2021 ACJC H2 Maths Promo Paper **Duration: 3 hrs** **Marks: 100**
Attempt all questions.

1 State a sequence of transformations that will transform the curve with equation $y = 2\sin(2x+\alpha)\cos(x)$ on to the curve with equation $y = -2\sin(4x+3\alpha)\cos(2x+\alpha)$, where α is a positive constant. [3]

2 Solve algebraically the inequality $\dfrac{x+3}{x^2+x-2} > -1$. [3]

Hence solve the inequality $\dfrac{x+3x^2}{1+x-2x^2} > -1$. [2]

3 A curve C has equation

$$\frac{x^2-4y^2}{x^2+xy^2+100} = \frac{1}{2}, \ x \in \mathbb{R}, \ x \neq -8.$$

Show that $\dfrac{dy}{dx} = \dfrac{2x-y^2}{2xy+16y}$. [2]

Hence, prove that curve C does not have any stationary point. [3]

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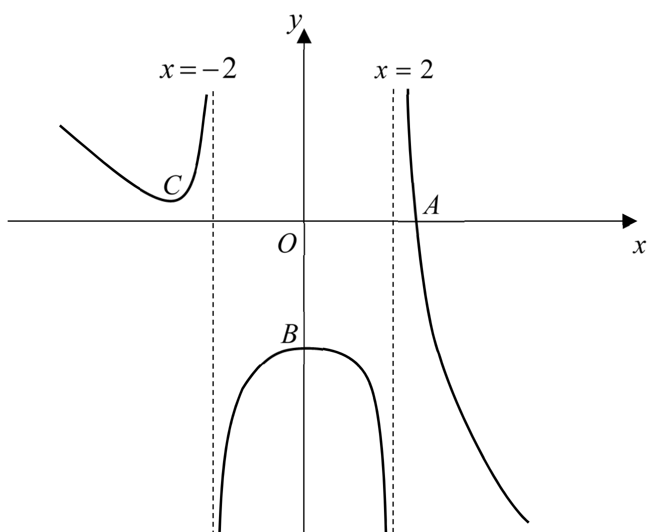


The diagram shows the curve $y = f(x)$. There are two vertical asymptotes with equations $x = -2$ and $x = 2$ respectively. The curve crosses the x-axis at the point A and has a maximum turning point at B where it crosses the y-axis.

The curve also has a minimum turning point at C. The coordinates of A, B and C are $(a, 0)$, $(0, -10)$ and (p, q) respectively, where a, p and q are constants.

Sketch the following curves and state the equations of the asymptotes, the coordinates of the turning points and of points where the curve crosses the axes, if any. Leave your answers in terms of a, p or q where necessary.

(i) $y = \dfrac{1}{f(x)}$, and [3]

(ii) $y = f(2-|x|)$. [3]

5 Referred to the origin O, points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. The modulus of $\mathbf{a}$ is 2 and $\mathbf{b}$ is a unit vector. The angle between $\mathbf{a}$ and $\mathbf{b}$ is 60°. Point C lies on AB, between A and B, such that $AC = kCB$, where $0 < k < 1$.

(i) Express $\overrightarrow{OC}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. [1]

(ii) Show that the length of projection of $\overrightarrow{OC}$ on $\overrightarrow{OA}$ is given by $\dfrac{k+4}{2(k+1)}$. [3]

(iii) Find, in terms of k, the area of triangle OAC. [3]

6 The Cartesian equation of line L_1 is $\dfrac{x-2}{a} = \dfrac{y+2}{b} = \dfrac{z-3}{c}$, where a, b, c are constants. The line L_2 is parallel to the vector $4\mathbf{i}+3\mathbf{j}$. The line L_3 passes through the origin and the point with position vector $\mathbf{j}+\mathbf{k}$.

(i) Given that L_1 is perpendicular to L_2, form an equation relating a and b. [1]

(ii) Given that L_1 intersects L_3, show that $5a+2b-2c=0$. [3]

(iii) Hence express a and b in terms of c. [1]

(iv) Find the acute angle between L_1 and L_3. [2]

7 The functions f and g are defined by

$$\mathrm{f}: x \mapsto \frac{1}{\left|1-x^2\right|},\ x \in \mathbb{R},\ \ -2 \le x < -1,$$

$$\mathrm{g}: x \mapsto -(x-2)^2 + k,\ x \in \mathbb{R},\ \ x \ge 0 \text{ where } k \text{ is a constant.}$$

(i) Sketch on the same diagram the graphs of

(a) $y = \mathrm{f}(x)$

(b) $y = \mathrm{f}^{-1}(x)$

(c) $y = \mathrm{f}^{-1}\mathrm{f}(x)$

stating the equations of any asymptotes and the coordinates of any endpoints. [3]

