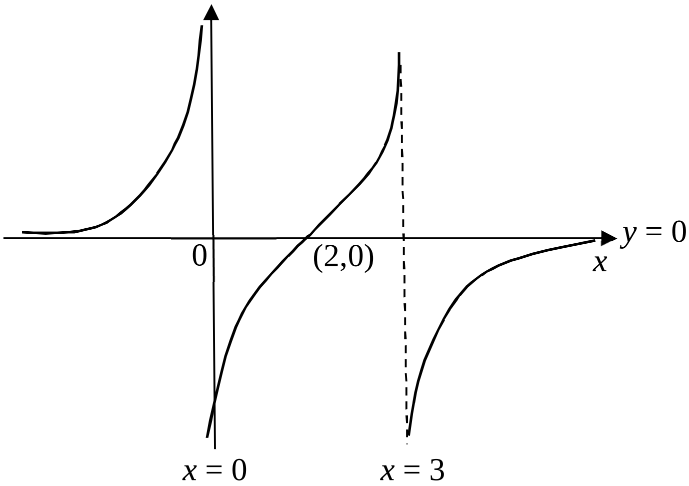
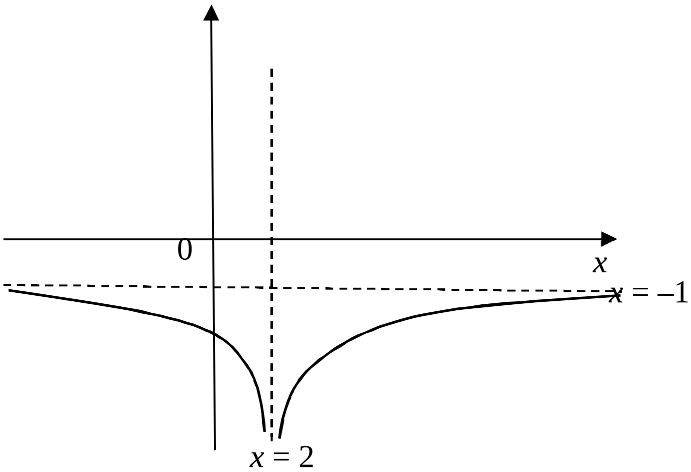
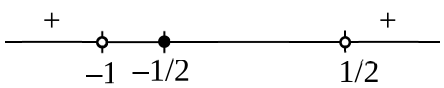
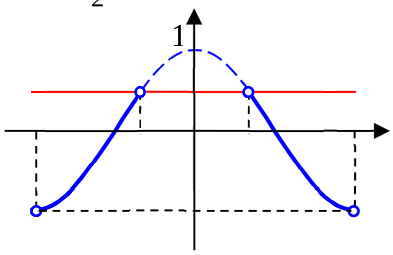
2021 HCI Prelim Paper 1 Suggested Solutions

1(i)		
1(ii)		
2(i)	$z = 2\mathrm{e}^{\mathrm{i}\theta}$, $w = r\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{2}\right)}$ $\frac{z}{w} = \frac{1}{32}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}\right) = \frac{1}{32}\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{3}\right)}$ $\left\lvert\frac{z}{w}\right\rvert = \frac{2}{r} = \frac{1}{32}$ $r = 64$ $\arg\left(\frac{z}{w}\right) = \arg z - \arg w = \theta + \frac{\pi}{2} = \frac{\pi}{3}$ $\theta = \frac{\pi}{3} - \frac{\pi}{2} = -\frac{\pi}{6}$	

2(ii)	$z=2\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{6}\right)}$ $\left(z^n\right)^*=\left(2^n\mathrm{e}^{\mathrm{i}\left(-\frac{\pi n}{6}\right)}\right)^*$ $=2^n\mathrm{e}^{\mathrm{i}\left(\frac{\pi n}{6}\right)}$ $\frac{n\pi}{6}=k\pi+\frac{\pi}{2}$ $n=6k+3$ $n=3,9,15$	
3(i)	$\frac{2x^2+3x}{2x^2+x-1}\le\frac{1}{2x+2}$ $\frac{2x^2+3x}{(2x-1)(x+1)}-\frac{1}{2(x+1)}\le 0$ $\frac{4x^2+6x-(2x-1)}{2(2x-1)(x+1)}\le 0$ $\frac{4x^2+4x+1}{2(2x-1)(x+1)}\le 0$ $\frac{(2x+1)^2}{2(2x-1)(x+1)}\le 0$  $\therefore\ -1<x<\frac{1}{2}$	

3(ii)	Replace x with $\cos x$, $\therefore\ -1<\cos x<\frac{1}{2}$  When $\cos x=\frac{1}{2}$, $\therefore\ x=\pm\frac{\pi}{3}$ Hence required values of x are $-\pi<x<-\frac{\pi}{3}$ or $\frac{\pi}{3}<x<\pi$	
4(a)	$\int\frac{6\tan 3x}{1+\cos 6x}\,\mathrm{d}x=\int\frac{6\tan 3x}{1+(2\cos^2 3x-1)}\,\mathrm{d}x$ $=\int\frac{6\tan 3x}{2\cos^2 3x}\,\mathrm{d}x$ $=\int 3\sec^2 3x\tan 3x\ \mathrm{d}x$ $=\frac{\tan^2 3x}{2}+C$ OR $\int\frac{3\sin 3x}{(\cos 3x)^3}\,\mathrm{d}x$ $=-\frac{(\cos 3x)^{-2}}{-2}+C$ $=\frac{1}{2}\sec^2 3x+C$	
4(b) (i)	$I_n=\int_1^{\mathrm{e}}(\ln x)^n\,\mathrm{d}x$ $=\left[x(\ln x)^n\right]_1^{\mathrm{e}}-\int_1^{\mathrm{e}}xn(\ln x)^{n-1}\left(\frac{1}{x}\right)\mathrm{d}x$ $=[\mathrm{e}(\ln\mathrm{e})^n-0]-n\int_1^{\mathrm{e}}(\ln x)^{n-1}\,\mathrm{d}x$ $=\mathrm{e}-nI_{n-1}$ (Shown)	
4(ii)	Required volume $=\pi\int_1^{\mathrm{e}}(\ln x)^4\,\mathrm{d}x=\pi I_4$ <u>Method 1</u>: Using the result in (b)(i),	

	$I_4 = \mathrm{e} - 4I_3$ $I_3 = \mathrm{e} - 3I_2$ $I_2 = \mathrm{e} - 2I_1$ where $I_1 = \int_1^{\mathrm{e}} \ln x \, \mathrm{d}x$ $= [x \ln x]_1^{\mathrm{e}} - \int_1^{\mathrm{e}} x\left(\frac{1}{x}\right) \mathrm{d}x$ $= \mathrm{e} - \int_1^{\mathrm{e}} 1 \, \mathrm{d}x$ $= \mathrm{e} - [x]_1^{\mathrm{e}}$ $= \mathrm{e} - (\mathrm{e} - 1)$ $= 1$ Using the result in (b)(i), $\therefore I_4 = \mathrm{e} - 4I_3$ $= \mathrm{e} - 4(\mathrm{e} - 3I_2)$ $= \mathrm{e} - 4\mathrm{e} + 12(\mathrm{e} - 2I_1)$ $= 9\mathrm{e} - 24I_1$ $= 9\mathrm{e} - 24$ Thus, the required volume is $(9\mathrm{e} - 24)\pi$ units^3.	
5(a)	$z_2 - z_1 = 2\left(\cos\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{6}\right) - 2\mathrm{i}$ $= 2\left(\frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}\right) - 2\mathrm{i}$ $= \sqrt{3} + \mathrm{i} - 2\mathrm{i}$ $= \sqrt{3} - \mathrm{i}$ $\frac{z_2 - z_1}{z_3} = \frac{\sqrt{3} - \mathrm{i}}{\frac{1}{7}\left(\cos\frac{2}{3}\pi + \mathrm{i}\sin\frac{2}{3}\pi\right)}$ $= \frac{2\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{6}\right)}}{\frac{1}{7}\mathrm{e}^{\mathrm{i}\left(\frac{2\pi}{3}\right)}}$ $= 14\mathrm{e}^{\mathrm{i}\left(-\frac{2\pi}{3} - \frac{\pi}{6}\right)}$ $= 14\mathrm{e}^{\mathrm{i}\left(-\frac{5\pi}{6}\right)}$	

5(b)	$2z^3+az^2+18z-9=0$ Given that $z=k\mathrm{i}$ is a root $2(k\mathrm{i})^3+a(k\mathrm{i})^2+18(k\mathrm{i})-9=0$ $-2k^3\mathrm{i}-ak^2+18k\mathrm{i}-9=0$ $\left(-2k^3+18k\right)\mathrm{i}-9-ak^2=0$ ------------------- (1) From (1) $-2k^3+18k=0$ $k(-2k^2+18)=0$ $k=0, k=-3$ or $k=3$ Since k is real and negative, $k=-3$ $-9-ak^2=0$ $-9-9a=0$ $a=-1$ $2z^3-z^2+18z-9=0$ Since the coefficients are real, the complex numbers occur in conjugate pairs. One other root will be $z=3\mathrm{i}$. $2z^3-z^2+18z-9=(z-3\mathrm{i})(z+3\mathrm{i})(2z+b)$ By comparison $-9=(-3\mathrm{i})(3\mathrm{i})(b)$ $b=-1$ The third root will be $z=\frac{1}{2}$.	

	The other roots of the equation $2z^3 - z^2 + 18z - 9$ are $z = 3\mathrm{i}$, and $z = \frac{1}{2}$	
6(i)	$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{2p}{t^3} - 1$ $\frac{\mathrm{d}y}{\mathrm{d}t} = p - \frac{1}{t^2}$ $\therefore\ \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} = \frac{\frac{pt^2-1}{t^2}}{\frac{-2p-t^3}{t^3}} = \frac{t(pt^2-1)}{-2p-t^3}$	
6(ii)	Gradient of normal $= -\frac{1}{\frac{\mathrm{d}y}{\mathrm{d}x}} = \frac{2p+t^3}{t(pt^2-1)}$ $\therefore$ equation of normal at $t=1$ is $y-(p+1) = \frac{2p+1}{p-1}(x-(p-1))$ Since normal passes through $A(3,-1)$, $\therefore\quad -1-(p+1) = \frac{2p+1}{p-1}(3-(p-1))$ $(-2-p)(p-1) = (2p+1)(4-p)$ $-2p+2-p^2+p = 8p-2p^2+4-p$ $p^2-8p-2=0$ $\therefore\ p = \frac{8\pm\sqrt{64+8}}{2} = \frac{8\pm6\sqrt{2}}{2}$ $= 4+3\sqrt{2}$ (reject $4-3\sqrt{2}$ since $p>0$)	
6(iii)	When $t=1$ and $x=0$, equation of normal is $y-(p+1) = \frac{2p+1}{p-1}(0-(p-1))$ $y-p-1 = -2p-1$ $\therefore\ y = -p = -4-3\sqrt{2}$ Hence $B(0,-4-3\sqrt{2})$. $\therefore\ AB = \sqrt{(3-0)^2+(-1-(-4-3\sqrt{2}))^2}$ $= 7.839377789$ $= 7.84$ units (3 s.f.)	
7(a)	Note that for any vector $\underset{\sim}{v}$, $\underset{\sim}{v}\cdot\underset{\sim}{v} = \lvert\underset{\sim}{v}\rvert^2$. Hence we have $\lvert\underset{\sim}{a}+\underset{\sim}{b}\rvert^2 = \lvert\underset{\sim}{a}-\underset{\sim}{b}\rvert^2$ $\Rightarrow (\underset{\sim}{a}+\underset{\sim}{b})\cdot(\underset{\sim}{a}+\underset{\sim}{b}) = (\underset{\sim}{a}-\underset{\sim}{b})\cdot(\underset{\sim}{a}-\underset{\sim}{b})$ $\Rightarrow \underset{\sim}{a}\cdot\underset{\sim}{a}+2\underset{\sim}{a}\cdot\underset{\sim}{b}+\underset{\sim}{b}\cdot\underset{\sim}{b} = \underset{\sim}{a}\cdot\underset{\sim}{a}-2\underset{\sim}{a}\cdot\underset{\sim}{b}+\underset{\sim}{b}\cdot\underset{\sim}{b}$ $\Rightarrow 2\underset{\sim}{a}\cdot\underset{\sim}{b}+2\underset{\sim}{a}\cdot\underset{\sim}{b} = \underset{\sim}{a}\cdot\underset{\sim}{a}+\underset{\sim}{b}\cdot\underset{\sim}{b}-\underset{\sim}{a}\cdot\underset{\sim}{a}-\underset{\sim}{b}\cdot\underset{\sim}{b}$ $\Rightarrow 4\underset{\sim}{a}\cdot\underset{\sim}{b} = 0$ $\Rightarrow \underset{\sim}{a}\cdot\underset{\sim}{b} = 0$	

