Class	Index Number

Name : ___________________________________

METHODIST GIRLS' SCHOOL

Founded in 1887

PRELIMINARY EXAMINATION 2021
Secondary 4

Friday | **ADDITIONAL MATHEMATICS** | **4049/01**

13 August 2021 | **Paper 1** | 2 h 15 min

Candidates answer on the Question Paper.
No Additional Materials are required.

READ THESE INSTRUCTIONS FIRST
Write your class, index number and name in the spaces at the top of this page.
Write in dark blue or black pen
You may use a HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** questions.
Give non-exact numerical answers correct to 3 significant figure, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved scientific calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

The total number of marks for this paper is 90.

	90

Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the quadratic equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Binomial Expansion

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\ldots(n-r+1)}{r!}$.

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

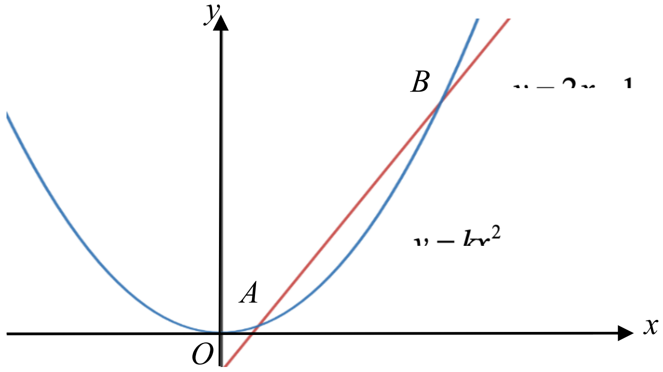
Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\Delta = \frac{1}{2} bc \sin A$$

1 The diagram shows the graphs of $y = 2x - 1$ and $y = kx^2$, where k is a positive constant. The graphs intersects at two distinct points A and B.



(i) Show that $k < 1$. [2]

$kx^2 = 2x - 1$

$kx^2 - 2x + 1 = 0$

Discriminant > 0

$(-2)^2 - 4(k)(1) > 0$ [M1]

$4 - 4k > 0$

$k < 1$ [A1]

(ii) Describe the relationship between the graphs $y = 2x - 1$ and $y = kx^2$ for $k = 1$. [2]

[Method 1]

When $k = 1$ **, discriminant =0** [A1]

The line $y = 2x - 1$ is a tangent to the curve $y = kx^2$ *or*

The line $y = 2x - 1$ touches the curve $y = kx^2$ [B1]

[Method 2]

When $k = 1$, $x^2 - 2x + 1 = 0$

$x = 1,\ y = 1$

The line $y = 2x - 1$ is a tangent to the curve $y = kx^2$ at (1, 1) *or*

The line $y = 2x - 1$ touches the curve $y = kx^2$ at (1, 1) [M1A1]

2 It is given that $\mathrm{f}(x)$ is such that $\mathrm{f}'(x) = 3\sin x\cos x$ and $\left(\frac{\pi}{2}, 1\right)$ is a point on $\mathrm{f}(x)$. Find an expression for $\mathrm{f}(x)$. [5]

$\mathrm{f}'(x) = 3\sin x\cos x$

$f(x) = \int (3\sin x\cos x)dx$ [M1 – attempt to integrate f'(x)]

$= \int \frac{3}{2}(2\sin x\cos x)dx$

$= \int \frac{3}{2}\sin 2x \; dx$ [B1]

$\mathrm{f}(x) = \frac{3}{2}(-\frac{\cos 2x}{2}) + c$ [M1]

At $\left(\frac{\pi}{2}, 1\right)$,

$1 = -\frac{3}{4}(-\cos\pi) + c$ [M1 – substitution & attempt to find c]

$\mathrm{c} = \frac{1}{4}$

$f(x) = -\frac{3}{4}\cos 2x + \frac{1}{4}$ [A1]

3 **(a)** Factorise $3x^3 - 24y^3$ completely. [2]

$$3x^3 - 24y^3 = 3(x^3 - 8y^3)$$
$$= 3[(x)^3 - (2y)^3] \quad \text{[M1]}$$
$$= 3(x-2y)(x^2 + 2xy + 4y^2) \quad \text{[B1]}$$

(b) Express $\dfrac{7x^2+19x+15}{(x+1)^2(x+2)}$ as partial fractions. [5]

Let $\dfrac{7x^2+19x+15}{(x+1)^2(x+2)} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{C}{x+2}$ [M1]

$$7x^2+19x+15 = A(x+1)(x+2) + B(x+2) + C(x+1)^2$$

Let $x=-1$

$7-19+15 = B$

$B=3$

$Let\ x=-2,$

$28-38+15 = C$

$C=5$

$let\ x=0,$

$15 = A(1)(2) + 3(2) + 5(1)^2$

$2A = 4$

$A = 2$

M1 [Subst of essential values of x/ Expansion & comparing coef]

A2 [Given if all 3 correct]

$$\frac{7x^2+19x+15}{(x+1)^2(x+2)} = \frac{2}{x+1} + \frac{3}{(x+1)^2} + \frac{5}{x+2} \quad \text{[A1]}$$

4 **(i)** Show that $\frac{\mathrm{d}}{\mathrm{d}x}(2x\ln x) = 2\ln x + 2$. [2]

[M1] [M1]

$$\frac{\mathrm{d}}{\mathrm{d}x}(2x\ln x) = 2x\left(\frac{1}{x}\right) + (\ln x)(2)$$
$$= 2\ln x + 2$$

(ii) A curve is such that the gradient of its tangent is $\ln x$ and it passes through the point $(e^2, 4)$. Using part (i), find the equation of the curve, leaving your answer in exact form. [4]

$$\frac{dy}{dx} = \ln x$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(2x\ln x) = 2\ln x + 2$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(x\ln x) = \ln x + 1$$

$\int (1+\ln x)dx = x\ln x + c$, where c is an arbitrary constant.

[M1- correct reverse process step for integration/ no penalization for omitting c]

$\int \ln x \, dx = x\ln x - x + d$, where d is an arbitrary constant.

[M1 – correct integration of $\int 1 \, dx$+ "d" shown]

Equation of curve: $y = x\ln x - x + d$

At $(e^2, 4)$,

$4 = e^2 \ln e^2 - e^2 + d$ [M1- correct substitution]

$d = 4 - e^2$

$y = x\ln x - x + 4 - e^2$ [A1]

5 Given that $\sin A = -\frac{4}{5}$, $\tan B = -\frac{5}{12}$ and $\cos A > 0$, where A and B are in different quadrants, evaluate without using calculators, the values of

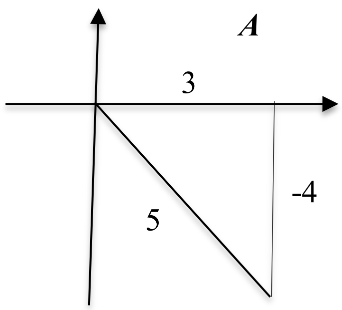
(i) $\cot A$, [1]

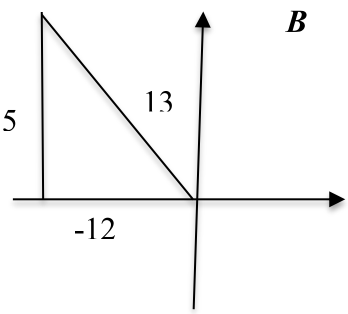
A is in the 4th quadrant and B is in the 2nd quadrant

$\tan A = -\frac{4}{3}$

$\cot A = -\frac{3}{4}$

[B1]





(ii) $\cos(A + B)$, [2]

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$= \left(\frac{3}{5}\right)\left(\frac{-12}{13}\right) - \left(-\frac{4}{5}\right)\left(\frac{5}{13}\right)$ [M1 – attempt with correct addition formulae]

$= -\frac{16}{65}$ [A1]

(iii) $\sin\left(\frac{B}{2}\right)$. [3]

$\cos B = 1 - 2\sin^2\left(\frac{B}{2}\right)$

$-\frac{12}{13} = 1 - 2\sin^2\left(\frac{B}{2}\right)$ [M1 – correct half angle use in double angle formulae]

$2\sin^2\left(\frac{B}{2}\right) = \frac{25}{13}$

$\sin^2\left(\frac{B}{2}\right) = \frac{25}{26}$ [A1]

$\sin\left(\frac{B}{2}\right) = \sqrt{\frac{25}{26}}$ (rej negative value since $0° < \frac{B}{2} < 90°$)

$= \frac{5}{\sqrt{26}}$

$= \frac{5\sqrt{26}}{26}$ [A1 – Accept non rationalised form]

6 **(a)** Find the range of values of x for which $(3x-2)(x+2)>3x-2$. [3]

$(3x-2)(x+2)>3x-2$
$(3x-2)(x+2)-(3x-2)>0$
$(3x-2)(x+2-1)>0$ or $3x^2+x-2>0$ [M1]
$(3x-2)(x+1)>0$
$x<-1$ $x>\dfrac{2}{3}$ [each correct A1 A1]

(b) The path of a diver, John, is modelled by the function $y=-3x^2+4.5x+10$, where y is the height, in metres, of John above the water and x is the horizontal distance, in metres, of John from the end of the diving board.

(i) Find the height of John above the water when he first left the diving board. [1]

When x= 0, y= 10
Height of John above the water = 10m [B1]

(ii) John said that he could reach a height of 12m above the water when executing his dive. Do you agree? Explain your answer. [3]

Method 1: $y=-3\left[x^2-1.5x+\left(\dfrac{3}{4}\right)^2\right]+10+\dfrac{27}{16}$ [M1]

$y=-3\left[x-\dfrac{3}{4}\right]^2+11.6875$ [A1]

I do not agree as his max height above water is 11.6875 < 12 m. [A1]

Method 2: $-3x^2+4.5x+10=12$
$-3x^2+4.5x-2=0$ [B1]
Discriminant = $(4.5)^2-4(-3)(-2)=-3.75<0$ [M1]
Since there are no **real** roots for x, I disagree with John as he will not reach 12. [A1]

Method 3 : $y=-3x^2+4.5x+10$
$\dfrac{dy}{dx}=-6x+4.5$
For max or min y, $\dfrac{dy}{dx}=0 \Rightarrow x=0.75$ [M1]
$\dfrac{d^2y}{dx^2}=-6<0 \Rightarrow \text{max } y$ [M1]
When $x=0.75$, max y = 11.6875 < 12, I disagree with John as he will not reach 12. [A1]

7 **(a)** Explain why all terms in the expansion of $\left(kx^3+x\right)^{12}$ do not contain any odd powers of x. [3]

General Term $= \begin{pmatrix}12\\r\end{pmatrix}\left(kx^3\right)^{12-r}x^r$ [M1]

$= \begin{pmatrix}12\\r\end{pmatrix}k^{12-r}x^{36-3r+r}$ [M1]

$= \begin{pmatrix}12\\r\end{pmatrix}k^{12-r}x^{2(18-r)}$

Since the index of x is a multiple of 2, the expansion will not contain any odd powers of x. [A1]

(b) Given that the coefficient of x^{16} in the expansion of $\left(1-\frac{x}{2}\right)^2\left(kx^3+x\right)^{12}$ is 258, find the integral value of k. [5]

$$\left(1-\frac{x}{2}\right)^2\left(kx^3+x\right)^{12} = \left(1-x+\frac{x^2}{4}\right)\left(...+..x^{14}....+...x^{16}+...\right)$$

[M1]

$2(18-r)=14$ $\quad$ $2(18-r)=16$

$r=11$ $\quad$ $r=10$

$T_{12}=\begin{pmatrix}12\\11\end{pmatrix}kx^{14}$ $\quad$ $T_{11}=\begin{pmatrix}12\\10\end{pmatrix}kx^{16}$

$=12kx^{14}$ $[M1]$ $\quad$ $=66kx^{16}$ $[M1]$

$(12k)\left(\frac{1}{4}\right)+66k^2=258$ **[M1]**

$66k^2+3k-258=0$

$22k^2+k=86=0$

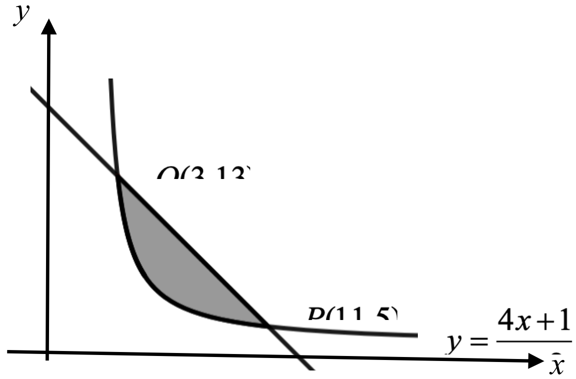
$\left(22k-43\right)\left(k+2\right)=0$

$k=\frac{43}{22}(rej)$ $\quad k=-2$

[A1]

8 The diagram shows part of the curve $y = \dfrac{4x+1}{x-2}$.

A line intersects the curve at points $P(11,5)$ and $Q(3,13)$.



By expressing $\dfrac{4x+1}{x-2}$ in the form $a+\dfrac{b}{x-2}$, where a and b are constants, find, showing full working, the area of the shaded region. [7]

$\dfrac{4x+1}{x-2} = 4 + \dfrac{9}{x-2}$ [A1]

$$\begin{array}{r|l} & 4 \\ x-2 & 4x+1 \\ & 4x-8 \\ \hline & 9 \end{array}$$

[M1]

Area $= \dfrac{1}{2}(13+5)(8) - \displaystyle\int_3^{11} \frac{4x+1}{x-2}\,dx$ [M1]

$= 4(18) - \displaystyle\int_3^{11} \left[4 + \frac{9}{x-2}\right] dx$

$= 72 - \underbrace{\left\{[4x]_3^{11} + [9\ln(x-2)]_3^{11}\right\}}_{[\text{A1}]}$

[A1]

$= 72 - [(44-12) - (9\ln 9 - 9\ln 1)]$ [M1]

$= 20.2$ units^2 or 40-9ln9 [A1]

9 The equation of a polynomial is given by $\text{p}(x) = 4x^3 + x - 5$.

(i) Find the remainder when $\text{p}(x)$ is divided by $(2x-1)$. [1]

$\text{p}(\frac{1}{2}) = 4\left[\frac{1}{2}\right]^3 + \left(\frac{1}{2}\right) - 5 = -4$ Remainder = -4 [B1]

(ii) Show that the equation $\text{p}(x) = 0$ has only one real root. [4]

By trial and error,

$\text{p}(1) = 4(1)^3 + 1 - 5 = 0$

(x-1) is a factor. [B1]

$p(x) = (x-1)(4x^2 + 4x + 5)$ [M1] [show long division M1]

$0 = (x-1)(4x^2 + 4x + 5)$

$0 = (x-1)$ $\quad 0 = (4x^2 + 4x + 5)$

$x = 1$ $\quad$ Discriminant = 16 – 4 (4)(5)

= -64 < 0

Therefore no real roots.

Hence, p(x) has only 1 real root, x =1.

(iii) Hence, solve the equation $2^{3y+2} + 2^y - 5 = 0$. [2]

$2^{3y+2} + 2^y - 5 = 0$

$2^{3y}2^2 + 2^y - 5 = 0$

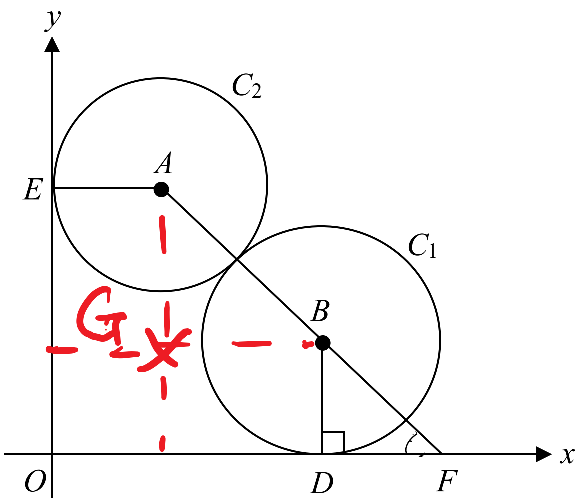
$4(2^y)^3 + 2^y - 5 = 0$ [M1]

$(2^y - 1)\left[4(2^y)^2 + 4(2^y) + 5\right] = 0$

$2^y = 1$ $\quad \left[4(2^y)^2 + 4(2^y) + 5\right] = 0$ (rej)

$y = 0$ [A1]

10 The figure shows two circles C_1 and C_2 which touch each other and lie in the xy-plane as shown below. C_1 has radius 4 units and touches the x-axis at D, C_2 has radius 3 units and touches the y-axis at E. The line AB, joining the centres of C_2 and C_1, meets the x-axis at F such that $\angle BFO = \theta°$.



(i) Obtain expressions for OD and OE in terms of θ and show that $ED^2 = 74 + 56\sin\theta + 42\cos\theta$. [3]

$ED^2 = EO^2 + OD^2$

$\left.\begin{array}{l} EO = AG + 4 = 7\sin\theta + 4 \\ OD = 3 + GB = 3 + 7\cos\theta \end{array}\right\}$ **[B1]**

$ED^2 = [7\sin\theta + 4]^2 + [3 + 7\cos\theta]^2$ **[M1]**

$= 49\sin^2\theta + 56\sin\theta + 16 + 9 + 42\cos\theta + 49\cos^2\theta$

$= 49 + 56\sin\theta + 25 + 42\cos\theta$

$= 74 + 56\sin\theta + 42\cos\theta$ **[A1]**

(ii) Express ED^2 in the form $74+R\cos(\theta-\alpha)$ where $R>0$ and $0°<\alpha<90°$. [3]

$42\cos\theta+56\sin\theta=R\cos\theta\cos\alpha+R\sin\theta\sin\alpha$

$R=\sqrt{56^2+42^2}=70$ [B1]

$\tan\alpha=\dfrac{56}{42}$

$\alpha=53.1°$ [M1]

$ED^2=74+70\cos(\theta-53.1°)$ [A1]

(iii) Find the maximum value of ED and the value of θ at which this occurs. [2]

maximum ED when $\cos(\theta-53.1°)=1$

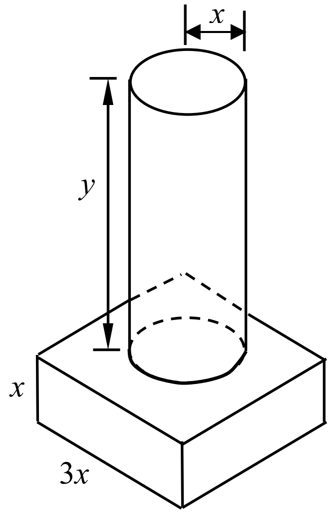
$\theta-53.1°=0°$

$\theta=53.1°$ [B1]

Max ED^2 = 74 +70 =144

Max ED = 12 [B1]

11 The diagram shows a cat scratch stand which consists of a solid cylinder fixed to a solid cuboid. The cylinder has a radius of x cm and a height of y cm. The cuboid has a square base of side $3x$ cm and a height of x cm.



(i) Given that the total volume of the wood material needed to make the scratch stand is 1300 cm^3, express y in terms of x. [2]

$$1300 = (3x)^2(x) + \pi x^2 y \quad \text{[M1]}$$

$$1300 = 9x^3 + \pi x^2 y$$

$$y = \frac{1300 - 9x^3}{\pi x^2} \quad \text{[A1]}$$

(ii) Show that the total surface area, $A \text{ cm}^2$, of the scratch stand is given by $A = \frac{2600}{x} + 12x^2$. [2]

$$A = x(3x)(4) + 2(3x)^2 + 2\pi(x)(y) \quad \text{[M1]}$$

$$A = 12x^2 + 18x^2 + 2\pi(x)\left(\frac{1300 - 9x^3}{\pi x^2}\right)$$

$$A = 30x^2 + \frac{2600}{x} - 18x^2$$

$$A = \frac{2600}{x} + 12x^2 \quad \text{[A]}$$

Given that x can vary,

(iii) find the stationary value of A, [3]