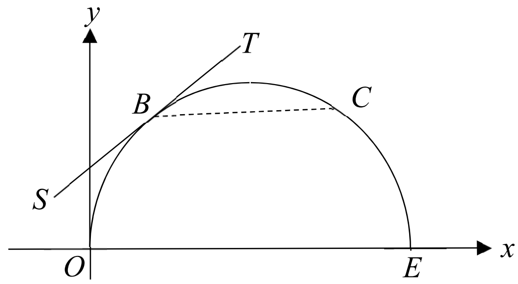
(ii) Find f^{-1} and state the domain of f^{-1}. [3]

(iii) Show that the composite function gf exists and find its range. [2]

8 The figure below shows a cross-section *OBCE* of a car headlight whose reflective surface is modelled in suitable units by the curve with parametric equations

$$x = a(\theta - \sin\theta), \quad y = a(1 - \cos\theta)$$

for $0 \le \theta \le 2\pi$, where a is a positive constant.



(i) Find in terms of a

(a) the length of OE, [2]

(b) the maximum height of the curve $OBCE$. [1]

(ii) Show that $\dfrac{dy}{dx} = \cot\dfrac{\theta}{2}$. [3]

Point B lies on the curve and has parameter β. TS is tangential to the curve at B and BC is parallel to the x-axis. Given that $\angle TBC = \dfrac{\pi}{6}$,

(iii) show that $\beta = \dfrac{2\pi}{3}$. [2]

(iv) Show that the equation of normal to the curve at the point B is

$$ky = -k^2 x + 2\pi a,$$

where k is an exact constant to be determined. [3]

9 **(a)** Given that $\sum_{r=1}^{n} r^2 = \dfrac{1}{6}n(n+1)(2n+1)$, find $\sum_{r=7}^{n+1}\left(2^r + r^2 - r\right)$ in terms of n. [4]

(b) **(i)** Use the method of differences to show that $\sum_{r=2}^{n} \dfrac{1}{r^2-1} = \dfrac{3}{4} + \dfrac{A}{n} + \dfrac{A}{n+1}$, where A is a constant to be determined. [3]

(ii) Explain why the series $\sum_{r=2}^{\infty} \dfrac{1}{r^2-1}$ converges, and write down its value. [2]

(iii) Hence deduce that $\dfrac{2}{2^2} + \dfrac{2}{3^2} + \dfrac{2}{4^2} + \ldots$ is less than $\dfrac{3}{2}$. [2]

10 Referred to the origin O, the points A, B and C have position vectors $4\mathbf{i} - 2\mathbf{j}$, $\alpha\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $-\mathbf{i} - 7\mathbf{j} + \beta\mathbf{k}$ respectively, where α and β are constants.

(i) Given that A, B and C are collinear, show that $\alpha = 5$, and find the value of β. [3]

The plane π contains the line L, which has equation $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} + \mu(2\mathbf{i} - \mathbf{j} + \mathbf{k})$. The plane π is also parallel to the line that passes through the points A and B.

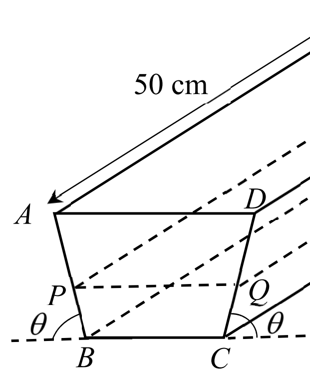
(ii) Find the shortest distance from point A to the line L. [2]

(iii) Show that the cartesian equation of the plane π is $x + y - z = 5$. [2]

(iv) Find the position vector of the foot of the perpendicular from point A to the plane π. [3]

(v) Hence find the reflection of the line that passes through points A and B about the plane π. [2]

11 The figure below shows a container with an open top. The uniform cross section $ABCD$ of the container is a trapezium with $AB = BC = CD = 10$ cm. AB and CD are each inclined to the line BC at an acute angle of θ radians.
The length of the container is 50 cm and the container is placed on a horizontal table.



(i) Show that the volume V of the container is given by

$$V = 5000\left(\sin\theta\right)(1+\cos\theta)\ \text{cm}^3. \qquad [2]$$

Hence using differentiation, find the exact maximum value of V, proving that it is a maximum. [5]

(ii) For the remaining part of the question, θ is fixed at $\dfrac{\pi}{4}$.

Water fills the container at a rate of $100\,\text{cm}^3\text{s}^{-1}$. At time t seconds, the depth of the water is h cm. The surface of the water is a rectangle $PQRS$. When $h = 3$ cm, find the rate of change of

(a) the depth of the water, h, [3]
(b) the surface area of the water $PQRS$. [2]

12 Mrs Tan plans to start a business which requires a start-up capital of \$700,000. She decided to first save \$200,000 by depositing money every month into a savings plan. For the remaining \$500,000, she intends to take a loan from a finance company.
She deposited \$3000 into the savings plan in the first month and on the first day of each subsequent month, she deposited \$100 more than the previous month. