2021 4E Add Math Preliminary Examination (4049/1)
Marking Scheme

1 A curve is such that $\frac{d^2y}{dx^2}=2(1-2x)$. The equation of the normal to the curve at the point $(-1,7)$ is $9y=x+64$. Find the equation of the curve. [6]

Working	Mark allocation	Remark
$\frac{d^2y}{dx^2}=2(1-2x)$ $\frac{dy}{dx}=-2x^2+2x+C$	M1	integration (with *C*)
Gradient of normal = $\frac{1}{9}$ Gradient of tangent = -9	M1	
Sub $\frac{dy}{dx}=-9$, $x=-1$, $-9=-2(-1)^2+2(-1)+C$ $C=-5$	M1	
$\frac{dy}{dx}=-2x^2+2x-5$	M1	
$y=-\frac{2x^3}{3}+x^2-5x+D$	M1	integration (with *D*)
Sub $(-1,7)$, $7=-\frac{2(-1)^3}{3}+(-1)^2-5(-1)+D$ $D=\frac{1}{3}$ Equation of curve: $y=-\frac{2}{3}x^3+x^2-5x+\frac{1}{3}$	A1	

2 **(i)** Show that $p+q$ is a factor of the expression $p^3-3p^2q+4q^3$. [2]

Working	Mark allocation	Remark
Let f(p) = $p^3-3p^2q+4q^3$ f($-q$) = $(-q)^3-3(-q)^2q+4q^3$ = 0 $\therefore$ $p+q$ is a factor. or Let f(q) = $p^3-3p^2q+4q^3$ f($-p$) = $p^3-3p^2(-p)+4(-p)^3$ = 0 $\therefore$ $p+q$ is a factor.	M1 A1 M1 A1	accept long division method

(ii) Factorise $p^3-3p^2q+4q^3$ completely. [2]

Working	Mark allocation	Remark
Let f(p) = $(p+q)(p^2-4pq+4q^2)$ = $(p+q)(p-2q)^2$	M1 A1	quadratic factor by long division

(iii) Hence solve the equation $(m+2)^3-3(m+2)^2(m-3)+4(m-3)^3=0$. [2]

Working	Mark allocation	Remark
$(m+2)^3-3(m+2)^2(m-3)+4(m-3)^3=0$ By comparison with f(p), $p=m+2$, $q=m-3$ $(m+2)^3-3(m+2)^2(m-3)+4(m-3)^3=0$ From (ii),		

$(m+2+m-3)(m+2-2(m-3))^2=0$ $(2m-1)(-m+8)^2=0$ $m=\frac{1}{2}$ or 8	M1 A1	 For both correct answers

3 **(a)** Solve $7\cos\left(\frac{\theta}{2}\right)=10\sin\theta$, for $0\le\theta\le 2\pi$. [5]

Working	Mark allocation	Remark
$7\cos\left(\frac{\theta}{2}\right)=10\sin\theta$		
$7\cos\frac{\theta}{2}=10\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)$	M1	Use of sine double angle
$20\sin\frac{\theta}{2}\cos\frac{\theta}{2}-7\cos\frac{\theta}{2}=0$		
$\cos\frac{\theta}{2}\left(20\sin\frac{\theta}{2}-7\right)=0$	M1	factorising
$\cos\frac{\theta}{2}=0$ $\frac{\theta}{2}=\frac{\pi}{2}$		
$\theta=\pi$	A1	
or $\sin\frac{\theta}{2}=\frac{7}{20}$ $\alpha=0.35757$ $\frac{\theta}{2}=0.35757$ or 2.7840	A1, A1	
$\theta=0.715$ (3 sf) or 5.57 (3 sf)		

(b) Prove that $\dfrac{\sec x\tan x+\sec^2 x}{(\tan x+\sec x)^2+1}=\dfrac{1}{2}$. [3]

Working	Mark allocation	Remark
LHS = $\dfrac{\sec x\tan x+\sec^2 x}{(\tan x+\sec x)^2+1}$	M1	expanding

$= \dfrac{\sec x \tan x + \sec^2 x}{\tan^2 x + 2\tan x \sec x + \sec^2 x + 1}$ $= \dfrac{\sec x \tan x + \sec^2 x}{2\tan x \sec x + \sec^2 x + \sec^2 x}$ $= \dfrac{\sec x(\tan x + \sec x)}{2\sec x(\tan x + \sec x)}$ $= \dfrac{1}{2}$ = RHS (proved)	M1 A1	use of identity: $1 + \tan^2 x = \sec^2 x$ completion of proof

4 **(i)** Write down and simplify, in descending powers of x, the first three terms in the expansion of $\left(x^5 + \dfrac{2}{x^6}\right)^n$, where $n > 0$. [2]

Working	Mark allocation	Remark
$\left(x^5 + \dfrac{2}{x^6}\right)^n$ $= x^{5n} + nx^{5n-5}\left(\dfrac{2}{x^6}\right) + \dbinom{n}{2}\left(x^{5n-10}\right)\left(\dfrac{2}{x^6}\right)^2 + \ldots$ $= x^{5n} + 2nx^{5n-11} + 2n(n-1)x^{5n-22} + \ldots$	M1 A1	

(ii) When the third term of the expansion is divided by the second term, $\dfrac{8}{x^{11}}$ is obtained. Show that $n - 9$. [2]

Working	Mark allocation	Remark
$\dfrac{2n(n-1)}{2n} = 8$ $n - 9$ (shown)	M1 A1	

(iii) Using the value of n found in **(ii)**, without expanding $\left(x^5 + \dfrac{2}{x^6}\right)^n$, show that there is no constant term in the expansion. [3]

Working	Mark allocation	Remark

$T_{r+1} = \binom{9}{r}\left(x^5\right)^{9-r}\left(\frac{2}{x^6}\right)^r$ $= \binom{9}{r}\left(2^r\right)x^{45-11r}$	M1	
Power of x = $45-11r$ For constant term⇒ power of x = 0		
$45-11r=0$ $r=\frac{45}{11}$ Since r is not an integer, there is no constant term in the expansion.	M1 A1	equating power of x to 0

5 **(a)** Solve the equation $3\left(9^k\right)+2\left(4^k\right)=5\left(6^k\right)$. [5]

Working	Mark allocation	Remark
$3\left(9^k\right)+2\left(4^k\right)=5\left(6^k\right)$ $3\left(3^k\right)^2+2\left(2^k\right)^2=5\left(3^k 2^k\right)$ Let $x = 3^k$ and $y = 2^k$,		
$3x^2+2y^2=5xy$	M1	
$3x^2-5xy+2y^2=0$		
$(3x-2y)(x-y)=0$	M1	factorising
$3x=2y$ or $x=y$	M1	
$3\left(3^k\right)=2\left(2^k\right)$ or $3^k = 2^k$		
$\left(\frac{3}{2}\right)^k=\frac{2}{3}$	A1, A1	
$k=-1$ or $k=0$		

(b) Show that $\frac{\sqrt{\left(a-b^2\right)^3\left(a+b^2\right)}}{\left(\sqrt{a}+b\right)\sqrt{a^2-b^4}}$ can be expressed in the form $m\sqrt{a}+nb$ where m and n are constants to be determined. [4]

Working	Mark allocation	Remark

$\dfrac{\sqrt{\left(a-b^2\right)^3\left(a+b^2\right)}}{\left(\sqrt{a}+b\right)\sqrt{a^2-b^4}}$ $=\dfrac{\sqrt{\left(a-b^2\right)\left(a+b^2\right)\left(a-b^2\right)^2}}{\left(\sqrt{a}+b\right)\sqrt{a^2-b^4}}$	M1, M1	$\sqrt{a^2-b^4}$ in numerator $a-b^2$ in numerator
$=\dfrac{\sqrt{a^2-b^4}\left(a-b^2\right)}{\left(\sqrt{a}+b\right)\sqrt{a^2-b^4}}$		
$=\dfrac{\left(a-b^2\right)}{\left(\sqrt{a}+b\right)}$	M1	rationalising denominator
$=\dfrac{\left(a-b^2\right)\left(\sqrt{a}-b\right)}{\left(\sqrt{a}+b\right)\left(\sqrt{a}-b\right)}$		
$=\dfrac{\left(a-b^2\right)\left(\sqrt{a}-b\right)}{a-b^2}$	A1	
$=\sqrt{a}-b$		

