2021 Preliminary Examination
PU3 MATHEMATICS 9758/02
Solutions

Section A: Pure Mathematics

Qn	Solution
1 **[4]**	Given that θ is a sufficiently small angle, $\frac{1}{\sin 2\theta+\cos\theta}$ $\approx \frac{1}{2\theta+1-\frac{\theta^2}{2}}$ $\approx \left(1+2\theta-\frac{\theta^2}{2}\right)^{-1}$ $=1+(-1)\left(2\theta-\frac{\theta^2}{2}\right)+\frac{(-1)(-2)}{2!}\left(2\theta-\frac{\theta^2}{2}\right)^2+\ldots$ $=1-2\theta+\frac{\theta^2}{2}+(2\theta)^2+\ldots$ $\approx 1-2\theta+\frac{9}{2}\theta^2$ where $a=-2,\ \ b=\frac{9}{2}$

Qn	Solution
2 **[6]**	$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y^2-2y+5}{y-2}$ $\frac{y-2}{y^2-2y+5}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)=1$ $\int \frac{y-2}{y^2-2y+5}\ \mathrm{d}y=\int 1\ \mathrm{d}x$ $\int \frac{1}{2}\left[\frac{2(y-2)}{y^2-2y+5}\right]\mathrm{d}y=\int 1\ \mathrm{d}x$ $\int \frac{1}{2}\left[\frac{2y-4}{y^2-2y+5}\right]\mathrm{d}y=\int 1\ \mathrm{d}x$ $\frac{1}{2}\int\left[\frac{2y-2}{y^2-2y+5}-\frac{2}{y^2-2y+5}\right]\mathrm{d}y=\int 1\ \mathrm{d}x$ $\frac{1}{2}\int\left[\frac{2y-2}{y^2-2y+5}-\frac{2}{(y-1)^2+2^2}\right]\mathrm{d}y=\int 1\ \mathrm{d}x$

Qn	Solution
	$\frac{1}{2}\left[\ln\left\|y^2-2y+5\right\|-2\left(\frac{1}{2}\right)\tan^{-1}\left(\frac{y-1}{2}\right)\right]=x+c$ $\frac{1}{2}\left[\ln\left(y^2-2y+5\right)-\tan^{-1}\left(\frac{y-1}{2}\right)\right]=x+c$ When $x=0$, $y=1$, $\frac{1}{2}\left[\ln 4-\tan^{-1}(0)\right]=0+c$ $c=\frac{1}{2}\ln 4=\ln 2$ $\frac{1}{2}\left[\ln\left(y^2-2y+5\right)-\tan^{-1}\left(\frac{y-1}{2}\right)\right]=x+\ln 2$

Qn	Solution
3(i) **[3]**	$\mathrm{f}(x)=\ln(2+2\sin x)$ $\mathrm{f}'(x)=\frac{2\cos x}{2+2\sin x}$ $=\frac{\cos x}{1+\sin x}$ **<u>Method 1: Apply Quotient Rule</u>** $\mathrm{f}''(x)=\frac{(1+\sin x)(-\sin x)-(\cos x)(\cos x)}{(1+\sin x)^2}$ $=\frac{-\sin x-\sin^2 x-\cos^2 x}{(1+\sin x)^2}$ $=\frac{-\sin x-\left(\sin^2 x+\cos^2 x\right)}{(1+\sin x)^2}$ $=\frac{-\sin x-1}{(1+\sin x)^2}$ $=\frac{-(\sin x+1)}{(1+\sin x)^2}$ $=\frac{-1}{1+\sin x}$ Therefore, $k=-1$. -- **<u>Method 2: Apply Product Rule</u>**

Qn	Solution
	$f'(x)=\dfrac{\cos x}{1+\sin x}=(\cos x)(1+\sin x)^{-1}$ $f''(x)=(\cos x)(1+\sin x)^{-1}$ $=(\cos x)\left[-1(1+\sin x)^{-2}(\cos x)\right]+(1+\sin x)^{-1}(-\sin x)$ $=\dfrac{-\cos^2 x}{(1+\sin x)^2}-\dfrac{\sin x}{1+\sin x}$ $=\dfrac{-\cos^2 x-\sin x(1+\sin x)}{(1+\sin x)^2}$ $=\dfrac{-\cos^2 x-\sin x-\sin^2 x}{(1+\sin x)^2}$ $=\dfrac{-\sin x-(\cos^2 x+\sin^2 x)}{(1+\sin x)^2}$ $=\dfrac{-(\sin x+1)}{(1+\sin x)^2}$ $=\dfrac{-1}{1+\sin x}$ Therefore, $k=-1$.
3(ii) **[4]**	$f''(x)=\dfrac{-1}{1+\sin x}=-(1+\sin x)^{-1}$ $f'''(x)=(1+\sin x)^{-2}(\cos x)$ When $x=0$, $f(0)=\ln(2+2\sin 0)=\ln 2$ $f'(0)=\dfrac{\cos 0}{1+\sin 0}=1$ $f''(0)=\dfrac{-1}{1+\sin 0}=-1$ $f'''(0)=(1+\sin 0)^{-2}(\cos 0)=1$ Therefore, $f(x)=\ln(2+2\sin x)$ $=\ln 2+x-\dfrac{1}{2}x^2+\dfrac{1}{6}x^3+\ldots$
3(iii) **[1]**	$\int_0^2 f(x)\ dx\approx\int_0^2\left(\ln 2+x-\dfrac{1}{2}x^2+\dfrac{1}{6}x^3\right)\ dx$ $\approx 2.7196=2.72$ (3 s.f.) (from graphing calculator)

