Name:		Centre/Index Number:		Class:	

DUNMAN HIGH SCHOOL
Preliminary Examination 2021
Year 6

MATHEMATICS

9758/02

Paper 2

22 September 2021

3 hours

Candidates answer on the Question Paper.

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your Centre number, index number, name and class on the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the Question Paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

For teachers' use:

Qn	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	Total
Score												
Max Score	5	7	8	10	10	10	7	8	11	12	12	100

Section A: Pure Mathematics [40 marks]

1 **Do not use a graphing calculator in answering this question. Show all workings clearly.**

The complex numbers w, z_1 and z_2 are given by $w = 3 - 4\text{i}$, $z_1 = \text{Im}\left(\frac{w}{w^*}\right)$ and $z_2 = w - w^*$.

(i) Find z_1 and z_2. [2]

(ii) Given that z_1 and z_2 are roots of the equation $bz^3 + cz^2 + dz + 1536 = 0$, find the values of the real numbers b, c and d. [3]

2 **(i)** By expressing $\frac{1}{2^{k+1}-3} - \frac{1}{2^k - 3}$ as a single fraction, find in terms of N, an expression for $\sum_{k=1}^{N} \frac{2^{k+1}}{\left(2^{k+1}-3\right)\left(2^k - 3\right)}$. [4]

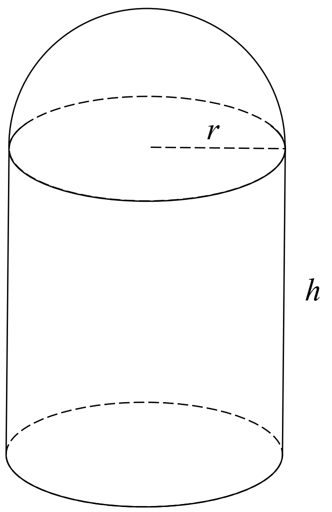
(ii) Use your result in **(i)**, or otherwise, to evaluate

$$\frac{16}{(61)(29)} + \frac{32}{(125)(61)} + \frac{64}{(253)(125)} + \ldots + \frac{2^{k-1}}{\left(2^{k+1}-3\right)\left(2^k-3\right)} + \ldots,$$

leaving your answer as a fraction in its lowest terms. [3]

3 [It is given that a sphere of radius r has surface area $4\pi r^2$ and volume $\frac{4}{3}\pi r^3$.]

The solid below is made up of a hemisphere, with radius r cm, placed on top of a cylinder with radius r cm and height h cm. The total surface area of the solid is kept at a constant k cm^2.



(i) Find r when the volume of the solid is at its maximum, leaving your answer in terms of k and π. [5]

(ii) Find this maximum volume in terms of k and π, simplifying your answer. Find also the ratio of $r : h$ when this occurs. [3]

4 **(a)** Find $\int \sin^{-1} 4x \, \mathrm{d}x$. [3]

(b) A curve C is defined by the following parametric equations, $x = a\left(t - \frac{1}{3}t^3\right)$, $y = at^2$, $t \geq 0$, for a positive constant a.

(i) Find the equation of the tangent to the curve at the point $P\ \left(a\left(p - \frac{1}{3}p^3\right),\ ap^2\right)$. [3]

The arc length between two points on C, where $t = t_1$ and $t = t_2$ is given by the formula

$$\int_{t_1}^{t_2} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t.$$

(ii) The tangent at P meets the x-axis at Q. Prove that the arc length OP is twice the length of OQ where O is the origin that C passes through. [4]

5 The points A, B and R have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{r}$ respectively.

(a) The point C has position vector $\frac{2}{7}\mathbf{a} - \frac{3}{7}\mathbf{b}$ and the point D is such that the origin O is the midpoint of the line segment CD. The point R lies on BD extended such that the ratio of BD to BR is $4:7$. Show that the points A, O and R are collinear and state the ratio of OA to OR. [4]

(b) It is given that the point R has position vector $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, and that $\mathbf{a} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$, and $\mathbf{b} = \begin{pmatrix} -1 \\ 5 \\ 3 \end{pmatrix}$.

(i) Determine the exact area of the triangle AOB. [2]

(ii) Give the geometrical interpretation of the point R, given that $\mathbf{r} \cdot (\mathbf{a} \times \mathbf{b}) = 0$. [2]

(iii) Find the shortest distance between the point $(-8, -2, 9)$ and the collection of all points R satisfying $\mathbf{r} \cdot (\mathbf{a} \times \mathbf{b}) = 0$. [2]

Section B: Probability and Statistics [60 marks]

6 There are ten cards of which 4 are black, 3 are blue, 2 are green and 1 is yellow. Cards of the same colour are indistinguishable.

(a) For parts **(a)(i)** to **(a)(iii)**, four cards are to be selected from these ten cards and the order of selection is not relevant.

(i) Find the number of possible selections that can be made if exactly 3 cards are of the same colour. [2]

(ii) Find the number of possible selections that can be made if at least 2 of the cards are of the same colour. [3]

(iii) Find the number of possible selections that can be made if either at least one green card is selected or yellow card is selected or both. [2]

(b) Find the number of arrangements of all ten cards in a row if the blue cards are not next to one another. [3]

7 Mike plays a game at a fun-fair. Each game costs \$3 to play. The probability distribution of the amount of money \$$X$, that Mike wins from a game is given by

$$\mathrm{P}\left(X=x\right)=\begin{cases} kx & \text{for } x=1,\ 2,\ 3, \\ \frac{1}{3}k(x-1) & \text{for } x=4,\ 5,\ 6, \\ 0 & \text{otherwise,} \end{cases}$$

where k is a constant.

