CHIJ St. Theresa's Convent
Preliminary Examination 2021
Secondary 4 Express
Additional Mathematics Paper 2
Marking Scheme

1(a) Show that the curve $y = -3x^2 + kx + 2$ intersects the line $y = x - 5$ for all real values of k. [3]

$$-3x^2 + kx + 2 = x - 5 \quad \text{[M1]}$$
$$3x^2 + (1-k)x - 7 = 0 \quad \text{--- (*)}$$
$$\text{Discriminant} = (1-k)^2 - 4(3)(-7) \quad \text{[M1]}$$
$$= (1-k)^2 + 84$$
$$\geq 84 > 0 \quad \text{for all real values of } k$$

[A1 – correct expression and get $D > 0$]

so the equation (*) has real roots $\Rightarrow$ curve intersects line for all real values of k.

(b) Determine the values of k for the line $y = x + 5$ to be a tangent to the curve $y = -3x^2 + kx + 2$. [3]

$$-3x^2 + kx + 2 = x + 5 \quad \text{[M1]}$$
$$3x^2 + (1-k)x + 3 = 0 \quad \text{--- (*)}$$
$$\text{Discriminant} = (1-k)^2 - 4(3)(3)$$
$$= (1-k)^2 - 36$$

For the line to be a tangent to the curve, the quadratic equation has real and equal roots so $D = 0$. [M1]

$$(1-k)^2 - 36 = 0 \Rightarrow (1-k)^2 = 36$$
$$\Rightarrow 1 - k = \pm 6$$
$$\Rightarrow k = -5 \text{ or } k = 7 \quad \text{[A1]}$$

(c) Solve the equation $x = \sqrt{1-x} - 5$, showing clearly if the answers obtained are correct. [4]

$$x = \sqrt{1-x} - 5 \Rightarrow x + 5 = \sqrt{1-x}$$
$$\Rightarrow (x+5)^2 = 1 - x \quad \text{[M1 – after shifting 5 to LHS]}$$
$$\Rightarrow x^2 + 10x + 25 = 1 - x$$
$$\Rightarrow x^2 + 11x + 24 = 0$$
$$\Rightarrow (x+3)(x+8) = 0 \quad \text{[M1 – factorise]}$$
$$\Rightarrow x = -3 \text{ or } x = -8 \quad \text{[A1 – both answers correct]}$$

Check: [A1 – reject $x = -8$ with working shown]

When $x = -3$, LHS $= -3$

RHS $= \sqrt{1-(-3)} - 5 = -3$

Since LHS = RHS, $x = -3$ is a solution.

2 When the polynomial $f(x) = 2x^3 + ax^2 + bx + 4$ is divided by $x - 2$, the quotient and the remainder are $Q(x)$ and R respectively, where a, b and R are constants.

(i) Form an equation relating $f(x)$, $Q(x)$ and R. [1]

$f(x) = (x-2)Q(x) + R$ [B1]

It is given that $R = 8$.
It is also given that the gradient of the curve $y = f(x)$ at the point $x = -1$ is 0.

(ii) Show that $a = 0$ and $b = -6$. [4]

Substitute $x = 2$:

$f(2) = 8$ so $2(2)^3 + a(2)^2 + b(2) + 4 = 8$

$\Rightarrow$ $4a + 2b = -12$

$\Rightarrow$ $2a + b = -6$ - - - (1) [M1]

$\frac{dy}{dx} = 6x^2 + 2ax + b$

When $x = -1$, $6(-1)^2 + 2a(-1) + b = 0$

$\Rightarrow$ $-2a + b = -6$ - - - (2) [M1]

(1) + (2): $2b = -12$

$\Rightarrow$ $b = -6$ [A1]

(1) – (2): $a = 0$ [A1]

(iii) Factorise $f(x)$ completely. [3]

$f(x) = 2x^3 - 6x + 4 = 2(x^3 - 3x + 2)$

By inspection, $f(1) = 0$ so $x - 1$ is a factor of $f(x)$. [A1]

$f(x) = 2(x^3 - 3x + 2) = 2(x-1)(x^2 + x - 2)$ [M1]

$= 2(x-1)(x-1)(x+2)$ [A1]

3(a) Solve $(\lg x)^2 = 3 - 2\lg x$. [4]

Let $u = \lg x$

$(\lg x)^2 = 3 - 2\lg x \Rightarrow u^2 = 3 - 2u$

$\Rightarrow u^2 + 2u - 3 = 0$ [M1]

$\Rightarrow (u-1)(u+3) = 0$

$\Rightarrow u = -3$ or $u = 1$ [A1]

$\Rightarrow \lg x = -3$ or $\lg x = 1$

$\Rightarrow x = 10^{-3}$ or $x = 10$ [A2]

(b) Use a change of base to find $\frac{\mathrm{d}}{\mathrm{d}x}\left(\lg 3x^2\right)$. [3]

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\lg 3x^2\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\ln 3x^2}{\ln 10}\right) \text{ [M1]}$$

$$= \frac{1}{\ln 10}\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln 3x^2\right)$$

$$= \frac{1}{\ln 10}\left(\frac{1}{3x^2}\right)(6x) \text{ [M1]}$$

$$= \frac{2}{x\ln 10} \text{ [A1]}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\lg 3x^2\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\lg 3 + \lg x^2\right)$$

$$= \frac{\mathrm{d}}{\mathrm{d}x}\left(\lg 3 + 2\lg x\right)$$

$$= 2\frac{\mathrm{d}}{\mathrm{d}x}\left(\lg x\right)$$

$$= 2\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\ln x}{\ln 10}\right)$$

$$= \frac{2}{\ln 10}\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln x\right)$$

$$= \frac{2}{x\ln 10}$$

(c) **(i)** Find $\frac{\mathrm{d}}{\mathrm{d}x}\left(e^x \sin x\right)$. [2]

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(e^x \sin x\right) = e^x \sin x + e^x \cos x \text{ [B2]}$$

(ii) Hence find $\int_0^{\frac{\pi}{3}} e^x(\sin x + \cos x)\,\mathrm{d}x$. [2]

$$\int_0^{\frac{\pi}{3}} e^x(\sin x + \cos x)\,\mathrm{d}x = \left[e^x \sin x\right]_0^{\frac{\pi}{3}} \text{ [B1]}$$

$$= e^{\frac{\pi}{3}}\sin\frac{\pi}{3} - e^0 \sin 0$$

$$= e^{\frac{\pi}{3}}\sin\frac{\pi}{3} \text{ [B1]}$$

4(i) Find the coordinates of the stationary point of the curve $y = \dfrac{21-20x}{x^2-2x+1}$ where $x \neq 1$. [5]

$$\frac{dy}{dx} = \frac{(x^2-2x+1)(-20)-(21-20x)(2x-2)}{(x^2-2x+1)^2}$$ [M1][A1]

$$= \frac{-20x^2+40x-20-(42x-42-40x^2+40x)}{(x^2-2x+1)^2}$$

$$= \frac{20x^2-42x+22}{(x^2-2x+1)^2}$$

At the stationary point, $\dfrac{dy}{dx} = 0$ so $20x^2-42x+22 = 0$ [M1]

$10x^2-21x+11 = 0$

$(x-1)(10x-11) = 0$

$x = 1$ (reject, given $x \neq 1$) or $x = 1.1$ [A1]

When $x = 1.1$, $y = \dfrac{21-20(1.1)}{(1.1)^2-2(1.1)+1} = -100$

So the coordinates of the stationary point are (1.1, −100). [A1]

(ii) Use the **First Derivative Test** to determine the nature of this stationary point. [3]

$$\frac{dy}{dx} = \frac{20x^2-42x+22}{(x^2-2x+1)^2}$$

x	1.09	1.1	1.11
$\dfrac{dy}{dx}$	−	0	+
shape	\	—	/

[M1 - suitable values of x]
[A1 – correct values for $\dfrac{dy}{dx}$]

By the First Derivative Test, the stationary point is a minimum point. [A1]

5(i) Given that $\left(\dfrac{x}{x-1}\right)^2 = A + \dfrac{B}{x-1} + \dfrac{C}{(x-1)^2}$, where A, B and C are constants, find the value of A, of B and of C. [4]

$$\left(\frac{x}{x-1}\right)^2 = A + \frac{B}{x-1} + \frac{C}{(x-1)^2} = \frac{A(x-1)^2 + B(x-1) + C}{(x-1)^2}$$

$$x^2 = A(x-1)^2 + B(x-1) + C \qquad \text{[M1]}$$

When $x = 1$, $1 = C$ [M1 – substitution method]

When $x = 0$, $0 = A - B + C \Rightarrow A - B = -1$ - - - (1)

When $x = 2$, $4 = A + B + C \Rightarrow A + B = 3$ - - - (2)

(1) + (2): $2A = 2 \Rightarrow A = 1$

From (2), $B = 2$

[A1 – one correct][A1 – all correct]

(ii) Hence find $\displaystyle\int_2^3 \left(\frac{x}{x-1}\right)^2 \mathrm{d}x$. [3]

$$\int_2^3 \left(\frac{x}{x-1}\right)^2 \mathrm{d}x = \int_2^3 1 + \frac{2}{x-1} + \frac{1}{(x-1)^2}\,\mathrm{d}x$$

$$= \left[x + 2\ln(x-1) - \frac{1}{x-1}\right]_2^3 \qquad \text{[B2]}$$

$$= \left[3 + 2\ln(3-1) - \frac{1}{3-1}\right] - \left[2 + 2\ln(2-1) - \frac{1}{2-1}\right]$$

$$= \frac{3}{2} + 2\ln 2 \qquad \text{[A1]}$$

6 The variables x and y are related by the equation $y = 10^{-a}b^{x}$, where a and b are constants. The table below shows values of x and y.

x	0	5	10	15	20	25
y	0.16	0.4	0.8	1.6	3.3	6.5

(i) Plot $\lg y$ against x for the given data and draw a straight line graph. [2]

x	0	5	10	15	20	25
y	0.16	0.4	0.8	1.6	3.3	6.5
$\lg y$	–0.80	–0.40	–0.10	0.20	0.52	0.81

[B1 – accurate point of at least 4 points]
[B1 – reasonably good best fit line]

(ii) Use your graph to estimate

(a) the value of a and of b, [3]

$$y = 10^{-a}b^{x} \quad \Rightarrow \quad \lg y = \lg(10^{-a}b^{x})$$
$$= \lg(10^{-a}) + \lg(b^{x})$$
$$= (\lg b)x - a$$ [M1 for eqn]

Gradient $= \lg b = \dfrac{0.6-(-0.4)}{21.25-5.5} = 0.063492$ (from the graph)

$b = 1.16$ (3 s.f.) [A1 – acceptable values: $1.05 \le b \le 1.27$]

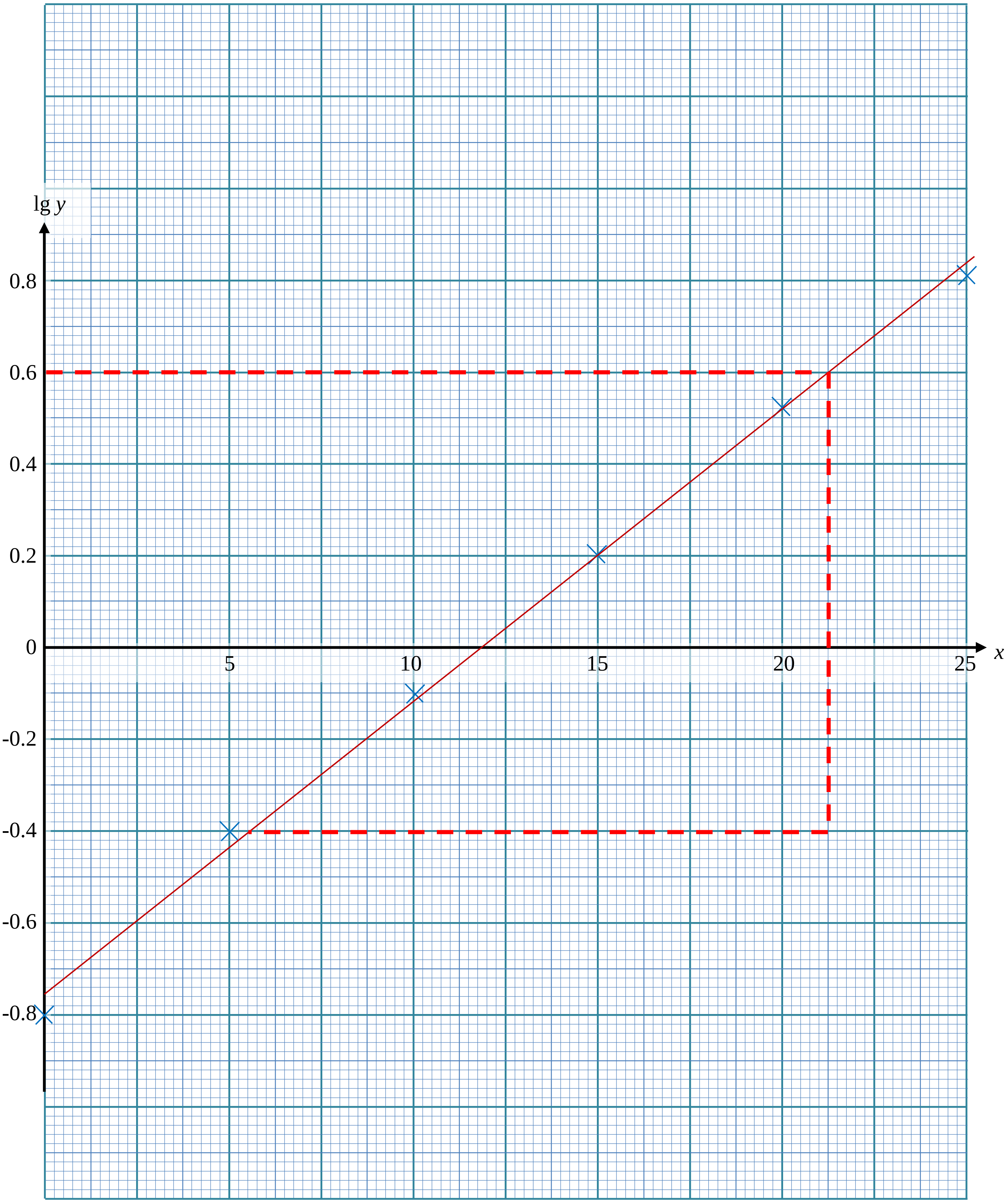
Vertical intercept $= (-a) = -0.75$ (from the graph)

$a = 0.75$ [A1 – acceptable values: $0.70 \le a \le 0.80$]