$A = 2600x^{-1} + 12x^2$

$\frac{dA}{dx} = -2600x^{-2} + 24x$ [M1]

$0 = -2600x^{-2} + 24x$

$2600 = 24x^3$

$x^3 = \frac{2600}{24}$

$x = 4.767098$ [M1]

Stationary Value of A = 818 cm^2 (3sf) [A1]

(iv) determine whether this stationary value of A is a maximum or a minimum. [2]

$\frac{d^2A}{dx^2} = \frac{5200}{x^3} + 24$ [M1]

At $x = 4.767098$,

$\frac{d^2A}{dx^2} = 72 > 0$

A = 818 cm^2 is a min value. [A1]

12 A curve has the equation $y = \dfrac{2\sin 3x}{2\cos 3x + 5}$.

(i) Find the value of a and of b for which $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{a + b\cos 3x}{(2\cos 3x + 5)^2}$. [3]

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(2\cos 3x + 5)(6\cos 3x) - (2\sin 3x)(-6\sin 3x)}{(2\cos 3x + 5)^2} \quad \text{[M1]}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{12\cos^2 3x + 30\cos 3x) + 12\sin^2 3x}{(2\cos 3x + 5)^2}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{12 + 30\cos 3x}{(2\cos 3x + 5)^2} \quad \text{[M1]}$$

a= 12 b = 30 [Both correctA1]

(ii) Hence, find the x-coordinates, where $0 \le x \le \pi$, of the points at which the normal to the curve is parallel to the y-axis. [5]

Gradient of tangent = 0

$\frac{dy}{dx} = 0$

$\frac{12+30\cos 3x}{(2\cos 3x+5)^2} = 0$ [M1]

$30\cos 3x = -12$

$\cos 3x = -\frac{2}{5}$

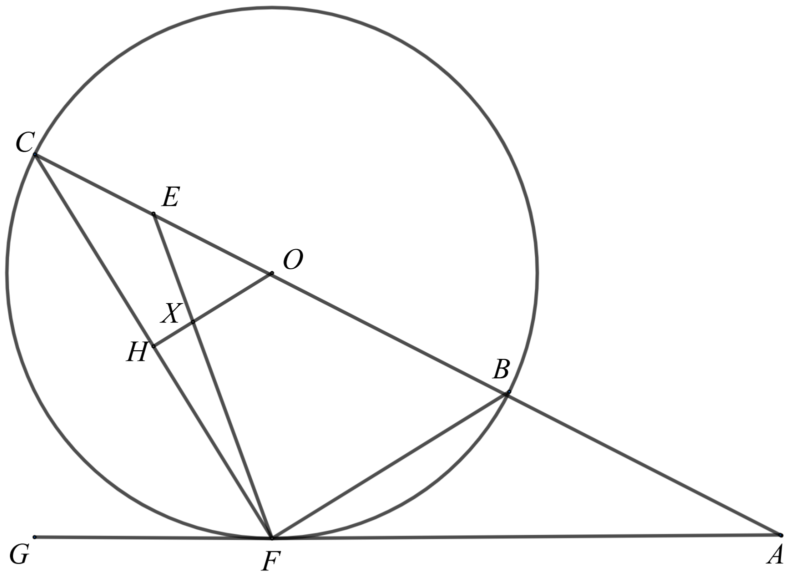
$\alpha = 1.15927$ [M1]

$3x = \pi-\alpha, \pi+\alpha, 3\pi-\alpha$ [M1]

$= 1.98231, \quad 4.30087, \quad 8.2655$

$x = 0.661, \quad 1.43, \quad 2.76$ [A2]

13 In the figure, BC is a diameter of the circle with O as the centre. H is the midpoint of CF. ABC is a straight line and AG *is* a tangent to the circle at point F. The line EF intersects OH at point X and E is the midpoint of CO.



(i) Prove that triangles ABF and AFC are similar. [2]

(ii) Show that $AF^2 = AB^2 + AB \times BC$. [2]

(iii) Prove that $OX : XH = 2:1$. [4]

End of Paper.

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