<table>
<tr><th>Qn</th><th>Solution</th></tr>
<tr><td>4 (i)
[2]</td><td>

Plane p is parallel to

$$\begin{pmatrix}0\\2\\-1\end{pmatrix}-\begin{pmatrix}-k\\2\\5\end{pmatrix}=\begin{pmatrix}k\\0\\-6\end{pmatrix} \text{ and } \begin{pmatrix}-\frac{1}{2}\\3\\-1\end{pmatrix}-\begin{pmatrix}0\\2\\-1\end{pmatrix}=\begin{pmatrix}-\frac{1}{2}\\1\\0\end{pmatrix}.$$

Normal of p is parallel to $\begin{pmatrix}k\\0\\-6\end{pmatrix}\times\begin{pmatrix}-\frac{1}{2}\\1\\0\end{pmatrix}=\begin{pmatrix}6\\3\\k\end{pmatrix}$.

Hence equation of p is

$$\mathbf{r}\bullet\begin{pmatrix}6\\3\\k\end{pmatrix}=\begin{pmatrix}0\\2\\-1\end{pmatrix}\bullet\begin{pmatrix}6\\3\\k\end{pmatrix}=(0)(6)+(2)(3)+(-1)(k)=6-k.$$

Cartesian equation is $6x+3y+kz=6-k$. (shown)

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<tr><td>4 (ii)
[2]</td><td>

$l: \ \dfrac{x+2}{-3}=y-2=\dfrac{z-4}{k}$

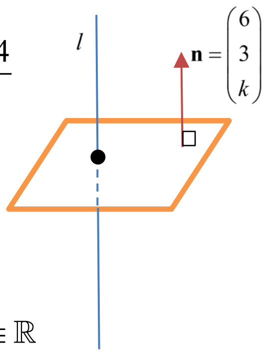
Let $\lambda=\dfrac{x+2}{-3}=y-2=\dfrac{z-4}{k}$

$x=-2-3\lambda$

$y=2+\lambda$

$z=4+k\lambda$

$$l: \ \mathbf{r}=\begin{pmatrix}-2\\2\\4\end{pmatrix}+\lambda\begin{pmatrix}-3\\1\\k\end{pmatrix}, \ \lambda\in\mathbb{R}$$



<u>Method 1</u>: Show l is not parallel to normal of p.

Suppose l is perpendicular to p, then l is parallel to the normal of p, i.e. $\begin{pmatrix}-3\\1\\k\end{pmatrix}=t\begin{pmatrix}6\\3\\k\end{pmatrix}$, for some $t\in\mathbb{R}$.

$$\Rightarrow\begin{cases}-3=6t\\1=3t\\k=tk\end{cases}\Rightarrow\begin{cases}t=-0.5\\t=\dfrac{1}{3}\\t=1\end{cases}$$

Since there is no unique value of t, l is not parallel to the normal of p, i.e. l cannot be perpendicular to p. (shown)

<u>Method 2</u>: Show l is not perpendicular to a direction parallel to p.

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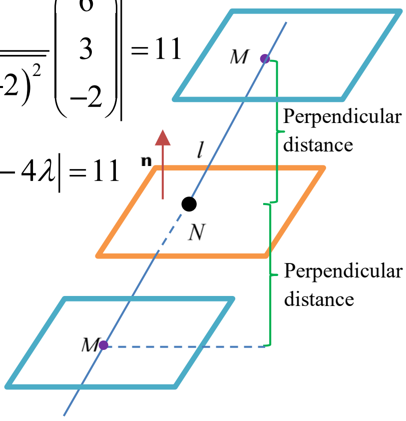
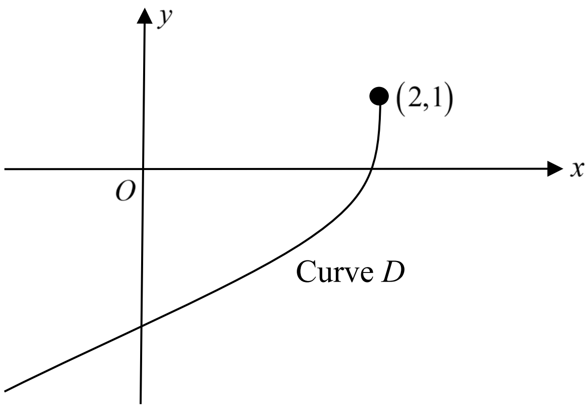
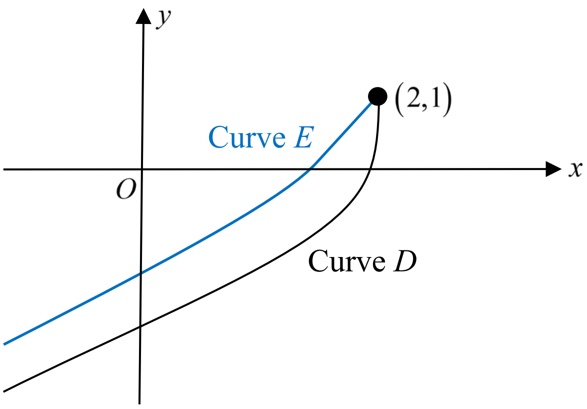
Qn	Solution
	Suppose l is perpendicular to p, then l is perpendicular to the $\begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix}$. [from **(i)**] Since $\begin{pmatrix} -3 \\ 1 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} -0.5 \\ 1 \\ 0 \end{pmatrix} = 1.5+1 = 2.5 \neq 0$, l is not perpendicular to the $\begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix}$ $\Rightarrow$ l cannot be perpendicular to p. (shown)
4(iii) **[3]**	$l: \ \mathbf{r} = \begin{pmatrix} -2 \\ 2 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 1 \\ -2 \end{pmatrix}, \ \lambda \in \mathbb{R}$ $p: \ \mathbf{r} \bullet \begin{pmatrix} 6 \\ 3 \\ -2 \end{pmatrix} = 8$ $\begin{pmatrix} -2-3\lambda \\ 2+\lambda \\ 4-2\lambda \end{pmatrix} \bullet \begin{pmatrix} 6 \\ 3 \\ -2 \end{pmatrix} = 8$ $-12-18\lambda+6+3\lambda-8+4\lambda = 8$ $-11\lambda = 22$ $\lambda = -2$ position vector of $N = \begin{pmatrix} -2-3(-2) \\ 2+(-2) \\ 4-2(-2) \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 8 \end{pmatrix}$ Coordinates of $N(4, 0, 8)$.
4(iv) **[3]**	<u>**Method 1**</u>: Length of Projection Let the point of intersection of l and π be M. Since M lies on l, $\overrightarrow{OM} = \begin{pmatrix} -2-3\lambda \\ 2+\lambda \\ 4-2\lambda \end{pmatrix}$ for some $\lambda \in \mathbb{R}$ $\overrightarrow{MN} = \begin{pmatrix} 4 \\ 0 \\ 8 \end{pmatrix} - \begin{pmatrix} -2-3\lambda \\ 2+\lambda \\ 4-2\lambda \end{pmatrix} = \begin{pmatrix} 6+3\lambda \\ -2-\lambda \\ 4+2\lambda \end{pmatrix}$.

Qn	Solution
	$\left\| \begin{pmatrix} 6+3\lambda \\ -2-\lambda \\ 4+2\lambda \end{pmatrix} \cdot \frac{1}{\sqrt{6^2+3^2+(-2)^2}} \begin{pmatrix} 6 \\ 3 \\ -2 \end{pmatrix} \right\| = 11$ $\frac{1}{7}\left\|36+18\lambda-6-3\lambda-8-4\lambda\right\| = 11$ $\left\|22+11\lambda\right\| = 77$ $22+11\lambda = 77 \text{ or } -77$ $11\lambda = 55 \text{ or } -99$ $\lambda = 5 \text{ or } -9$  $\overrightarrow{OM} = \begin{pmatrix} -2-3(5) \\ 2+(5) \\ 4-2(5) \end{pmatrix} \text{ or } \begin{pmatrix} -2-3(-9) \\ 2+(-9) \\ 4-2(-9) \end{pmatrix}$ $= \begin{pmatrix} -17 \\ 7 \\ -6 \end{pmatrix} \text{ or } \begin{pmatrix} 25 \\ -7 \\ 22 \end{pmatrix}$ Hence, possible points of intersections between l and π are $(-17, 7, -6)$ and $(25, -7, 22)$. -- **<u>Method 2</u>**: Distance between two planes Let the equation of π be $\mathbf{r} \cdot \begin{pmatrix} 6 \\ 3 \\ -2 \end{pmatrix} = a$ Distance between p and π $= \left\| \frac{8}{\sqrt{6^2+3^2+(-2)^2}} - \frac{a}{\sqrt{6^2+3^2+(-2)^2}} \right\|$ $= \left\|\frac{8-a}{7}\right\|$ $\left\|\frac{8-a}{7}\right\| = 11$ $\left\|8-a\right\| = 77$ $8-a = 77 \text{ or } -77$ $a = -69 \text{ or } 85$ Hence possible equations of π are $\mathbf{r} \cdot \begin{pmatrix} 6 \\ 3 \\ -2 \end{pmatrix} = -69 \text{ or } \mathbf{r} \cdot \begin{pmatrix} 6 \\ 3 \\ -2 \end{pmatrix} = 85$ To find the point of intersection between l and π :

Qn	Solution
	For π: $\mathbf{r}\bullet\begin{pmatrix}6\\3\\-2\end{pmatrix}=-69$, $\begin{pmatrix}-2-3\lambda\\2+\lambda\\4-2\lambda\end{pmatrix}\bullet\begin{pmatrix}6\\3\\-2\end{pmatrix}=-69$ $-12-18\lambda+6+3\lambda-8+4\lambda=-69$ $-11\lambda=-55$ $\lambda=5$ $\overrightarrow{OM}=\begin{pmatrix}-2-3(5)\\2+(5)\\4-2(5)\end{pmatrix}=\begin{pmatrix}-17\\7\\-6\end{pmatrix}$ Similarly, for π: $\mathbf{r}\bullet\begin{pmatrix}6\\3\\-2\end{pmatrix}=85$, $\begin{pmatrix}-2-3\lambda\\2+\lambda\\4-2\lambda\end{pmatrix}\bullet\begin{pmatrix}6\\3\\-2\end{pmatrix}=85$ $-12-18\lambda+6+3\lambda-8+4\lambda=85$ $-11\lambda=99$ $\lambda=-9$ $\overrightarrow{OM}=\begin{pmatrix}-2-3(-9)\\2+(-9)\\4-2(-9)\end{pmatrix}=\begin{pmatrix}25\\-7\\22\end{pmatrix}$. Hence, possible points of intersections between l and π are $(-17, 7, -6)$ and $(25, -7, 22)$.

Qn	Solution
5(i) **[3]**	Substitute $x=t+\frac{1}{t}+4,\ y=t-\frac{1}{t}+1$ into $y=x-2-\frac{1}{x-3}$, We have $t-\frac{1}{t}+1=\left(t+\frac{1}{t}+4\right)-2-\frac{1}{\left(t+\frac{1}{t}+4\right)-3}$ $\Rightarrow t-\frac{1}{t}+1=t+\frac{1}{t}+2-\frac{1}{t+\frac{1}{t}+1}$

	$\Rightarrow -\frac{2}{t}=1-\frac{t}{t^2+t+1}=\frac{t^2+1}{t^2+t+1}$ $\Rightarrow -2t^2-2t-2=t^3+t$ $\Rightarrow t^3+2t^2+3t+2=0$ **<u>Method 1</u>**: Use graphing calculator From graphing calculator, $t=-1$ is the only real solution. Hence, curve D and E intersect only once at $t=-1$. **<u>Method 2</u>**: Factorise $t^3+2t^2+3t+2=0$ $(t+1)(t^2+t+2)=0$. $t=-1$ or $t^2+t+2=0$ (no real solution) Hence, curve D and E intersect only once at $t=-1$. When $t=-1$, $x=2$ and $y=1$. The coordinate at $t=-1$ is $(2,1)$.
5(ii) **[2]**	
5(iii) **[3]**	Using $x=t+3$, $y=(t+3)-2-\frac{1}{(t+3)-3}=t+1-\frac{1}{t}$. $x\le 2 \Rightarrow t+3\le 2 \Rightarrow t\le -1$ Curve E: $x=t+3$, $y=t+1-\frac{1}{t}$, $t\le -1$ 

5(iv) **[4]**	**Method 1**: Using x-axis (both curve in parametric form) For the Curve D: $x = t + \frac{1}{t} + 4 \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = 1 - \frac{1}{t^2}$ When $x = 0,\ t = -\sqrt{3} - 2$ (given) When $x = 2,\ t = -1$ For the Curve E: $x = t + 3 \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = 1$ When $x = 0,\ t = -3$ When $x = 2,\ t = -1$ Area $= \int_{-3}^{-1} \left(t - \frac{1}{t} + 1\right)(1)\,\mathrm{d}t - \int_{-\sqrt{3}-2}^{-1} \left(t - \frac{1}{t} + 1\right)\left(1 - \frac{1}{t^2}\right)\mathrm{d}t$ $= 1.3929$ units^2. (4 d.p.) -- **Method 2**: Using x-axis (1 cartesian, 1 parametric) For the Curve D: $x = t + \frac{1}{t} + 4 \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = 1 - \frac{1}{t^2}$ When $x = 0,\ t = -\sqrt{3} - 2$ (given) When $x = 2,\ t = -1$ Area $= \int_{0}^{2} x - 2 - \frac{1}{x-3}\,\mathrm{d}x - \int_{-\sqrt{3}-2}^{-1} \left(t - \frac{1}{t} + 1\right)\left(1 - \frac{1}{t^2}\right)\mathrm{d}t$ $= 1.3929$ units^2. (4 d.p.) -- **Method 3**: Using y-axis For the Curve D: $y = t - \frac{1}{t} + 1 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t} = 1 + \frac{1}{t^2}$ At y-intercept ($x = 0$), $t = -\sqrt{3} - 2$ (given) When $y = 1,\ t = -1$ For the Curve E: $y = t + 1 - \frac{1}{t} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t} = 1 + \frac{1}{t^2}$ At y-intercept ($x = 0$), $t = -3$ When $y = 1,\ t = -1$ Area $= \int_{-\sqrt{3}-2}^{-1} \left(t + \frac{1}{t} + 4\right)\left(1 + \frac{1}{t^2}\right)\mathrm{d}t - \int_{-3}^{-1} (t+3)\left(1 + \frac{1}{t^2}\right)\mathrm{d}t$ $= 1.3929$ units^2. (4 d.p.)

Section B: Probability and Statistics

Qn	Solution
6(i) **[1]**	$\mathrm{P}(B)=\mathrm{P}(A\cup B)-\mathrm{P}(A\cap B')$ $=0.8-0.55$ $=0.25$
6(ii) **[1]**	Probability that neither A nor B occurs $=\mathrm{P}(A'\cap B')$ $=1-\mathrm{P}(A\cup B)$ $=1-0.8$ $=0.2$
6(iii) **[2]**	<u>**Method 1**</u> $\mathrm{P}(B'\cap C)$ $=\mathrm{P}(B')\times\mathrm{P}(C)$ $\because$ B and C are independent, B' and C are independent, $=(1-0.25)(0.6)$ $=0.45$ <u>**Method 2**</u> $\mathrm{P}(B'\cap C)$ $=\mathrm{P}(C)-\mathrm{P}(B\cap C)$ $=\mathrm{P}(C)-\mathrm{P}(B)\mathrm{P}(C)$ $\because$ B and C are independent $=0.6-(0.25)(0.6)$ $=0.45$
6(iv) **[3]**	Let $\mathrm{P}(A\cap B'\cap C)$ be x. Since $\mathrm{P}(B'\cap C)=0.45$, then $\mathrm{P}(A'\cap B'\cap C)=0.45-x$. Therefore, $0.45-x\geq 0\Rightarrow x\leq 0.45$ Since $\mathrm{P}(A\cup B)=0.8$, then $0.45-x\leq 0.2$. $\Rightarrow x\geq 0.25$. Therefore, the range of values of $\mathrm{P}(A\cap B'\cap C)$ is $0.25\leq x\leq 0.45$.
7(i) **[2]**	The sample size should be at least 30 so that it is **large enough for the <u>sample mean</u> mass of almond flour per packet to follow a normal distribution <u>approximately</u>**. These packets should be **randomly** selected.
7(ii) **[4]**	Unbiased estimate of the population variance, $s^2=\dfrac{40}{39}(4.3)^2=18.964$ (5 s.f.)

<table>
<tr><th>Qn</th><th>Solution</th></tr>
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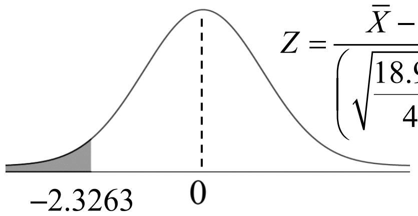
Let X be the mass, in grams of a randomly chosen packet of almond flour,
μ be the population mean mass of almond flour per packet

$H_0 : \mu = \mu_0$
$H_1 : \mu < \mu_0$
This is the population mean.

Under H_0, since n = 40 is large, by Central Limit Theorem, $\overline{X} \sim N\left(\mu_0, \frac{18.964}{40}\right)$ approximately.

Test statistic, $Z = \frac{\overline{X} - \mu_0}{\left(\sqrt{\frac{18.964}{40}}\right)} \sim N(0,1)$.

1-tail z-test is used at $\alpha = 0.01$:


$Z = \frac{\overline{X} - \mu_0}{\left(\sqrt{\frac{18.964}{40}}\right)} \sim N(0,1)$
“Concluded that the mean mass is not overstated” infers that H_0 is not rejected.
Since H_0 is not rejected, the test statistic value $\frac{248.5 - \mu_0}{\sqrt{\frac{18.964}{40}}}$ does not lie inside the rejection (critical) region.

$\frac{248.5 - \mu_0}{\sqrt{\frac{18.964}{40}}} > -2.3263$
$248.5 - \mu_0 > -2.3263\left(\sqrt{\frac{18.964}{40}}\right)$
$-\mu_0 > -250.10$
$\mu_0 < 250.10 \quad \text{(2 d.p.)}$
</td></tr>
<tr><td>7(iii)
[1]</td><td>“at the 1% level of significance” means 0.01 is the probability of concluding that the population mean mass of almond flour per packet is overstated when it is in fact μ_0 grams.</td></tr>
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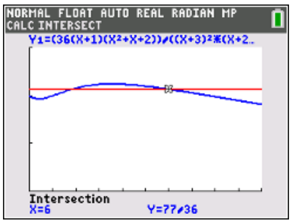
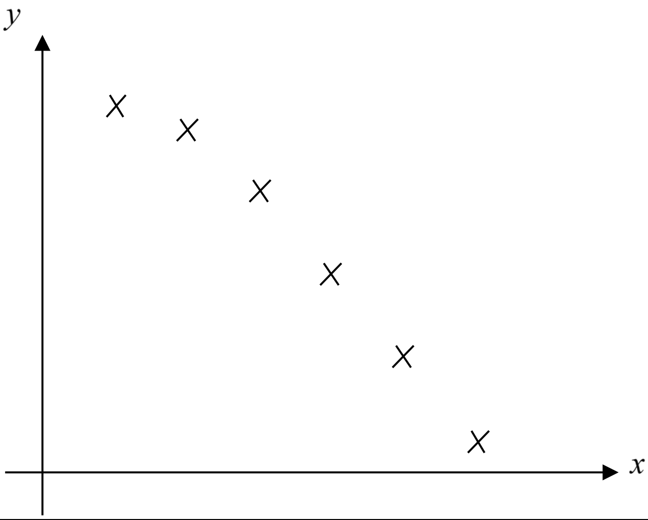
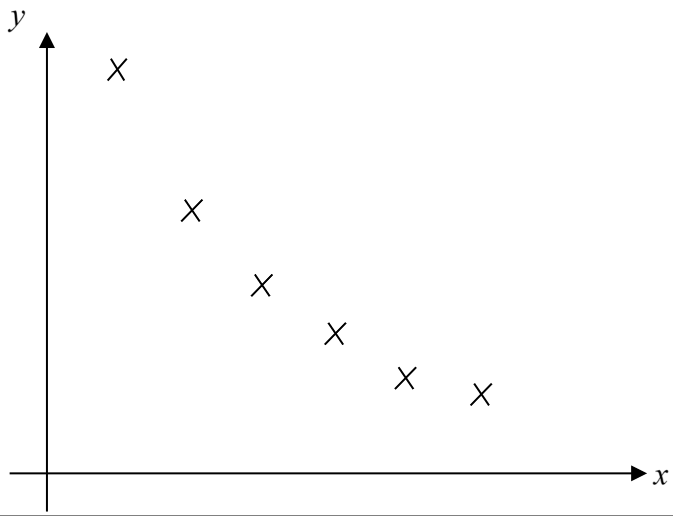
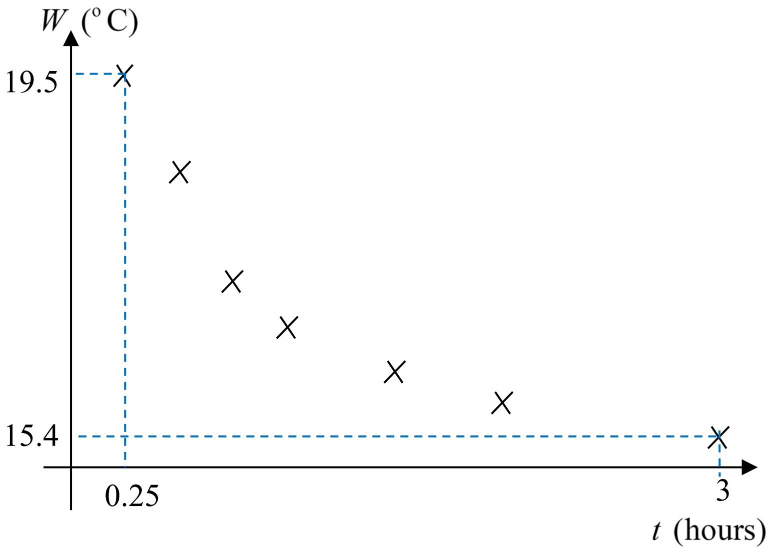
Qn	Solution
8(i) **[1]**	Let X be the number of patients who reported symptomatic relief after consuming the medication, out of 12, in a clinic. $X \sim B(12,\ p)$.

Qn	Solution
	$\mathrm{P}(X=10)={}^{12}\mathrm{C}_{10}p^{10}(1-p)^2$ $=66p^{10}(1-p)^2$.
8(ii) **[4]**	Since the mode is 10, $\mathrm{P}(X=10)>\mathrm{P}(X=9)$ $66p^{10}(1-p)^2>{}^{12}\mathrm{C}_9p^9(1-p)^3$ $66p^{10}(1-p)^2>220p^9(1-p)^3$ Divide throughout by $p^9(1-p)^3$: $66p>220(1-p)$ $66p>220-220p$ $286p>220$ $p>\frac{10}{13}$ $\mathrm{P}(X=10)>\mathrm{P}(X=11)$ $66p^{10}(1-p)^2>{}^{12}\mathrm{C}_{11}p^{11}(1-p)^1$ $66p^{10}(1-p)^2>12p^{11}(1-p)$ Divide throughout by $p^{10}(1-p)$: $66(1-p)>12p$ $66-66p>12p$ $-66p-12p>-66$ $-78p>-66$ $p<\frac{11}{13}$ Therefore, the possible range of values of p is $\frac{10}{13}<p<\frac{11}{13}$.
8(iii) **[1]**	Let Y be the number of patients who reported symptomatic relief after consuming the medication, out of 15, in another clinic. $Y\sim \mathrm{B}(15,\ 0.8)$. $\mathrm{P}(Y=15)\approx 0.035184=0.0352$. (3 s.f.)
8(iv) **[3]**	Required probability $=\mathrm{P}(Y\geq 8)\times(\mathrm{P}(Y=15))^2\times 3$ $=(1-\mathrm{P}(Y\leq 7))\times(0.035184)^2\times 3$ $=0.00370$ (to 3 s.f.)

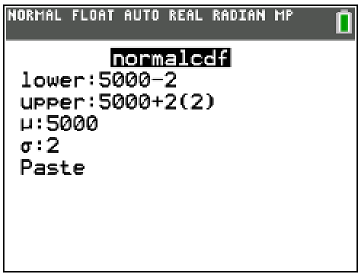
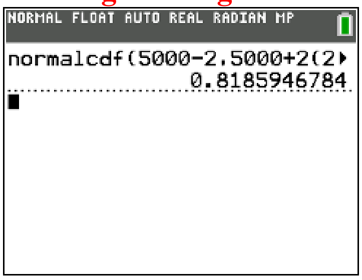
Qn	Solution
9(i) **[5]**	Possible values of M: 0, 1, 3, 4 Total number of notes: $n+3$ $\mathrm{P}(M=0)=\mathrm{P}(\text{draw 2 \$1 notes})+\mathrm{P}(\text{draw 2 \$2 notes})$ $=\frac{n}{n+3}\left(\frac{n-1}{n+2}\right)+\frac{2}{n+3}\left(\frac{1}{n+2}\right)$ $=\frac{n^2-n+2}{(n+3)(n+2)}$ $\mathrm{P}(M=1)=\mathrm{P}(\text{draw \$1 note and \$2 note})$ $=\frac{n}{n+3}\left(\frac{2}{n+2}\right)\times 2$ $=\frac{4n}{(n+3)(n+2)}$ $\mathrm{P}(M=3)=\mathrm{P}(\text{draw \$2 note and \$5 note})$ $=\frac{2}{n+3}\left(\frac{1}{n+2}\right)\times 2$ $=\frac{4}{(n+3)(n+2)}$ $\mathrm{P}(M=4)=\mathrm{P}(\text{draw \$1 note and \$5 note})$ $=\frac{n}{n+3}\left(\frac{1}{n+2}\right)\times 2$ $=\frac{2n}{(n+3)(n+2)}$
9(ii) **[5]**	$\mathrm{E}(M)$ $=0+\frac{4n}{(n+3)(n+2)}+3\left(\frac{4}{(n+3)(n+2)}\right)+4\left(\frac{2n}{(n+3)(n+2)}\right)$ $=\frac{12n+12}{(n+3)(n+2)}=\frac{12(n+1)}{(n+3)(n+2)}.$ $\mathrm{E}(M^2)$ $=0+1^2\left(\frac{4n}{(n+3)(n+2)}\right)+3^2\left(\frac{4}{(n+3)(n+2)}\right)+4^2\left(\frac{2n}{(n+3)(n+2)}\right)$ $=\frac{4n+36+32n}{(n+3)(n+2)}=\frac{36(n+1)}{(n+3)(n+2)}.$

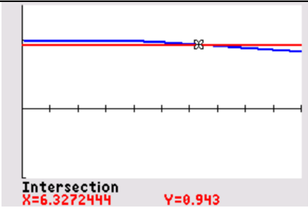
Qn	Solution
	$\mathrm{Var}(M)$ $= \mathrm{E}(M^2) - [\mathrm{E}(M)]^2$ $= \dfrac{36(n+1)}{(n+3)(n+2)} - \left[\dfrac{12(n+1)}{(n+3)(n+2)}\right]^2$ $= \dfrac{36(n+1)}{(n+3)(n+2)}\left[1 - \dfrac{4(n+1)}{(n+3)(n+2)}\right]$ $= \dfrac{36(n+1)}{(n+3)(n+2)}\left[\dfrac{n^2+5n+6-4n-4}{(n+3)(n+2)}\right]$ $= \dfrac{36(n+1)}{(n+3)(n+2)}\left[\dfrac{n^2+n+2}{(n+3)(n+2)}\right]$ $= \dfrac{36(n+1)(n^2+n+2)}{(n+3)^2(n+2)^2}$
9(iii) **[1]**	Given that $\mathrm{Var}(M) = \dfrac{77}{36}$, $\dfrac{36(n+1)(n^2+n+2)}{(n+3)^2(n+2)^2} = \dfrac{77}{36}$  Using GC graph, since $n \geq 2$, $n = 6$

Qn	Solution
10(i) **[2]**	(A) $y = ax^2 + b$, where a is negative and b is positive 

Qn	Solution
	(B) $y = c\ln x + d$, where c is negative and d is positive 
10(ii) **[2]**	 As the points do not lie close to a straight line, a linear model is not appropriate. [Accept: As W decreases at a decreasing rate, a linear model is not appropriate.]
10(iii) **[3]**	**(a)** r-value for t and W is -0.8675 (4 dp) **(b)** r-value for t^2 and W is -0.7294 (4 dp) **(c)** r-value for $\ln t$ and W is -0.9822 (4 dp)
10(iv) **[3]**	Since for the <u>scatter diagram for case (B), y **decreases at a decreasing rate**</u> and <u>$\|r\|$ **is closest to 1** for the case **(c)**</u>, hence the most appropriate model is $W = e\ln t + f$. Regression line: $W = -1.7295\ln t + 16.929$ $W = -1.73\ln t + 16.9$ (3 s.f.)
10(v) **[2]**	When $W = 16$,

Qn	Solution
	$16 = -1.7295\ln t + 16.929$ $t = 1.7111$ $= 1.71$ (3 s.f.) Time taken is 1.71 hours. Since $W = 16$ **lies within** the given **data range** of W and $\lvert r \rvert$ **is close to 1**, this estimate is **reliable**.

Qn	Solution
11(i) **[2]**	<u>**Method 1: Standardise**</u> $L \sim \mathrm{N}(5000, \sigma^2)$ $\mathrm{P}(5000-\sigma < L < 5000+2\sigma)$ $= \mathrm{P}\left(\frac{5000-\sigma-5000}{\sigma} < Z < \frac{5000+2\sigma-5000}{\sigma}\right)$ $= \mathrm{P}(-1 < Z < 2)$ $= 0.81859 = 0.819$ (to 3 s.f.) <u>**Method 2: Use GC**</u> **Note:** By the "68-95-99.7" rule of all normal distributions, it does not matter what the unknown σ value is in order to find $\mathrm{P}(5000-\sigma < L < 5000+2\sigma)$. Hence, we can key in any σ value into the GC to obtain the answer. (In the example below, $\sigma = 2$ is used. You can check that you will obtain the same answer regardless of which σ value you use.) **This only works when the probability we are finding involves the random variable being an integer value of σ away from the mean.**  Using GC, $\mathrm{P}(5000-\sigma < L < 5000+2\sigma) = 0.81859 = 0.819$ (3s.f.)
11(ii) **[2]**	<u>**Method 1:**</u> Use GC (Graph) $L \sim \mathrm{N}(5000, \sigma^2)$ $\mathrm{P}(L > 4990) = 0.943$ <u>Method 1: GC</u>

Qn	Solution
	 From GC: $\sigma = 6.3272$. Therefore, $\sigma^2 = 6.3272^2 = 40.033 = 40.0$ (to 3 s.f.) . **<u>Method 2:</u>** Standardise $\mathrm{P}(L > 4990) = 0.943$ $\mathrm{P}\left(Z > \frac{4990-5000}{\sigma}\right) = 0.943$ $\mathrm{P}\left(Z > \frac{-10}{\sigma}\right) = 0.943$ Using GC inverse norm, $\frac{-10}{\sigma} = -1.5805$ $1.5805\sigma = 10$ $\sigma = \frac{10}{1.5805} = 6.3271$ $\Rightarrow \sigma^2 = 6.3271^2 = 40.032 = 40.0$ (to 3 s.f.)
11(iii) **[1]**	$S \sim \mathrm{N}(1000, 25)$ $\mathrm{P}(S \le 1002)$ $= 0.65542 = 0.655$ (to 3 s.f.)
11(iv) **[2]**	Let A be the number of small cans, out of 20, with the amount of oil in the can to be at most 1002 millilitres. $A \sim \mathrm{B}(20, 0.65542)$ $\mathrm{P}(A < 10)$ $= \mathrm{P}(A \le 9)$ $= 0.047670 = 0.0477$ (to 3 s.f.)
11(v) **[3]**	The required probability is $\mathrm{P}(L > 5S + 30) = \mathrm{P}(L - 5S > 30)$. $\mathrm{E}(L - 5S) = 5000 - 5(1000) = 0$ $\mathrm{Var}(L - 5S) = 40 + 5^2(25) = 665$ Therefore, $L - 5S \sim \mathrm{N}(0, 665)$.

Qn	Solution
	$\mathrm{P}(L-5S>30)$ $=0.12234=0.122$ (3 s.f.)
11(vi) **[4]**	Let $C=0.13\left(L_1+L_2+\ldots+L_6\right)+0.05\left(S_1+S_2\right)$, where C refers to the total cost of 6 large cans of Type A industrial lubricating oil and 2 small cans of Type B industrial lubricating oil. We want to find $\mathrm{P}\left(C\geq 3995\right)$. $\mathrm{E}\left(C\right)$ $=\mathrm{E}\left(0.13\left(L_1+L_2+\ldots+L_6\right)+0.05\left(S_1+S_2\right)\right)$ $=\left(0.13\right)\left(6\right)\left(5000\right)+\left(0.05\right)\left(2\right)\left(1000\right)$ $=4000$ $\mathrm{Var}\left(C\right)$ $=\mathrm{Var}\left(0.13\left(L_1+L_2+\ldots+L_6\right)+0.05\left(S_1+S_2\right)\right)$ $=0.13^2\left(6\right)\left(40\right)+0.05^2\left(2\right)\left(25\right)$ $=4.181$ (exact) Therefore, $C\sim \mathrm{N}(4000,4.181)$. $\mathrm{P}\left(C\geq 3995\right)$ $=0.99276=0.993$ (to 3 s.f.)