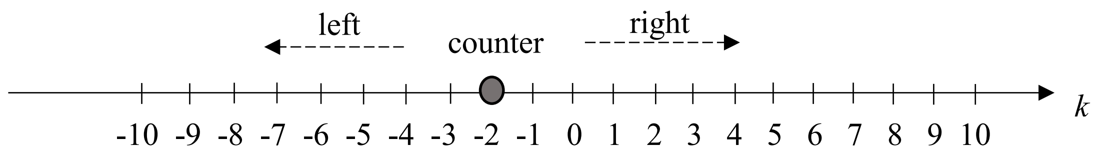
(i) Show that $k=\frac{1}{10}$. [1]

(ii) Find the expectation and variance of his net gain for one game. [3]

(iii) Assuming that each game is independent of one another, find the probability that his total net gain in 30 games is more than \$20. [3]

8 In a computer game, a counter moves along a straight line and is originally placed at $k = 0$. At each stage, it takes one step to the right with probability p or one step to the left with probability q, where $q = 1 - p$. Each step is of length 1 unit and the step taken at each stage is independent of one another.

For illustration purpose, the counter is seen to be at $k = -2$ in the diagram below.



The counter takes 10 consecutive steps.

(i) Explain why the counter cannot possibly end at $k = 7$. [1]

(ii) Show that the probability that the counter ends at $k = 6$ is $45p^8q^2$. [2]

(iii) Given that the most probable end-point of the counter is $k = 6$, find exactly the range of values of p in the form $p_1 < p < p_2$ where p_1 and p_2 are constants to be determined. [4]

(iv) Write down the most probable end-point(s) of the counter if $p = 1 - p_1$. [1]

9 A manufacturer produces sweets in 2 flavours, lime and mint. On average, a proportion p of the sweets are mint. The sweets are packed randomly in bags of 12 and the random variable X denotes the number of mint sweets in each bag.

(i) State, in context, two assumptions needed for the number of mint sweets in a bag to be well-modelled by a binomial distribution. [2]

Assume now that the number of mint sweets in a bag has a binomial distribution.

(ii) Explain why the variance of X does not exceed 3. [1]

(iii) Find the range of values of p such that in a bag, the probability that there are 5 mint sweets is less than 0.1. Give your answer to 2 decimal places. [2]

Take $p = 0.65$ for the rest of the question.

(iv) Find the probability that in a randomly selected bag, at least half of the sweets are mint. [2]

The bags are now randomly packed into boxes. Each box contains 5 bags.

(v) Find the probability that in each of the bags in a randomly selected box, at least half of the sweets are mint. [1]

(vi) Find the probability that there are at least 30 mint sweets in a randomly selected box. [2]

(vii) Explain why the answer to part **(vi)** is greater than that in **(v)**. [1]

10 The queue time for customers, in minutes, at a particular hair salon has a mean of 15 minutes and a standard deviation of 10 minutes.

(i) Explain why a normal model would not be suitable for the queue times. [1]

The time duration of a haircut for customers, in minutes, at 2 different hair salons, Qcut, an express service, and SPcut, a specialist service, are normally distributed with mean and standard deviation as given in the table.

	Mean	Standard Deviation
Qcut	9.2	σ
SPcut	20.7	3.1

(ii) If the probability that a Qcut customer having a haircut duration of more than 10 minutes does not exceed 0.35, find the range of values σ can take up. [2]

Use $\sigma = 1.5$ for the rest of the question.

(iii) Given that the difference between a haircut duration of a randomly chosen Qcut customer and its mean has a 95% chance of being larger than k, find the value of k. [2]

(iv) Find the probability that the mean haircut duration of two Qcut customers and three SPcut customers is less than 15 minutes. [3]

(v) State an assumption needed for your calculations in part **(iv)** to be valid. [1]

The salons fee-charging system comprises 2 components. Qcut charges a fixed component of \$3 and a variable component of \$0.50 per minute. Similarly, SPcut charges a fixed component of \$10 and a variable component of \$2 per minute.

(vi) Find the probability that the fees paid by a randomly chosen SPcut customer is at least 6 times the fees paid by a randomly chosen Qcut customer. [3]

11 In this question you should state clearly the variables and the values of the parameters of any distribution you use.

A pharmaceutical company has 300 employees from the research department and 330 employees from the manufacturing department.

(i) Billy wishes to find out about the stress level of employees in the research department, so he sends a survey to the 300 employees in the research department. Explain whether these 300 employees form a sample or a population. [1]

(ii) Charlie wishes to conduct a survey regarding the employees' experiences with a new facility. 100 of them are randomly selected from the research department and the other 100 are randomly selected from the manufacturing department. Explain whether the method will form a random sample. [2]

The pharmaceutical company produces pills against a particular bacteria. The manager claims that the average time taken to manufacture a batch of pills is less than 12 hours. The time taken in hours, x was measured on 80 occasions and the results are summarised as follows:

$$\sum(x-12)=-12.8, \quad \sum(x-12)^2=16.19.$$

(iii) Find the unbiased estimates of the population mean and variance of the time taken to manufacture a batch of pills. [2]

(iv) Carry out a hypothesis test on the manager's claim. Find the p-value for this test and explain what it indicates about the manager's claim. [4]

It was found that the testing process had not been rigorous enough and the same hypothesis test was conducted in another random sample of 80 observations. The sample mean was found to be k hours and the sample standard deviation was found to be 1.5 hours. It was found that the manager's claim was not supported at 1% level of significance.

(v) Find the range of values of k, correct to 3 decimal places. [3]