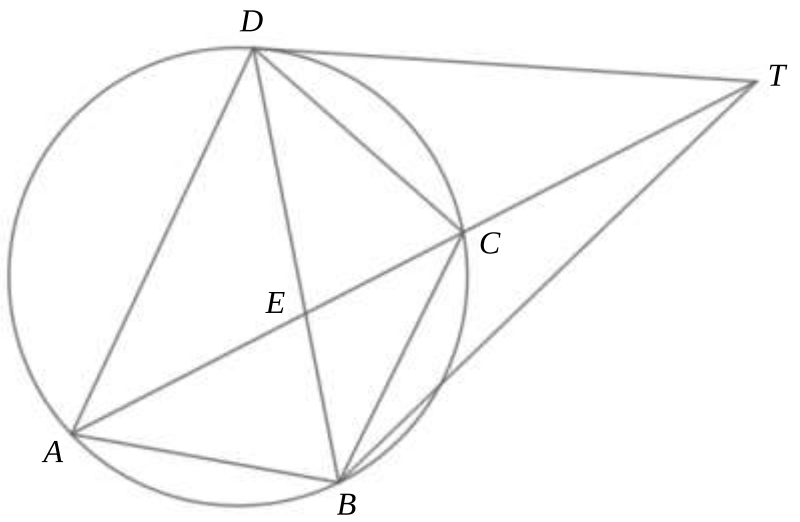
(b) the value of x when $y = 4$. [1]

When $y = 4$, $\lg y = 0.60$ (2 d.p.)

From the graph, $x = 21.25$ [B1 – acceptable values: $20 \le x \le 22$]


lg y
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
5
10
15
20
25
x

7



The diagram shows a point *D* on a circle, and *TD* is a tangent to the circle. Points *A*, *B*, and *C* lie on the circle such that *DC* bisects angle *TDB* and *TCA* is a straight line. The lines *TA* and *DB* intersects at *E*.

(i) Prove that *CB* = *CD*. [3]

$\angle$TCA = $\angle$TDC	(DC bisects $\angle$TDB)	[B1]
$\angle$TDC = $\angle$DBC	(Alternate Segment Theorem)	[B1]
$\therefore$ $\angle$TCA = $\angle$DBC		
$\therefore$ ΔBCA is an isosceles Δ.		[AG1]
$\therefore$ BC = DC (proved)		

(ii) Prove that *EA* bisects angle *DAB*. [4]

$\angle$CDB = $\angle$CAB	($\angle$s in the same segment)	[B1]
$\angle$TDC = $\angle$DAC	(Alternate Segment Theorem)	[B1]
Given $\angle$TDC = $\angle$CDB (CD bisects $\angle$TDB),		
$\therefore$ $\angle$CAB = $\angle$DAC		[B1]
Since $\angle$CAB = $\angle$DAC and E lies on AC,		[AG1]
EA bisects $\angle$DAB (proved)		

8(i) Express $4\sin 2x + 3\cos 2x + 1$ in the form $R\sin(2x + \alpha) + k$, where R, α and k are constants with $R > 0$ and α an acute angle.

[2]

$4\sin 2x + 3\cos 2x + 1 = R\sin(2x + \alpha) + k$

where $R = \sqrt{4^2 + 3^2} = 5$ [B1]

$\alpha = \tan^{-1}\left(\frac{3}{4}\right) = 36.870°$ (5 s.f.)

[B1]

and $k = 1$

$\therefore \quad 4\sin 2x + 3\cos 2x + 1 = 5\sin(2x + 36.870°) + 1$

(ii) Solve $4\sin 2x + 3\cos 2x + 1 = 0$ for $0 \le x \le 180°$. [4]

$4\sin 2x + 3\cos 2x + 1 = 0$

$\Rightarrow \quad 5\sin(2x + \sqrt{4^2 + 3^2}) + 1 = 0$

$\Rightarrow \quad \sin(2x + \tan^{-1}(\frac{3}{4})) = -\frac{1}{5}$ [M1]

