estimates for population mean and population variance,

$$\bar{x}=\frac{511.5}{75}=6.82$$

$$s^2=\frac{1}{n-1}\left[\sum x^2-\frac{\left(\sum x\right)^2}{n}\right]=\frac{1}{74}\left[4027.89-\frac{511.5^2}{75}\right]=7.29$$

10(ii)

Given μ denote mean blood glucose level,

$\mathrm{H}_0:\mu=6.0$

To test : $\mathrm{H}_1:\mu>6.0$ at 5% level of significance

	Under H_0, since sample size of 75 is large, using Central Limit Theorem, $\bar{X} \sim \mathrm{N}\left(6, \frac{7.29}{75}\right)$ approximately, and test statistic, $Z = \frac{\bar{X}-6}{\sqrt{7.29/75}} \sim \mathrm{N}(0,1)$ Critical Region: Reject H_0 if p-value ≤ 0.05 Calculations : Using GC, $z_{\text{cal}} = 2.630$ and p-value = 0.00427 Conclusion: Since p-value < 0.05, we reject H_0. There is sufficient evidence, at 5% level of significance, that Natalie's average blood glucose level is higher than 6.0.
10(iii)	Readings at weekend may be biased by different life style, so results may not be valid.
10(iv)	To test $\mathrm{H}_0 : \mu = 6.0$ $\mathrm{H}_1 : \mu < 6.0$ at 10% level of significance Under H_0, since n = 75 is large by Central Limit Theorem, $\bar{X} \sim \mathrm{N}\left(6, \frac{s^2}{75}\right)$ approximately, and test statistic, $Z = \frac{\bar{X}-6}{\sqrt{S^2/75}} \sim \mathrm{N}(0,1)$ Reject H_0 if $z_{\text{cal}} \le -1.55477$ Calculations : $\bar{x} = \frac{420}{75} = 5.6$ and using $z_{\text{cal}} = \frac{5.6-6}{s/\sqrt{75}}$ Since H_0 is rejected, $z_{\text{cal}} = \frac{5.6-6}{s/\sqrt{75}} \le -1.55477 \quad \Rightarrow \quad s^2 \le 4.96$ Required range is $s^2 \le 4.96$

10(v)

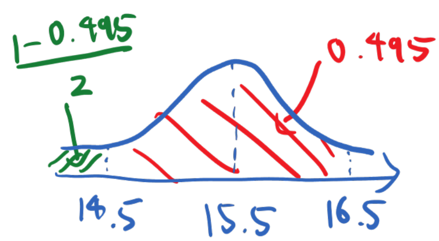
Since sample sizes are large enough, Natalie may use Central Limit Theorem to approximate the distribution of the sample mean to be normal. Therefore, no need to know anything about the population distribution of the glucose blood levels.

11(i)

Let C denote the preparation time of a randomly chosen order for NY Pasta Brava.

$C \sim \mathrm{N}\left(\mu, \sigma^2\right)$

Using symmetry, $\mu = \dfrac{13+18}{2} = 15.5$



$\mathrm{P}(14.5 < C < 16.5) = 0.495$

$\Rightarrow \mathrm{P}(C < 14.5) = \dfrac{1-0.495}{2} = 0.2525$

$\Rightarrow \mathrm{P}(Z < \dfrac{14.5-15.5}{\sigma}) = 0.2525$

Using InvNorm Left,

$-\dfrac{1}{\sigma} = -0.6666433049$

$\Rightarrow \sigma = 1.5001 \text{ (5 s.f.)} = 1.50 \text{ (3 s.f.)}$

Alternatively,

$\mathrm{P}(14.5 < C < 16.5) = 0.495$

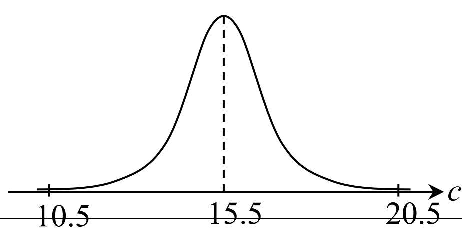
$\Rightarrow \mathrm{P}(\dfrac{14.5-15.5}{\sigma} < Z < \dfrac{16.5-15.5}{\sigma}) = 0.495$

