H

CANDIDATE NAME

CLASS

ADMISSION NUMBER

2021 Preliminary Examination
Pre-University 3

MATHEMATICS 9758/01

Paper 1

14 September 2021

3 hours

Candidates answer on the Question Paper.

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your admission number, name and class on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give your answers in the spaces provided in the Question Paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

Qn No.	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	*	Total
Score													
Max Score	**5**	**7**	**7**	**8**	**8**	**9**	**8**	**10**	**12**	**12**	**14**		**100**

This document consists of **27** printed pages and **1** blank pages.

1 **(i)** Find the derivative of $\dfrac{1}{4-x^2}$. [1]

(ii) Hence find $\displaystyle\int \frac{x^2}{\left(4-x^2\right)^2}\,\mathrm{d}x$. [4]

2 **(a)** The curve $y=\mathrm{f}\left(x\right)$ cuts the axes at $\left(a,0\right)$ and $\left(0,b\right)$. State, if it is possible to do so, the coordinates of the points where the following curves cut the axes.

(i) $y=\mathrm{f}\left(x\right)-b$ [1]

(ii) $y=\mathrm{f}\left(ax\right)$ [1]

(b) The functions g and h are defined by

$$\begin{aligned}&\mathrm{g}:x\mapsto 3-\frac{1}{x-1},\quad x\in\mathbb{R},\ x\neq 1,\\&\mathrm{h}:x\mapsto 2-x,\quad x\in\mathbb{R}.\end{aligned}$$

(i) Show that the composite function hg exists. [2]

(ii) Find an expression for hg(x) and hence find $\left(\mathrm{hg}\right)^{-1}\left(3\right)$. [3]

3 **(i)** It is given that $x^2\dfrac{\mathrm{d}y}{\mathrm{d}x}-3xy+4=0$. Using the substitution $y=ux^3$, show that the differential equation can be transformed to $\dfrac{\mathrm{d}u}{\mathrm{d}x}=\dfrac{c}{x^5}$, where c is a constant to be determined. [3]

(ii) Hence given that $y=3$ when $x=1$, solve the differential equation $x^2\dfrac{\mathrm{d}y}{\mathrm{d}x}-3xy+4=0$ to find y in terms of x. [4]

4 With respect to the origin O, the position vectors of the points A, B and C are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively. Point C lies on AB such that $AC:CB=1:2$. It is given that $\mathbf{a}$ is a unit vector and the length of OB is 2 units.

(i) Give a geometrical interpretation of $|\mathbf{a}\bullet\mathbf{c}|$. [1]

(ii) It is given that the angle AOB is $60°$. By considering $(2\mathbf{a}-\mathbf{b})\bullet(2\mathbf{a}-\mathbf{b})$, find $|2\mathbf{a}-\mathbf{b}|$. [3]

(iii) Find $\mathbf{c}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. [1]

(iv) Hence by considering cosine of angle AOC and cosine of angle COB, determine if the line segment OC bisects the angle AOB. [3]

5 **(i)** By using the substitution $x = \tan\theta$, show that $\int \frac{1}{\sqrt{x^2+1}}\,\mathrm{d}x$ can be written as $\int \sec\theta\,\mathrm{d}\theta$. Hence find $\int \frac{1}{\sqrt{x^2+1}}\,\mathrm{d}x$. [4]

(ii) The finite region R is bounded by the curve $y=\sqrt{\frac{1}{(x-1)^2}-1}$, the line $y=\frac{1}{\sqrt{3}}$ and the y-axis. By referring to your answer in part **(i)**, find the exact volume of the solid generated when R is rotated through 2π radians about the y-axis. [4]

6 **(i)** Sketch the curve with equation $y=\left|\frac{1}{a-x}\right|$, where a is a positive constant. State, in terms of a, the equations of the asymptotes and coordinates of any intersections with the x-axis and y-axis. On the same diagram, sketch the line with equation $y=b(x-a)$, where b is a positive constant. [4]

(ii) Find, in terms of a and b, the root of the equation $\left|\frac{1}{a-x}\right|=b(x-a)$. [3]

(iii) Hence solve the inequality $\left|\frac{1}{a-x}\right|>b(x-a)$. [2]

7 A sequence $u_1,\ u_2,\ u_3,\ldots$ is such that $u_n = a^{n+1} - (n+1)^a$, where a is a constant and $n \geq 1$.

(i) Given that $u_1 = 0$, find u_3. [2]

For the rest of this question, let $a = 2$. It is given that $\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$.

(ii) Find $\sum_{r=1}^{n} u_r$ in terms of n. (You need not simplify your answer.) [4]

(iii) Another sequence $v_1,\ v_2,\ v_3,\ldots$ is such that $\sum_{r=1}^{9}(v_{r+1} - v_r) = \sum_{r=1}^{10} u_r$. Given that $v_1 = u_1$, find the value of v_{10}. [2]

8 A curve C has parametric equations

$$x = \cot t + 2, \qquad y = \sec t, \qquad -\frac{\pi}{2} < t < 0.$$

(i) Sketch the graph of C, indicating the equations of any asymptotes. [2]

(ii) Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\sin^3 t}{\cos^2 t}$. Hence explain why C is increasing for $-\frac{\pi}{2} < t < 0$. [3]

(iii) Find the equation of normal to C when $t = -\frac{\pi}{4}$. [2]

(iv) The point P on C has coordinates $(\cot p + 2,\ \sec p)$. Given that the point R is the midpoint of P and the point with coordinates $(-2,\ 0)$, find the cartesian equation of the curve traced by R as p varies. [3]

9 **Do not use a calculator in answering this question.**

(a) Two complex numbers are $w = 1 - \sqrt{3}\,\mathrm{i}$ and $z = \sqrt{2}\left(\cos\frac{3}{4}\pi - \mathrm{i}\sin\frac{3}{4}\pi\right)$.

(i) Find $w^2 z^*$ in the form $r(\cos\theta + \mathrm{i}\sin\theta)$, where $r > 0$ and $-\pi < \theta \le \pi$. [4]

(ii) Show that there is no integer value of n for which the real part of w^n is zero. [2]

(b) **(i)** One of the roots of the equation $3z^3 + 13z^2 + 20z + 14 = 0$ is $-1 + \mathrm{i}$. Find the other roots of the equation in cartesian form, $p + \mathrm{i}q$, showing your working. [4]

(ii) Hence find the roots of the equation $w^3 + 13w^2 + 60w + 126 = 0$. [2]

10 A RC series circuit comprises a power source of V volts in series with a resistor of R ohms and capacitor of C farads. When the power is switched on and power is supplied to the capacitor, the charge builds up in the capacitor. At t seconds after the power is switched on, the charge on the capacitor is q coloumbs and the current in the circuit is I amps. It is given that R and C are constants.

A differential equation for a RC series circuit is $RI + \frac{q}{C} = V$, where $I = \frac{\mathrm{d}q}{\mathrm{d}t}$.

(i) Find the maximum value of q in terms of C and V. (You do not need to prove that it is a maximum.) [2]

(ii) Show that, under certain conditions on V which should be stated, $R\frac{\mathrm{d}I}{\mathrm{d}t} + \frac{I}{C} = 0$. [2]

(iii) In a particular circuit, $I = \frac{V}{R}$ when $t = 0$. Solve the differential equation in part **(ii)** and find I in terms of R, C, V and t. [5]

(iv) Sketch the graph of I against t. [2]

(v) Describe what happens to the current in the circuit after a long time. [1]

11 Bank A offers a study loan to students enrolled in an undergraduate course of study. The key features of the loan are:

- interest-free during the course of study,
- fixed monthly interest of 0.3% upon graduation,
- minimum monthly repayment of \$100.

Ali decides to take a study loan of \$50000 from Bank A on 1 January 2021 at the start of his 3-year undergraduate course.

(a) During his course of study, Ali pays \$200 at the end of January 2021 and on the last day of each subsequent month, he pays \$10 more than in the previous month. Thus on 28 February 2021, he pays \$210 and on 31 March 2021, he pays \$220, and so on. How much does Ali owe the bank at the end of his course of study? [2]

(b) Bank A charges interest on any outstanding amount of the loan on the first day of each month, starting on the month right after a student graduates.

(i) Upon graduation, Ali immediately found a job and decides to pay \$900 to bank A on the last day of each month, starting on the month right after he graduated. Show that Ali owes the bank \$34121 (to the nearest dollar) on the last day of the 3rd month after he graduated. [2]

(ii) Use the formula for the sum of a geometric progression to find an expression for the amount owed by Ali on the last day of the nth month after he graduated. Hence find in which month Ali pays off his study loan. [5]

(iii) Find the total amount of interest that Ali paid. [2]

(iv) If Ali decides to pay off his study loan within 3 years upon graduation, i.e. at the end of December 2026, what is the minimum amount, to the nearest dollar, that he needs to pay per month after graduation? [3]

End of Paper