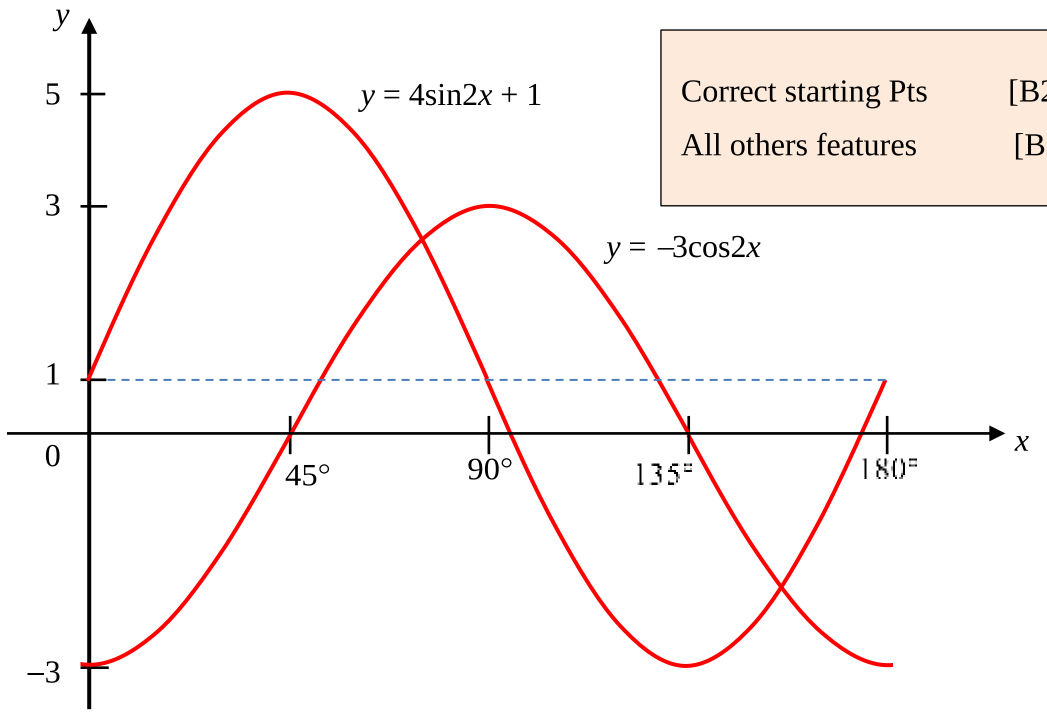
Basic angle $= \sin^{-1}\left(\frac{1}{5}\right) = 11.537°$ (5 s.f.) [M1]

Since the sine ratio is negative, the angle is in the 3rd or 4th quadrants. [M1]

$2x + 36.870° = 180° + 11.537°$ or $360° - 11.537°$

$x = 77.334°$ or $155.797°$ (5 s.f.) [A1 – both correct]

(iii) Sketch, in the same diagram, the graphs of $y = 4\sin 2x + 1$ and $y = -3\cos 2x$ for $0 \le x \le 180°$. [4]



Correct starting Pts	[B2]
All others features	[B2]

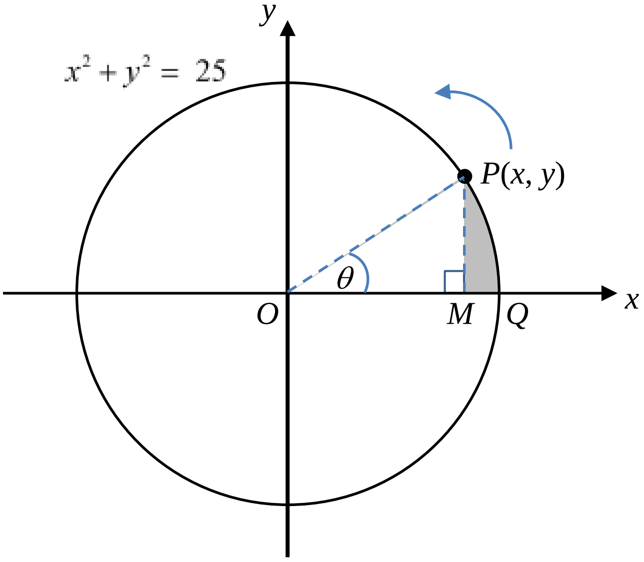
(iv) Find the coordinates of the points of intersection of the two graphs. [2]

When $x = 77.334°$, $y = -3\cos 2(77.334°) = 2.71$ (3 s.f.)