	Hence $\underset{\sim}{a}$ is perpendicular to $\underset{\sim}{b}$.	
7(b)	$\dfrac{\underset{\sim}{a}}{\lvert\underset{\sim}{a}\rvert} = \dfrac{\underset{\sim}{b}}{\lvert\underset{\sim}{b}\rvert}$ $\Rightarrow \hat{\underset{\sim}{a}} = \hat{\underset{\sim}{b}}$ (the unit vector of $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are the same) Hence $\underset{\sim}{a}$ and $\underset{\sim}{b}$ have the same direction (or they are parallel to each other).	
7(c)(i)	$(2\underset{\sim}{a}+\underset{\sim}{b})\cdot\underset{\sim}{c} = \begin{pmatrix}0\\-5\\1\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\2\end{pmatrix}$ $\Rightarrow 2\underset{\sim}{a}\cdot\underset{\sim}{c}+\underset{\sim}{b}\cdot\underset{\sim}{c}=12$ $\Rightarrow 2(5)+\underset{\sim}{b}\cdot\underset{\sim}{c}=12$ $\Rightarrow \underset{\sim}{b}\cdot\underset{\sim}{c}=2$	
7(c)(ii)	$\cos\theta = \dfrac{\underset{\sim}{b}\cdot\underset{\sim}{c}}{\lvert\underset{\sim}{b}\rvert\lvert\underset{\sim}{c}\rvert} = \dfrac{2}{3}$, $\quad \lvert\underset{\sim}{c}\rvert = \sqrt{1^2+2^2+2^2}=3$, $\quad \lvert\underset{\sim}{b}\rvert = 1$ $\Rightarrow \cos\theta = \dfrac{2}{3}$ (hence θ is acute) $\therefore \sin\theta = \sqrt{1-\cos^2\theta} = \dfrac{\sqrt{5}}{3}$	
7(c)(iii)	$\lvert\underset{\sim}{b}\times\underset{\sim}{c}\rvert = \lvert\underset{\sim}{b}\rvert\lvert\underset{\sim}{c}\rvert\sin\theta = \sqrt{5}$ Let $\overrightarrow{OB}=\underset{\sim}{b}$ and $\overrightarrow{OC}=\underset{\sim}{c}$. And note that $\underset{\sim}{b}$ is unit vector. Geometrical meaning 1: $\lvert\underset{\sim}{b}\times\underset{\sim}{c}\rvert$ means the parallelogram with adjacent sides OB and OC has area $\sqrt{5}$ units square. Or Geometrical meaning 2: The perpendicular distance of the point $C(1,-2,2)$ to the line passing through origin O and with direction vector $\underset{\sim}{b}$ is $\sqrt{5}$ units.	

8(i)

$$y = \frac{2x+6}{x-2}$$

$$xy - 2y = 2x + 6$$

$$x(y-2) = 2y + 6$$

$$x = \frac{2y+6}{y-2}$$

$$\mathrm{f}^{-1}(x) = \frac{2x+6}{x-2}, x > 2$$

$$\mathrm{f}^{-1}(x) \to \frac{2x+6}{x-2}, x > 2$$

8(ii)

Notice from part (ii) that $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$, i.e. self-inverse.

$\mathrm{f}^2(x) = x$

Apply inverse on both sides:

$\Rightarrow \mathrm{f}(x) = \mathrm{f}^{-1}(x)$

$\Rightarrow$ Solution is when $D_\mathrm{f} = R_\mathrm{f}$ since it is self-inverse

The solution is $\{x \in \mathbb{R} : x > 2\}$.

8(iii)

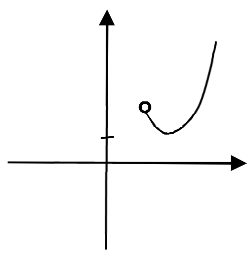
fg exists $\Leftrightarrow R_\mathrm{g} \subseteq D_\mathrm{f}$

$R_\mathrm{g} = [k, \infty)$

$D_\mathrm{f} = (2, \infty)$

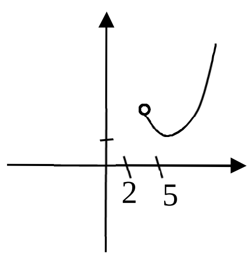
Since $R_\mathrm{g} \subseteq D_\mathrm{f}$, as $k > 3 > 2$,

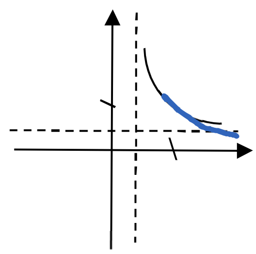
fg exists.



8(iv)

$$\mathrm{g}(x) = (x-5)^2 + k, x > 2$$





$$(2,\infty) \xrightarrow{g} [k,\infty) \xrightarrow{f} (2, \frac{2k+6}{k-2}]$$

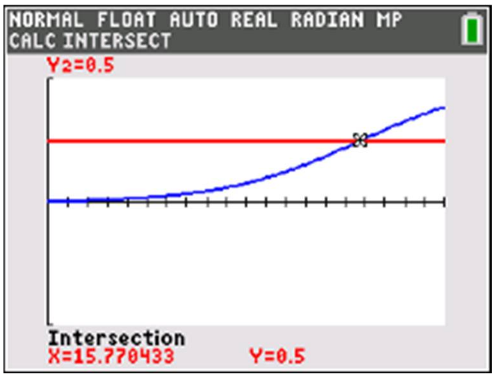
$$R_{fg} = (2, \frac{2k+6}{k-2}]$$

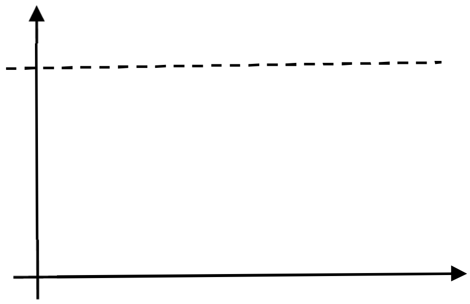
8(v)	$y=\dfrac{2x+a}{x-2}$ $2x+a\neq k\left(x-2\right)$ If $a=-4$, we have $\mathrm{f}\left(x\right)=2$, which is not a 1-1 function, thus ϕ^{-1} will not exist, a contradiction. Thus $a\neq-4$.	
8(vi)	$a>-4$	
9	$y=(\tan x+\sec x)^2$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=2(\tan x+\sec x)(\sec^2 x+\sec x\tan x)$ $\cos x\dfrac{\mathrm{d}y}{\mathrm{d}x}=2(\tan x+\sec x)(\sec x+\tan x)$ $\cos x\dfrac{\mathrm{d}y}{\mathrm{d}x}=2y$	
9(i)	Diff wrt x: $-\sin x\dfrac{\mathrm{d}y}{\mathrm{d}x}+\cos x\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=2\dfrac{\mathrm{d}y}{\mathrm{d}x}$ Diff wrt x: $-\cos x\dfrac{\mathrm{d}y}{\mathrm{d}x}-\sin x\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+\cos x\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}-\sin x\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=2\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ Sub $x=0$ $y=1,\ \dfrac{\mathrm{d}y}{\mathrm{d}x}=2,\ \dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=4,\ \dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}=10$ Let $y=\mathrm{f}(x)$ $y=\mathrm{f}(0)+\mathrm{f}\ '(0)x+\dfrac{\mathrm{f}\ ''(0)}{2}x^2+\dfrac{\mathrm{f}\ '''(0)}{3!}x^3+\ldots$ $=1+2x+2x^2+\dfrac{5}{3}x^3+\ldots$	
9(ii)	$1^{\mathrm{o}}=\dfrac{\pi}{180}$ Sub $x=\dfrac{\pi}{180}$ $\tan 1^{\mathrm{o}}+\sec 1^{\mathrm{o}}=\sqrt{1+2(\dfrac{\pi}{180})+2(\dfrac{\pi}{180})^2+\dfrac{5}{3}(\dfrac{\pi}{180})^3}$ $=\sqrt{1.03552468}$ $=1.0176$(4d.p.)	

