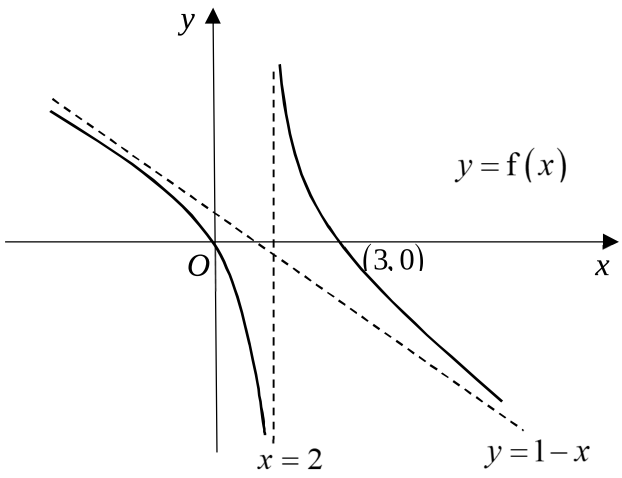
2021 HCI Prelim Paper 1

1 The diagram shows the graph of $y = \mathrm{f}(x)$. The lines $x = 2$ and $y = 1 - x$ are asymptotes to the curve, and the graph passes through the points $(0,0)$ and $(3,0)$.



Sketch the following graphs, indicating clearly the coordinates of any axial intercepts (where applicable) and the equations of any asymptotes.

(i) $y = \dfrac{1}{\mathrm{f}(x)}$, [3]

(ii) $y = \mathrm{f}'(x)$ [3]

2 **(i)** The complex number z has modulus 2 and argument θ and the complex number w has modulus r and argument $-\dfrac{\pi}{2}$, where r is a positive constant and $-\dfrac{\pi}{2} < \theta < 0$. Given that $\dfrac{z}{w} = \dfrac{1}{32}\left(\sin\dfrac{\pi}{6} + \mathrm{i}\cos\dfrac{\pi}{6}\right)$, find the values of r and θ.

[3]

(ii) Hence find the 3 smallest positive integers n such that $\left(z^n\right)^*$ is purely imaginary. [3]

3 **(i)** Without using a calculator, solve the inequality $\dfrac{2x^2 + 3x}{2x^2 + x - 1} \le \dfrac{1}{2x + 2}$. [4]

(ii) Using your answer to part **(i)**, deduce the values of x for the inequality $\dfrac{2\cos^2 x + 3\cos x}{2\cos^2 x + \cos x - 1} \le \dfrac{1}{2\cos x + 2}$, where $-\pi \le x \le \pi$, leaving your answer in exact form. [3]

4 **(a)** Find $\displaystyle\int \frac{6\tan 3x}{1 + \cos 6x}\,\mathrm{d}x$. [3]

(b) It is given that $I_n = \displaystyle\int_1^{\mathrm{e}} (\ln x)^n \,\mathrm{d}x$, for $n \in \mathbb{Z}^+$.

(i) Use integration by parts to show that

$$I_n = \mathrm{e} - nI_{n-1}.$$ [2]

(ii) The region bounded by the curve $y=(\ln x)^2$, the x-axis and the line $x=\mathrm{e}$ is rotated through 2π radians about the x-axis. Using the result in **(b)(i)**, find the exact volume of the solid formed. [3]

Do not use a calculator for this question.

5 **(a)** The three complex numbers z_1, z_2 and z_3 are given as $z_1=2\mathrm{i}$, $z_2=2\mathrm{e}^{\frac{\pi}{6}\mathrm{i}}$ and $z_3=\frac{1}{7}\left(\cos\frac{2}{3}\pi+\mathrm{i}\sin\frac{2}{3}\pi\right)$. Find $\frac{1}{z_3}(z_2-z_1)$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\le\pi$. [3]

(b) The complex number z satisfies the equation $2z^3+az^2+18z-9=0$, where a is a real number. It is given that one root is of the form $z=k\mathrm{i}$, where k is real and negative. Find the values of a and k, and the other roots of the equation. [6]

6 A curve C is defined by the parametric equations

$$x=\frac{p}{t^2}-t,\quad y=pt+\tfrac{1}{t};$$

where t is a parameter, $t>0$, and p is a positive constant.

(i) Show that $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{t\left(pt^2-1\right)}{-2p-t^3}$. [3]

The normal to C at $t=1$ passes through the point $A(3,-1)$.

(ii) Find the exact value of p. [3]

(iii) Given that the normal to C at $t=1$ meets the y axis at point B, find the length of AB. [3]

7 It is given that $\mathbf{a}$ and $\mathbf{b}$ are non zero vectors.

(a) Given that $|\mathbf{a}+\mathbf{b}|=|\mathbf{a}-\mathbf{b}|$, by considering suitable scalar product, comment on the relationship between $\mathbf{a}$ and $\mathbf{b}$. [2]

(b) Given that $\frac{\mathbf{a}}{|\mathbf{a}|}=\frac{\mathbf{b}}{|\mathbf{b}|}$, what can you say about the relationship between $\mathbf{a}$ and $\mathbf{b}$? [1]

(c) Given that $2\mathbf{a}+\mathbf{b}=-5\mathbf{j}+\mathbf{k}$, $\mathbf{c}=\mathbf{i}-2\mathbf{j}+2\mathbf{k}$, $\mathbf{a.c}=5$ and $\mathbf{b}$ is a unit vector,

(i) find the value of $\mathbf{b.c}$. [2]

(ii) find the sine of the angle between $\mathbf{b}$ and $\mathbf{c}$. [2]

(iii) find $|\mathbf{b}\times\mathbf{c}|$, and state the geometrical meaning of this result. [2]

8 The function f is defined by $\mathrm{f}: x \mapsto \dfrac{2x+6}{x-2},\ x \in \mathbb{R}, x > 2$.

(i) Find f^{-1} in a similar form. [3]

(ii) Find the solution set for $\mathrm{f}^2(x) = x$. [1]

It is given that k is a constant such that $k > 3$.

The function g is defined by $\mathrm{g}: x \mapsto (x-5)^2 + k, x \in \mathbb{R}, x > 2$.

(iii) Show that fg exists. [1]

(iv) Find the range of values of fg, in terms of k. [2]

The function ϕ is defined by $\phi: x \mapsto \dfrac{2x+a}{x-2}$, where a is a constant, $x \in \mathbb{R}, x > 2$.

(v) Given that ϕ^{-1} exists, state the value that a cannot take, justifying your answer. [2]

A function h is said to be self-inverse if $\mathrm{h}(x) = \mathrm{h}^{-1}(x)$ for all x in the domain of h.

(vi) State the range of values of a such that ϕ is a self-inverse function. [1]

9 Given that $y = (\tan x + \sec x)^2$, show that $\cos x \dfrac{\mathrm{d}y}{\mathrm{d}x} = 2y$. [2]

(i) By repeat differentiation, find the Maclaurin series for y up to and including the term in x^3. [4]

(ii) Using the result obtained in (i) estimate the value of $\tan 1^{\circ} + \sec 1^{\circ}$ to 4 decimal places. [2]

(iii) By expressing y in terms of sine and cosine, use the standard series in MF26 to find the series expansion for y up to and including the term in x^3. [3]

(iv) Comment on the results obtained from part **(i)** and part **(iii)**. [1]

10 A flu virus is spreading in a community which has a fixed population of N people. Scientists discover that at time t weeks from the beginning, the rate of change of the number of people who are infected is proportional to the product of the number of people infected, I, and the number of people who are not infected.

(i) Write down a differential equation relating I and t. [1]

It is noted that at the beginning, 1% of the people in the community is infected with the virus. 12 weeks later, the proportion of people infected in the community has risen to 25%.

(ii) By solving the differential equation obtained in part **(i)**, show that

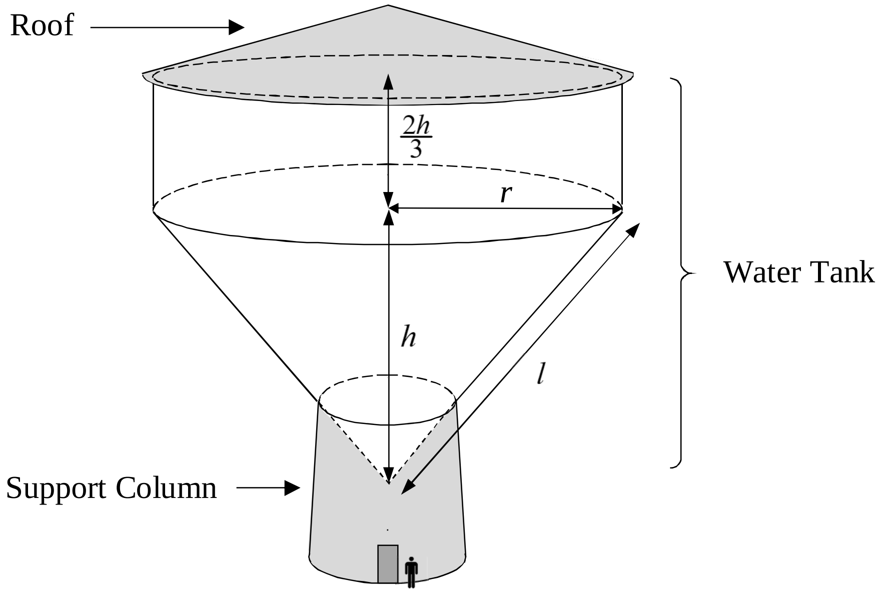
$$I = \frac{N}{99(33^{-\frac{t}{12}}) + 1}.$$ [6]

(iii) Find the least number of weeks needed for the proportion of people infected to exceed 50%, giving your answer to the nearest integer. [2]

(iv) Sketch the graph of I against t. Explain, with justification, what happens to the number of people infected if this situation continues indefinitely. [2]

(v) State a possible limitation of the model in part **(i)**. [1]

11 [It is given that a cone of radius r, slant height l and vertical height h has curved surface area πrl and volume $\frac{1}{3}\pi r^2 h$.]



A water tank is being constructed as a service reservoir supplying fully treated potable water to electronic chip manufacturing industries nearby. The water tank, held by a support column and covered by a roof, has a composite body made up of an inverted cone with radius r m, slant height l m and vertical height h m, and a cylinder of radius r m and height $\frac{2h}{3}$ m, as shown in the diagram.

It is assumed that the water tank is constructed with material of negligible thickness. It is given that the volume of the water tank is a fixed value v m^3 and the external surface area of the water tank, formed by the slanted surface of the inverted cone and the curved surface of the cylinder, is A m^2.

(i) Show that $A = \dfrac{4v}{3r} + \dfrac{1}{r}\sqrt{\pi^2 r^6 + v^2}$. [3]

In order to keep the cost of construction down, A must be a minimum.

(ii) Use differentiation to show that when A is a minimum, $r^6 = k\left(\dfrac{v}{\pi}\right)^2$, where k is an exact constant to be determined. (You need not show that your answer gives a minimum.) [6]

(iii) Hence find the value of $\dfrac{h}{r}$ when A is a minimum. Comment on the significance of this value with reference to the shape of the water tank. [3]