When $x = 155.797°$, $y = -3\cos 2(155.797°) = -1.99$ (3 s.f.)

$\therefore$ the coordinates are $(77.3°, 2.71)$ and $(155.8°, -1.99)$. [B2]

9



The point $P(x, y)$ starts from the point Q with coordinates (5, 0) and it moves along the circle $x^2 + y^2 = 25$ in an anticlockwise sense. The angle QOP is θ and it increases at a constant rate of 0.04 radians per second.

The point M on the x-axis is such that PM is perpendicular to x-axis.

(i) Find the area of triangle POM, giving your answer in the form $k\sin 2\theta$ where k is a constant. [4]

$x = 5\cos\theta$ [B1]

$y = 5\sin\theta$ [B1]

$$\text{area of } \Delta POM = \frac{1}{2}xy$$

$$= \frac{1}{2}(5\cos\theta)(5\sin\theta) \quad \text{[M1]}$$

$$= \frac{25}{2}\sin\theta\cos\theta$$

$$= \frac{25}{4}(2\sin\theta\cos\theta)$$

$$= \frac{25}{4}\sin 2\theta \quad \text{[A1]}$$

(ii) The area of the shaded region PMQ is A unit^2. Obtain an expression for A in terms of θ. [2]

A = (area of sector POQ) – (area of ΔPOM)

$$= \frac{1}{2}(5^2)\theta - \frac{25}{4}\sin 2\theta$$ [B1 – for area of sector]

$$= \frac{25}{2}\theta - \frac{25}{4}\sin 2\theta$$ [M1]

(iii) Find, in terms of θ, the rate at which A is changing. [3]

$$\frac{dA}{dt} = \frac{dA}{d\theta} \times \frac{d\theta}{dt}$$ [M1]

$$= \frac{d}{d\theta}\left(\frac{25}{2}\theta - \frac{25}{4}\sin 2\theta\right) \times 0.04$$

$$= \left[\frac{25}{2} - \frac{25}{4}(2\cos 2\theta)\right] \times 0.04$$ [A1]

$$= \frac{1}{2}(1 - \cos 2\theta)$$ [A1]

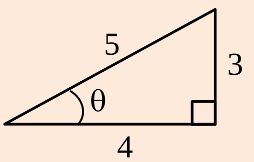
(iv) Hence find the **exact** value of this rate when P passes through the point (4, 3). [2]

$$\frac{dA}{dt} = \frac{1}{2}(1 - \cos 2\theta)$$

$$= \frac{1}{2}\left[1 - \left(1 - 2\sin^2\theta\right)\right]$$

$$= \sin^2\theta$$ [M1]

At the point (4, 3), $\tan\theta = \frac{3}{4}$



so $\sin\theta = \frac{3}{5}$

$$\therefore \quad \frac{dA}{dt} = \left(\frac{3}{5}\right)^2 = \frac{9}{25} \text{ unit}^2/\text{s}$$ [A1 – only from exact fraction]

10 A particle moves in a straight line so that t seconds after passing through a fixed point O, its velocity v m/s is given by $v = 7\sin 2t$ where $0 \le t \le \pi$. Find

(i) the initial acceleration of the particle, [2]

$v = 7\sin 2t \quad \Rightarrow \quad \frac{dv}{dt} = 7(\cos 2t) \times 2 = 14\cos 2t$ [B1]

When $t = 0$, acceleration $= 14\cos 0 = 14$ m/s^2 [B1]

(ii) the value of t when the particle first comes to instantaneous rest, [1]

$v = 0 \quad \Rightarrow \quad \sin 2t = 0$

$\Rightarrow \quad 2t = 0$ or $2t = \pi$

so the particle first comes to instantaneous rest when $t = 0$ or $\frac{\pi}{2}$ [B1]

(iii) the maximum displacement of the particle, [3]

Displacement, s, is maximum when $\frac{ds}{dt} = v = 0$, i.e. at $t = \frac{\pi}{2}$.

$s = \int v\,dt = \int 7\sin 2t\,dt = \frac{-7\cos 2t}{2} + C$ [M1]

When $t = 0$, $s = 0$ so $\frac{-7\cos 0}{2} + C = 0 \quad \Rightarrow \quad C = \frac{7}{2}$ [A1]

When $t = \frac{\pi}{2}$, $s = \frac{-7\cos 2(\frac{\pi}{2})}{2} + \frac{7}{2} = 7$ [A1]

(iv) the distance travelled by the particle in the first 2.5 seconds after passing through O. [4]

Distance travelled

$$= \begin{pmatrix}\text{area of shaded region} \\ \text{from } t=0 \text{ to } t=\frac{\pi}{2}\end{pmatrix}$$

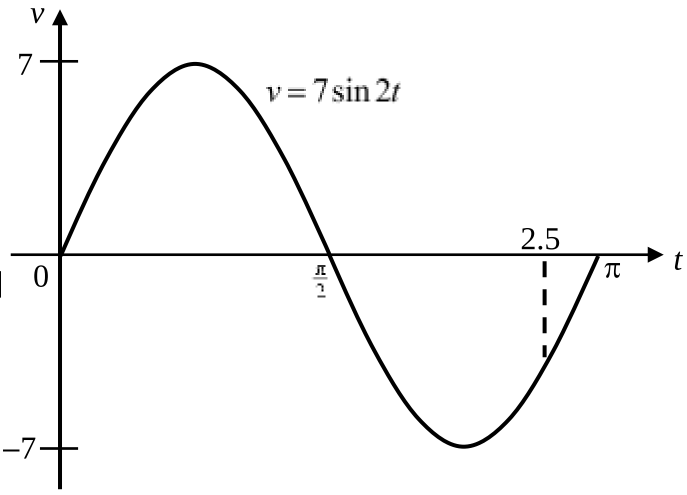
$$+ \begin{pmatrix}\text{area of shaded region} \\ \text{from } t=\frac{\pi}{2} \text{ to } t=2.5\end{pmatrix} \quad \text{[M1]}$$

$$= 7 + \left[-\int_{\frac{\pi}{2}}^{2.5} 7\sin 2t \, dt\right] \quad \text{[M1]}$$

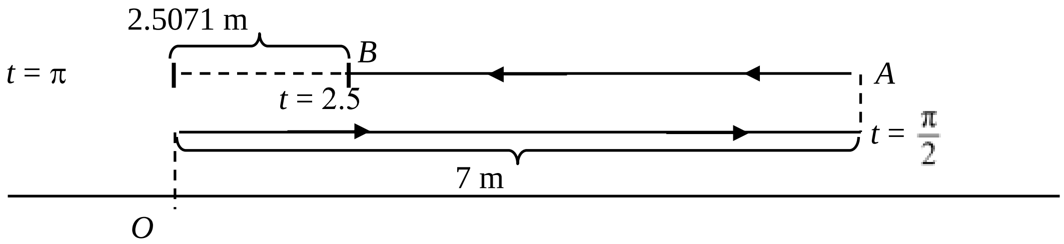
$$= 7 + \left[-\left(\frac{-7\cos 2t}{2}\right)\right]_{\frac{\pi}{2}}^{2.5} \quad \text{[M1]}$$

$$= 7 + \left[\frac{7\cos 5}{2} - \frac{7\cos \pi}{2}\right]$$

$$= 11.5 \text{ m (3 s.f.)} \quad \text{[A1]}$$



<u>Alternative solution</u>



When $t = 2.5$, $s = \dfrac{-7\cos 2(2.5)}{2} + \dfrac{7}{2} = 2.5071$ [M1]

$\therefore$ distance travelled $= 7 + (7 - 2.5071)$ [M1]

$= 11.5$ m (3 s.f.) [A1]

~ ~ End of Paper 2 ~ ~