Using InvNorm Center,

$-\dfrac{1}{\sigma} = -0.6666433049 \quad \& \quad \dfrac{1}{\sigma} = -0.6666433049$

$\Rightarrow \sigma = 1.5001 \text{ (5 s.f.)} = 1.50 \text{ (3 s.f.)}$

11(ii)



11(iii)	Let D denote the delivery time of a randomly chosen order for NY Pasta Brava. $D \sim \mathrm{N}\left(24, 6^2\right)$ Let $\overline{T} = \dfrac{0.85(C_1 + C_2 + ... + C_{85}) + (D_1 + D_2 + ... + D_{85})}{85}$ $\mathrm{E}\left(\overline{T}\right) = \dfrac{0.85\left[85\mathrm{E}(C)\right] + \left[85\mathrm{E}(D)\right]}{85} = 37.175$ $\mathrm{Var}\left(\overline{T}\right) = \dfrac{0.85^2\left[85\mathrm{Var}(C)\right] + \left[85\mathrm{Var}(D)\right]}{85^2} = 0.44625$ (5 s.f.) $\overline{T} \sim \mathrm{N}\left(37.175, \left(\sqrt{0.44625}\right)^2\right)$ $\mathrm{P}(\overline{T} < 38) = 0.89251$ (5 s.f.) $= 0.893$ (3 s.f.)
11(iv)	Let M denote the preparation time of a randomly chosen order for Merlion Pasta Bar. If $M \sim \mathrm{N}\left(13.5, 7^2\right)$, $\mathrm{P}(M < 0) = 0.0269$ (3 s.f.) which is not possible OR $\mathrm{P}(13.5 - 3(7) < M < 13.5 + 3(7)) \approx 0.997$ but $13.5 - 3(7) = -7.5 < 0$ which is not possible

11(v)	Let $\overline{C}=\frac{0.85(C_1+C_2+...+C_n)}{n}$ & $\overline{M}=\frac{M_1+M_2+...+M_n}{n}$ $\mathrm{E}\left(\overline{C}\right)=\frac{0.85\left[n\mathrm{E}\left(C\right)\right]}{n}=13.175$ $\mathrm{Var}\left(\overline{C}\right)=\frac{0.85^2\left[n\mathrm{Var}\left(C\right)\right]}{n^2}=\frac{1.625625}{n}$ $\overline{C}\sim\mathrm{N}\left(13.175,\left(\sqrt{\frac{1.625625}{n}}\right)^2\right)$ $\mathrm{E}\left(\overline{M}\right)=\mathrm{E}\left(M\right)=13.5$ $\mathrm{Var}\left(\overline{M}\right)=\frac{\mathrm{Var}\left(M\right)}{n^2}=\frac{7^2}{n}$ Since n is large, by CLT, $\overline{M}\sim\mathrm{N}\left(13.5,\left(\frac{7}{\sqrt{n}}\right)^2\right)$ approximately $\mathrm{E}\left(\overline{C}-\overline{M}\right)=-0.325$ $\mathrm{Var}\left(\overline{C}-\overline{M}\right)=\frac{50.625625}{n}$ $\therefore\overline{C}-\overline{M}\sim\mathrm{N}\left(-0.325,\left(\sqrt{\frac{50.625625}{n}}\right)^2\right)$ approximately $\mathrm{P}(\overline{C}<\overline{M})=\mathrm{P}(\overline{C}-\overline{M}<0)>0.8$ $\mathrm{P}(Z<\frac{0.325}{\sqrt{\frac{50.625625}{n}}})>0.8$ $\mathrm{P}(Z<0.045677062\sqrt{n})>0.8$ $0.045677062\sqrt{n}>0.8416212335$ $n>339.4978624$ $\therefore$ Least n is 340.