9(iii)	$y=(\frac{\sin x+1}{\cos x})^2$ $=(1+\sin x)^2(\cos x)^{-2}$ $=(1+x-\frac{x^3}{6})^2(1-\frac{x^2}{2})^{-2}$ $=[1+2(x-\frac{x^3}{6})+(x-\frac{x^3}{6})^2](1+x^2+...)$ $=(1+2x+x^2-\frac{1}{3}x^3)(1+x^2+...)$ $=1+2x+2x^2+\frac{5}{3}x^3+...$	
9(iv)	Answers from both parts are consistent	
10(i)	$\frac{\mathrm{d}I}{\mathrm{d}t}=kI(N-I)$	
10(ii)	$\frac{\mathrm{d}t}{\mathrm{d}I}=\frac{1}{kI(N-I)}$ <u>Method 1</u>: Using partial fractions $\frac{\mathrm{d}t}{\mathrm{d}I}=\frac{1}{kI(N-I)}=\frac{1}{kN}\left(\frac{1}{I}+\frac{1}{N-I}\right)$ $t=\int\frac{1}{kN}\left(\frac{1}{I}+\frac{1}{N-I}\right)\mathrm{d}I$ $=\frac{1}{kN}\int\left(\frac{1}{I}+\frac{1}{N-I}\right)\mathrm{d}I$ $=\frac{1}{kN}\left[\ln I-\ln(N-I)\right]+C\ (\because 0<I<N)$ $=\frac{1}{kN}\ln\left(\frac{I}{N-I}\right)+C$ $kN(t-C)=\ln\left(\frac{I}{N-I}\right)$ $\mathrm{e}^{kNt-kNC}=\frac{I}{N-I}$ $A\mathrm{e}^{kNt}=\frac{I}{N-I}$ $A\mathrm{e}^{kNt}(N-I)=I$ $A\mathrm{e}^{kNt}N=I(1+A\mathrm{e}^{kNt})$ $I=\frac{A\mathrm{e}^{kNt}N}{1+A\mathrm{e}^{kNt}}=\frac{AN}{\mathrm{e}^{-kNt}+A}$	

	When $t=0,\ I=0.01N,\ 0.01N=\dfrac{AN}{1+A}$ $\Rightarrow 0.01(A+1)=A$ $\Rightarrow A=\dfrac{1}{99}$ When $t=12,\ I=0.25N,\ 0.25N=\dfrac{\frac{1}{99}N}{\mathrm{e}^{-12Nk}+\frac{1}{99}}$ $0.25\left(\mathrm{e}^{-12Nk}+\dfrac{1}{99}\right)=\dfrac{1}{99}$ $\Rightarrow \mathrm{e}^{-12Nk}=\dfrac{1}{33}$ $\Rightarrow Nk=\dfrac{\ln 33}{12}$ $\therefore I=\dfrac{\frac{1}{99}N}{\mathrm{e}^{\frac{-t\ln 33}{12}}+\frac{1}{99}}$ $=\dfrac{N}{99\mathrm{e}^{\frac{-t\ln 33}{12}}+1}$ $=\dfrac{N}{99\mathrm{e}^{\ln 33^{-\frac{t}{12}}}+1}$ $=\dfrac{N}{99\left(33^{-\frac{t}{12}}\right)+1}$ <u>Method 2</u>: Using MF26 $\dfrac{\mathrm{d}t}{\mathrm{d}I}=\dfrac{1}{-k\left(I^2-NI+\left(\frac{N}{2}\right)^2-\left(\frac{N}{2}\right)^2\right)}$ $=\dfrac{1}{k\left(\left(\frac{N}{2}\right)^2-\left(I-\frac{N}{2}\right)^2\right)}$ $t=\dfrac{1}{kN}\ln\left(\dfrac{\frac{N}{2}+I-\frac{N}{2}}{\frac{N}{2}-I+\frac{N}{2}}\right)+C$ $t=\dfrac{1}{kN}\ln\left(\dfrac{I}{N-I}\right)+C$	

	$kN(t-C)=\ln\left(\dfrac{I}{N-I}\right)$ $e^{kNt-kNC}=\dfrac{I}{N-I}$ $Ae^{kNt}=\dfrac{I}{N-I}$ $Ae^{kNt}(N-I)=I$ $Ae^{kNt}N=I(1+Ae^{kNt})$ $I=\dfrac{Ae^{kNt}N}{1+Ae^{kNt}}=\dfrac{AN}{e^{-kNt}+A}$ When $t=0,\ I=0.01N,\ 0.01N=\dfrac{AN}{1+A}$ $\Rightarrow 0.01(A+1)=A$ $\Rightarrow A=\dfrac{1}{99}$ When $t=12,\ I=0.25N,\ 0.25N=\dfrac{\frac{1}{99}N}{e^{-12Nk}+\frac{1}{99}}$ $0.25\left(e^{-12Nk}+\dfrac{1}{99}\right)=\dfrac{1}{99}$ $\Rightarrow e^{-12Nk}=\dfrac{1}{33}$ $\Rightarrow Nk=\dfrac{\ln 33}{12}$ $\therefore I=\dfrac{\frac{1}{99}N}{e^{\frac{-t\ln 33}{12}}+\frac{1}{99}}=\dfrac{N}{99e^{\frac{-t\ln 33}{12}}+1}=\dfrac{N}{99e^{\ln 33^{-\frac{t}{12}}}+1}$ $=\dfrac{N}{99\left(33^{-\frac{t}{12}}\right)+1}$	
10(iii)	<u>Method 1</u>:	

	When $I > 0.50N$ $\dfrac{N}{99\left(33^{-\frac{t}{12}}\right)+1} > 0.5N$ $99\left(33^{-\frac{t}{12}}\right)+1 < 2$ $99\left(33^{-\frac{t}{12}}\right) < 1$ $\left(33^{-\frac{t}{12}}\right) < \dfrac{1}{99}$ $\dfrac{-t\ln 33}{12} < -\ln 99$ $t > 15.8$ The least number of weeks is 16. Method 2: Use GC $I = \dfrac{N}{99\left(33^{-\frac{t}{12}}\right)+1} > 0.5N$ $\dfrac{1}{99\left(33^{-\frac{t}{12}}\right)+1} - 0.5 > 0$  From the graph, $t > 15.8$ The least number of weeks is 16.	

10(iv)	 As $t \to \infty$, $\mathrm{e}^{-\frac{t\ln 33}{12}} \to 0 \Rightarrow I \to N$. If this situation continues indefinitely, almost everyone in the community will be infected by the virus.	
10(v)	One possible limitation is that the model assumes that everyone will get the virus which in actual fact, some may have natural immunity against the flu virus. OR: The death may have occurred during some weeks and N is no longer constant.	
11(i)	Volume of tank $v = \frac{1}{3}\pi r^2 h + \pi r^2 (\frac{2h}{3}) = \pi r^2 h$ $\therefore\ h = \frac{v}{\pi r^2}$ Let l m be slant height of cone. $\therefore\ l = \sqrt{r^2 + h^2}$ Let A m^2 be external surface area of water tank. $\therefore\ A = 2\pi r(\frac{2h}{3}) + \pi r l$ $= \frac{4}{3}\pi r h + \pi r\sqrt{r^2 + h^2}$ $= \frac{4}{3}\pi r(\frac{v}{\pi r^2}) + \pi r\sqrt{r^2 + (\frac{v}{\pi r^2})^2}$ $= \frac{4v}{3r} + \frac{1}{r}\sqrt{\pi^2 r^6 + v^2}$ (shown)	
11(ii)	$\frac{\mathrm{d}A}{\mathrm{d}r} = -\frac{4v}{3r^2} + \frac{r(\frac{1}{2})(\pi^2 r^6+v^2)^{-\frac{1}{2}}(6\pi^2 r^5)-(\pi^2 r^6+v^2)^{\frac{1}{2}}}{r^2}$ $= \frac{(-4v)(\sqrt{\pi^2 r^6+v^2})+9\pi^2 r^6-3(\pi^2 r^6+v^2)}{3r^2\sqrt{\pi^2 r^6+v^2}}$ When $\frac{\mathrm{d}A}{\mathrm{d}r} = 0$,	

	$\therefore (-4v)(\sqrt{\pi^2 r^6 + v^2}) + 6\pi^2 r^6 - 3v^2 = 0$ $\sqrt{\pi^2 r^6 + v^2} = \frac{6\pi^2 r^6 - 3v^2}{4v}$ $\pi^2 r^6 + v^2 = \frac{36\pi^4 r^{12} - 36\pi^2 v^2 r^6 + 9v^4}{16v^2}$ $16\pi^2 v^2 r^6 + 16v^4 = 36\pi^4 r^{12} - 36\pi^2 v^2 r^6 + 9v^4$ $36\pi^4 r^{12} - 52\pi^2 v^2 r^6 - 7v^4 = 0$ Or $\frac{\mathrm{d}A}{\mathrm{d}r} = -\frac{4v}{3r^2} + \frac{1}{2}\frac{1}{\sqrt{\pi^2 r^4 + \frac{v^2}{r^2}}}\left(4\pi^2 r^3 - \frac{2v^2}{r^3}\right)$ $= -\frac{4v}{3r^2} + \frac{2\pi^2 r^6 - v^2}{r^2\sqrt{\pi^2 r^6 + v^2}}$ $\frac{\mathrm{d}A}{\mathrm{d}r} = 0$ $\frac{2\pi^2 r^6 - v^2}{r^2\sqrt{\pi^2 r^6 + v^2}} = \frac{4v}{3r^2}$ $6\pi^2 r^6 - 3v^2 = 4v\sqrt{\pi^2 r^6 + v^2}$ $36\pi^4 r^{12} - 52\pi^2 v^2 r^6 - 7v^4 = 0$ $\therefore\ r^6 = \frac{52\pi^2 v^2 \pm \sqrt{2704\pi^4 v^4 + 1008\pi^4 v^4}}{72\pi^4}$ $= \frac{52\pi^2 v^2 \pm \sqrt{3712}\pi^2 v^2}{72\pi^4}$ $= \frac{52\pi^2 v^2 \pm 8\sqrt{58}\pi^2 v^2}{72\pi^4}$ $= (\frac{13+2\sqrt{58}}{18})\frac{v^2}{\pi^2}$ [$(\frac{13-2\sqrt{58}}{18})\frac{v^2}{\pi^2}$ rejected since $r > 0$]	
11(iii)	Since $h = \frac{v}{\pi r^2}$, $\therefore\ \frac{h}{r} = \frac{v}{\pi r^3} = \frac{v}{\pi(\sqrt{\frac{13+2\sqrt{58}}{18}}\frac{v}{\pi})}$ $= \frac{1}{\sqrt{\frac{13+2\sqrt{58}}{18}}}$ $= 0.7984889679$ $= 0.798$ (3 s.f.) For minimum A, the height of the inverted cone must be shorter than its radius, giving a water tank that is very wide at the top compared to its height.	