2021 HCI Prelim Paper 2
Section A: Pure Mathematics [40 marks]

1 A curve C has parametric equations

$$x = a\left(\frac{4}{\pi}t + \sin 2t\right) + 2a,$$
$$y = a\cos t,$$

where $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$ and a is a positive constant.

(i) Sketch C. State clearly the coordinates of any points where C meets the x-axis. [2]

(ii) Show that the area enclosed by C and the x-axis is given by

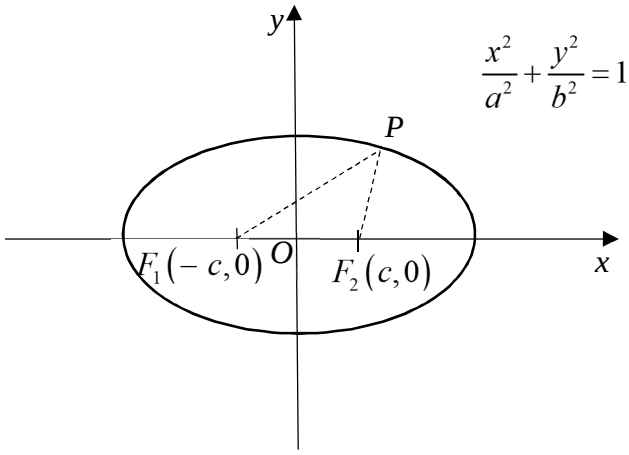
$$a^2\int_{\theta_1}^{\theta_2}\left(\frac{4}{\pi}\cos t + 2\cos t\cos 2t\right)\mathrm{d}t,$$

where θ_1 and θ_2 are values to be stated. [3]

(iii) Hence find, in terms of a, the exact area enclosed by C and the x-axis. [3]

2 Clear workings and explanations are required for this question.

(a) An ellipse of equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $0 < b < a$, has two points called foci $F_1(-c, 0)$ and $F_2(c, 0)$. The definition of the ellipse is such that for every point P on the ellipse, the sum of the distance of P to F_1 and F_2 is always a constant k.



(i) By considering one of the x-intercepts of the ellipse, determine the value of k in terms of a and/or b. [2]

(ii) By considering another suitable point on the ellipse, find c in terms of a and b. [2]

(b) A hyperbola with equation $(y-h)^2-1=\frac{1}{4}(x-k)^2$ has $y=\frac{1}{2}x+\frac{3}{2}$ as one of its asymptotes, and the point $(1,3)$ is on the hyperbola. Find the values of h and k. [4]

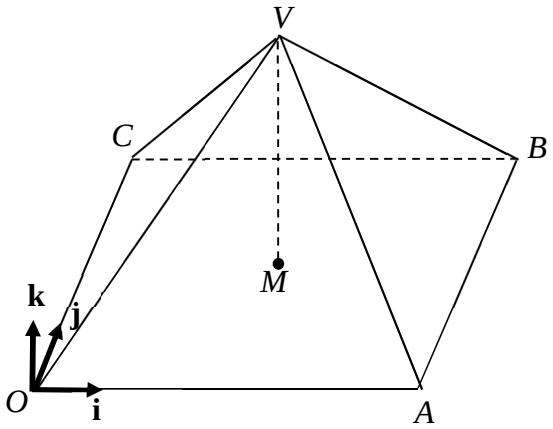
3 **(i)** Show that $\frac{4-x}{x(x-1)(x-2)}=\frac{2}{x}-\frac{3}{x-1}+\frac{1}{x-2}$. [1]

(ii) Hence find $\sum_{x=4}^{N}\frac{4-x}{x(x-1)(x-2)}$ in terms of N, giving your answer in the form $\mathrm{f}(N)-k$, where k is a constant to be determined. [3]

(iii) Show that, for all integers $N \geq 4$, $\sum_{x=4}^{N}\frac{4-x}{x(x-1)(x-2)}>-\frac{1}{6}$. [2]

(iv) Using your answer in part **(ii)**, find $\sum_{x=a}^{2a}\frac{3-x}{x(x+1)(x-1)}$, where a is an integer greater than 4, giving your answer in terms of a. (There is no need to express your answer as a single algebraic fraction.) [4]

4 Taking the point O as the origin, the diagram below shows a right pyramid with rectangular base $OABC$, and the base has its centre at M. The vertex of the pyramid is at V. The perpendicular unit vectors **i**, **j** and **k** are parallel to OA, OC and MV respectively. The length of OA, OC and MV are 8 units, 6 units and 10 units respectively.