6 **(i)** Express $\dfrac{4x^3+9x^2-17x-5}{(2x-3)\left(x^2+1\right)}$ in partial fractions. [6]

Working	Mark allocation	Remark
$\dfrac{4x^3+9x^2-17x-5}{(2x-3)\left(x^2+1\right)}$ $=2+\dfrac{15x^2-21x+1}{(2x-3)\left(x^2+1\right)}$	M1	
$\dfrac{15x^2-21x+1}{(2x-3)\left(x^2+1\right)}=\dfrac{A}{2x-3}+\dfrac{Bx+C}{x^2+1}$	M1	
$15x^2-21x+1=A\left(x^2+1\right)+(Bx+C)(2x-3)$		
Sub $x=\dfrac{3}{2}$: 3.25 = 3.25A A = 1	M1	(correct value of A)
Sub $x=0$: 1 = 1 – 3C C = 0	M1	(correct value of C)
Sub $x=1$: $-5=2-B$	M1	(correct value of B)

$B = 7$ $\frac{4x^3+9x^2-17x-5}{(2x-3)(x^2+1)} = 2+\frac{1}{(2x-3)}+\frac{7x}{(x^2+1)}$	A1	

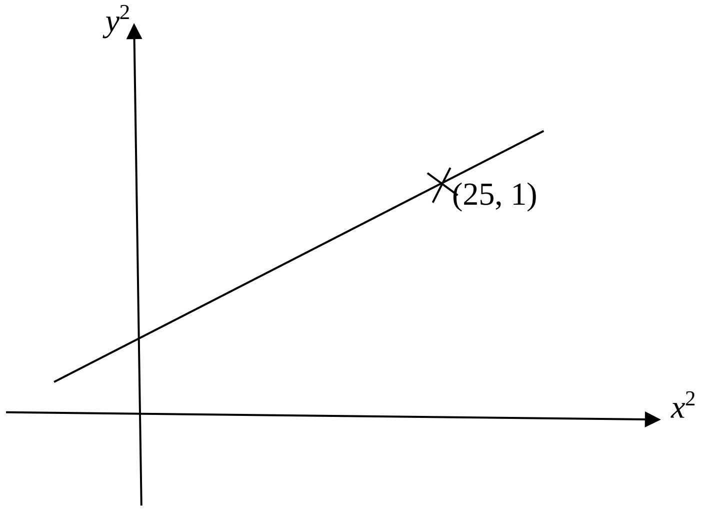
(ii) Differentiate $\ln\left(x^2+1\right)$ with respect to x. [1]

Working	Mark allocation	Remark
$\frac{\mathrm{d}}{\mathrm{d}x}\ln\left(x^2+1\right)$ $= \frac{2x}{x^2+1}$	B1	

(iii) Hence find $\int \frac{4x^3+9x^2-17x-5}{(2x-3)(x^2+1)}\,\mathrm{d}x$. [3]

Working	Mark allocation	Remark
$\int \frac{4x^3+9x^2-17x-5}{(2x-3)(x^2+1)}\,\mathrm{d}x$ $= \int 2\,\mathrm{d}x + \int \frac{1}{(2x-3)}\,\mathrm{d}x + \int \frac{7x}{(x^2+1)}\,\mathrm{d}x$ $= 2x+\frac{1}{2}\ln(2x-3)+\frac{7}{2}\ln\left(x^2+1\right)+C$	B1, B1, B1	Deduct 1 m if student did not put '*C*'.

7 **(a)**



Two variables x and y are related by the equation $\frac{x^2}{p^2} = 1 + \frac{3y^2}{q^2}$, where p and q are constants. When the graph of y^2 against x^2 is drawn, a straight line is obtained.

Given that the line passes through the point (25, 1) and has a gradient $\frac{1}{15}$, find

(i) the exact values of p and q. [4]

Working	Mark allocation	Remark
$\frac{x^2}{p^2}=1+\frac{3y^2}{q^2}$ $\frac{3y^2}{q^2}=\frac{x^2}{p^2}-1$ $y^2=\left(\frac{q^2}{3p^2}\right)x^2-\frac{q^2}{3}$	M1	manipulation of equation into linear form
Since gradient = $\frac{1}{15}$, $\frac{q^2}{3p^2}=\frac{1}{15}$ ------ (1) Sub $\frac{q^2}{3p^2}=\frac{1}{15}$ and (25, 1) into equation: $1=\left(\frac{1}{15}\right)(25)-\frac{q^2}{3}$ $\frac{q^2}{3}=\frac{2}{3}$ $q=\pm\sqrt{2}$	M1	correct substitution of gradient and point into equation
Sub $q^2=2$ into (1): $\frac{2}{3p^2}=\frac{1}{15}$ $p=\pm\sqrt{10}$		
$\therefore p=\sqrt{10}$, $q=\sqrt{2}$, $p=-\sqrt{10}$, $q=-\sqrt{2}$, $p=\sqrt{10}$, $q=-\sqrt{2}$, $p=-\sqrt{10}$, $q=\sqrt{2}$.	A1 for 1 correct pair A1 for 1 correct pair	

(ii) the values of x when $y=\sqrt{\frac{2}{5}}$. [2]

Working	Mark allocation	Remark

$y^2 = \frac{1}{15}x^2 - \frac{2}{3}$		
Sub $y = \sqrt{\frac{2}{5}}$,	M1	Substitution of *y* value
$\frac{2}{5} = \frac{1}{15}x^2 - \frac{2}{3}$	A1	
$x = +4$		

(b) Determine, showing your working, if it is possible for any line that passes through the point (2, 1) to be a tangent to the circle with equation $x^2 + y^2 - 6x + 8y - 11 = 0$. [5]

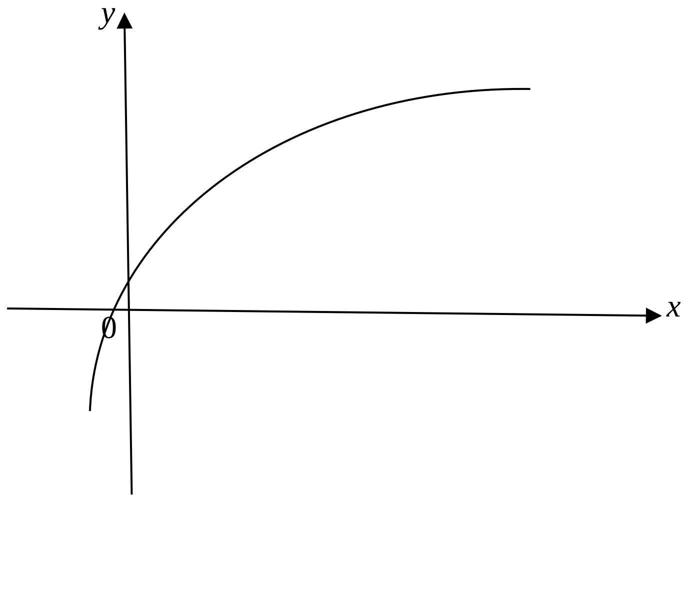
Working	Mark allocation	Remark
$x^2 - 6x + y^2 + 8y - 11 = 0$ $x^2 - 6x + (-3)^2 - (-3)^2 + y^2 + 8y + 4^2 - 4^2 - 11 = 0$ $(x-3)^2 + (y+4)^2 = 36$	M1	
Centre = $(3, \ -4)$	M1	
Radius = 6 units	M1	
Distance of centre of circle to (2, 1) = $\sqrt{(3-2)^2 + (-4-1)^2}$	M1	
= $\sqrt{26}$ < 6 units which is the radius of circle ∴ (2, 1) is inside the circle ∴ it is not possible for any line that passes through (2, 1) to be a tangent to the circle.	A1	

8 **(a)** A curve has the equation $y = \frac{4x+15}{x-9}$, where $x \neq 9$. Find the gradient function of the curve and determine, with explanation, whether *y* is an increasing or decreasing function. [4]

Working	Mark allocation	Remark
$\frac{dy}{dx} = \frac{(x-9)(4)-(4x+15)(1)}{(x-9)^2}$	M1	

$= -\frac{51}{(x-9)^2}$ For $x \neq 9$, $(x-9)^2 > 0$ => $\frac{dy}{dx} = -\frac{51}{(x-9)^2} < 0$ => y is a decreasing function.	A1 M1 A1	

(b) The diagram shows part of the curve $y = \ln \sqrt[3]{x}$.



Calculate the area of the region bounded by the curve, the line $x=3$ and the x-axis. [5]

Working	Mark allocation	Remark
$y=\ln\sqrt[3]{x}$ $x=e^{3y}$	M1	making x the subject
Area of required region $= 3\ln\sqrt[3]{3}-\int_0^{\ln\sqrt[3]{3}} e^{3y}\ dy$	M1	
$= \ln 3-\left[\frac{e^{3y}}{3}\right]_0^{\ln\sqrt[3]{3}}$	M1	integration of e^{3y}
$= \ln 3-\frac{1}{3}\left(e^{3\ln\sqrt[3]{3}}-1\right)$	M1	application of definite integral
$= \ln 3-\frac{2}{3}$ = 0.432 units^2 (3 sf)	A1	

9 **(a)** Write down the principal value, in radians as a multiple of π, of $\cos^{-1}\left(-\sin\frac{\pi}{3}\right)$. [1]

Working	Mark allocation	Remark
$\frac{5\pi}{6}$	B1	

(b) x, y and z are three angles of a triangle. Given that x and y are acute angles such that $\sin x=\frac{8}{17}$ and $\sin y=\frac{3}{5}$, find the exact value of $\tan z$ without the use of a calculator. [4]

Working	Mark allocation	Remark
$\tan z = \tan\left(180° - (x+y)\right)$	M1	
$= -\tan(x+y)$	M1	
$= -\left(\dfrac{\tan x + \tan y}{1 - \tan x \tan y}\right)$		
$= -\left(\dfrac{\frac{8}{15}+\frac{3}{4}}{1-\left(\frac{8}{15}\right)\left(\frac{3}{4}\right)}\right)$	M1	
$= -\dfrac{77}{36}$	A1	

10 A factory is tasked to design an open cylindrical container with a surface area of 243π cm^2. The radius and height of the cylinder is r cm and h cm respectively.

(i) Show that the volume, V cm^3, of the cylinder is $V = \dfrac{\pi}{2}\left(243r - r^3\right)$. [4]

Working	Mark allocation	Remark
$2\pi rh + \pi r^2 = 243\pi$	M1	
$h = \dfrac{243 - r^2}{2r}$	M1	
$V = \pi r^2\left(\dfrac{243 - r^2}{2r}\right)$	M1	
$= \dfrac{\pi}{2}\left(243r - r^3\right)$ (shown)	A1	

(ii) Find, in terms of π, the maximum volume of the cylinder. [6]

Working	Mark allocation	Remark
$\dfrac{dV}{dr} = \dfrac{\pi}{2}\left(243 - 3r^2\right)$	M1	
When volume of cylinder is maximum,		
$\dfrac{dV}{dr} = 0$		
$\dfrac{\pi}{2}\left(243 - 3r^2\right) = 0$	M1	
$r - 9$		

$\frac{d^2V}{dr^2} = -3\pi r$	M1 M1	
Sub $r = 7$: $\frac{d^2V}{dr^2} = -3\pi(9)$ $= -27\pi < 0$ $\therefore$ Volume is a maximum when $r = 9$	M1	
Maximum volume = $\frac{\pi}{2}\left(243(9) - 9^3\right)$ $= 729\pi$ cm^2	A1	