(i) A line l with equations $\frac{y}{4}=\frac{z+1}{3}, x=1$, cuts the plane $OABC$ at the point P. Find the coordinates of P. [3]

(ii) Find a vector that is perpendicular to the plane OCV. [1]

(iii) Find the shortest distance from *P* to the plane *OCV*. Hence, or otherwise, find shortest the distance from *P* to the plane *ABV*. [4]

(iv) Write down the equation of the line where plane *OCV* meets plane *ABV*. [1]

(v) A point *Q* with coordinates (a,b,c) lies on the plane *OCV*, what can you say about the values of *a*, *b* and *c*? [2]

(vi) A point *R* with coordinates $(h,k,3)$ lies on the line segment *CV*, find the values of *h* and *k*. [3]

Section B: Probability and Statistics [60 marks]

5 Correlation and Regression [7]

6 Tom has a bag of wooden rectangular blocks of identical size. The bag contains 1 blue block, *m* red blocks and $(m-1)$ yellow blocks, where $m>2$. Tom and Jerry play a game using Tom's bag of wooden blocks. Jerry draws 2 blocks at random, one at a time, without replacement. 3 points will be awarded if a yellow block is drawn, 2 points will be awarded if a red block is drawn, but no points will be awarded if a blue block is drawn. Jerry's final score is the product of the points awarded for the 2 blocks drawn.

Let the random variable *X* denotes Jerry's final score.

(i) Show that $\mathrm{P}(X=0)=\frac{1}{m}$ and hence, find the probability distribution of *X*. [4]

(ii) Find the value of *m* if Jerry's expected final score is 5. [2]

Tom pays Jerry \$5 if Jerry's final score is at least 5 and Jerry pays Tom \$*a* if his final score is less than 5.

(iii) Using the value of *m* found in **(ii)**, find the range of values of *a* if Tom is expected to make a profit. [2]

7 A toy factory manufactures gel beads which are polymer beads that increase in size when soaked in water. On average, 8% of the gel beads are defective. The gel beads are packed in bags of 500. A significant number of customers recently gave feedback that many of the gel beads they bought could not expand in water or cracked while expanding. The quality control department decides to take a random sample of 20 gel beads from each bag to test. If more than 4 gel beads are found to be defective in the sample of 20, the bag is rejected. Otherwise the bag is accepted.

(i) State, in context, two assumptions needed for the number of defective gel beads in

the sample to be well modelled by a binomial distribution. [2]

Assume now that the number of defective gel beads in a sample of 20 is modelled by a binomial distribution.

(ii) Find the probability that a randomly chosen bag of gel beads is rejected. [1]

(iii) An officer from the quality control department is in charge of inspecting 10 randomly chosen bags of gel beads. Find the probability that the last bag inspected is the second bag that is being rejected. [2]

(iv) A random sample of 20 gel beads is taken from a particular bag. Given that the bag is rejected, find the probability that there are more than 13 gel beads with no defects in the random sample of 20 gel beads. [3]

(v) The quality control department now decides to test 50 randomly chosen bags of gel beads. Find the probability that the mean number of defective gel beads found in the sample of each bag will not exceed 1.5. [2]

8 Mr Wong works as the Information Technology manager at a company. To boost the security of the network used by the company, he bought a breach detection system (BDS) which is a defensive tool designed to detect the activity of malware inside a network. The BDS sends out an alert to indicate that a breach has occurred. If there is malicious activity, there is a 90% chance that the BDS will correctly identify the activity as malicious. If the activity is not malicious, there is a 1% chance that the BDS will incorrectly identify the activity as malicious.

It is known that the BDS sends out alerts identifying 109 activities as malicious out of 10000 activities for a particular network system at a particular instance.

(i) Find the probability of an activity being malicious. [2]

(ii) Find the probability that the BDS is correct in detecting the type of activity in the network. Give your answer correct to 4 significant figures. [2]

(iii) Given that the BDS sends out an alert, find the probability that the activity is not malicious. Give your answer correct to 4 significant figures. [2]

(iv) Comment on the effectiveness of the BDS in the identification of a security breach. [1]

(v) The BDS is able to provide daily reports on the activities on the network which can only be accessible to Mr Wong. During the initial set-up, Mr Wong is required to set a 4-letter code formed from the letters of the word 'NINETEEN' as his password for authentication. Find the number of possible 4-letter codes that Mr Wong can choose from to be his password. [4]

9 In this question, you should state clearly all the distributions that you use, together with the values of the appropriate parameters.

Mr Tan manages a stall on his own, selling original egglets. Due to limited budget, he can only afford one egglet machine. As such, he can only prepare egglets one at a time. The time needed, in minutes, to make an original egglet follows the distribution $\mathrm{N}(4, 0.25)$.

(i) Sketch the distribution for the time needed to make an original egglet to be between 2 and 6 minutes. [2]

(ii) On a particular morning, Mr Tan sold 15 original egglets. Find the probability that there were exactly 6 original egglets which he took less than 4 minutes to make each of them. [2]

(iii) Find the probability that the time needed to make 3 randomly chosen original egglets differs from thrice the time needed to make a randomly chosen original egglet by not more than 5 minutes. [3]

Hoping to attract more customers, Mr Tan decides to introduce chicken floss egglets and soft drinks at his stall. The time needed, in minutes, to make a chicken floss egglet follows the distribution $\mathrm{N}(6, 1.2)$. The time needed, in minutes, to prepare a cup of soft drink is modelled as 20 % of the time needed to make an original egglet.

(iv) A customer orders an original egglet, a chicken floss egglet and a cup of soft drink. Given that the probability that the total time taken by Mr Tan to prepare these 3 items, one after another, in less than m minutes is at most 90%, find the range of possible values of m. [4]

(v) State an assumption needed for your calculations in part **(iv)** to be valid. [1]

10 Disposable face masks undergo the [1]Bacterial Filtration Efficiency (BFE) Test to assess how well a mask filters droplets containing biological agents such as bacteria or viruses. It is known that the higher the BFE, the more effective a mask is in preventing bacteria-containing droplets from reaching the wearer. For instance, a mask with a BFE of 95% will meet the requirements for medical and surgical masks since it blocks 95% of droplets it is exposed to.

A company that manufactures disposable face masks claimed that their masks are rated with BFE of at least 95%. An intern from the company wishes to check if the claim made by the company is valid. He is told that the BFE of the disposable face masks manufactured is distributed normally and that the standard deviation is 0.99%. He decides to carry out a hypothesis test at 5% level of significance on a random sample of 10 disposable face masks.

(i) Explain what is meant by a random sample in this context. [1]

(ii) State the hypotheses for the test, defining any symbols that you used, and find the set of possible mean BFE of the 10 randomly chosen disposable masks corresponding to the critical region. [4]

(iii) Given that the null hypothesis is rejected in the test conducted by the intern at 5% level of significance, comment if the same conclusion is obtained when the test is conducted at 1% level of significance. [1]

The company modified the manufacturing process to increase the BFE of the masks produced to be more than 98%. A quality control manager decides to perform a hypothesis test on a random sample of 70 disposable face masks produced by the modified manufacturing process to find out if this is the case.

The distribution of the BFE, y %, of the random sample of 70 disposable face masks produced by the modified manufacturing process are given as follows.

BFE, y %	95.8	96.8	97.5	98.3	98.5	98.7	98.8	99.2	99.5	99.8
Number of disposable face masks	3	8	11	10	10	9	5	5	7	2

(iv) Explain why the quality control manager takes a sample of 70 new masks but the intern only takes a sample of 10 masks. [2]

(v) Carry out a hypothesis test at 5% level of significance for the quality control manager. Give your conclusion in context. [4]

[1] A-STAR Explainer: Testing the Efficacy of Protective Face Masks.*https://www.a-star.edu.sg/News-and-Events/a-star-news/news/covid-19/*