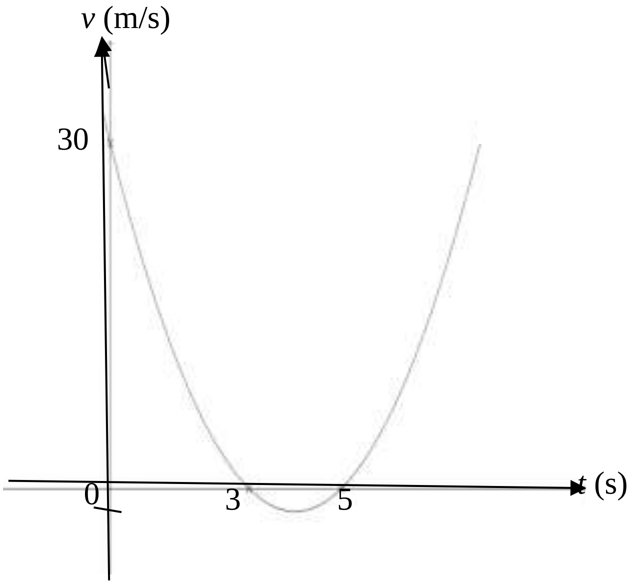
11 A particle moves in a straight line so that, t seconds after passing a fixed point O, its velocity, v m/s, is given by $v = 2t^2 - 16t + 30$.

(i) Find an expression, in terms of t, for the displacement of the particle. [2]

Working	Mark allocation	Remark
$v = 2t^2 - 16t + 30$ $s = \frac{2t^3}{3} - \frac{16t^2}{2} + 30t + D$ $= \frac{2t^3}{3} - 8t^2 + 30t + D$ Sub $t = 0$, $s = 0$: $D = 0$ $s = \frac{2t^3}{3} - 8t^2 + 30t$	M1 A1	integration of v

(ii) Sketch the velocity-time graph and hence find the range of times when the particle is travelling towards O. [3]

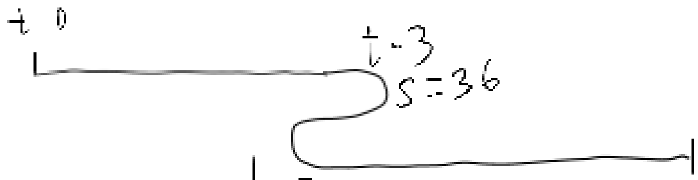
Working	Mark allocation	Remark
$v = 2t^2 - 16t + 30$ $= 2\left(t^2 - 8t + 15\right)$		

$= 2(t-5)(t-3)$  2 cases for particle to travel towards *O*: 1) $s > 0$ & $v < 0$ or 2) $s < 0$ & $v > 0$ To check if the particle will pass through *O* again: Sub $s = 0$: $\frac{2t^3}{3} - 8t^2 + 30t = 0$ $2t^3 - 24t^2 + 90t = 0$ $t^3 - 12t^2 + 45t = 0$ $t\left(t^2 - 12t + 45\right) = 0$ $t = 0$ or $t^2 - 12t + 45 = 0$ no solution since discriminant $= (-12)^2 - 4(45)$ $= -36 < 0$ ∴ particle doesn't pass through *O* again after $t = 0$ <u>OR</u> Sub $t = 5$ into displacement eqn: $s = -8(5)^2 + \frac{2(5)^3}{3} + 30(5)$ $= 33\frac{1}{3}$ m	B1 M1	If students just write $3 < t < 5$ without checking *s* award 1 m out of 2 m.

∴ particle is on the right side of O and it is moving away from O after $t = 5$. Since $s > 0$ for all t, the range of times when the particle is travelling towards O is when $v < 0$, i.e $3 < t < 5$ as seen from the graph.	A1	

(iii) Calculate the total distance travelled by the particle in the first 7 seconds. [4]

Working	Mark allocation	Remark
When $t = 3$, $s = -8(3)^2 + \frac{2(3)^3}{3} + 30(3)$ $= 36$ m	M1	
When $t = 5$, $s = -8(5)^2 + \frac{2(5)^3}{3} + 30(5) = 33\frac{1}{3}$ m		Students might have shown this in part (ii).
When $t = 7$, $s = -8(7)^2 + \frac{2(7)^3}{3} + 30(7)$ $= 46\frac{2}{3}$ m	M1	
Total distance travelled in 1st 7 seconds $= 36 + \left(36 - 33\frac{1}{3}\right) + \left(46\frac{2}{3} - 33\frac{1}{3}\right)$ $= 52$ m OR	M1	
Total distance travelled in 1st 7 seconds $= 46\frac{2}{3} + 2\left(36 - 33\frac{1}{3}\right)$ = 52 m	A1	



$t = 5$

$s = 33\frac{1}{3}$

$t = 7$

$s = 46\frac{2}{3}$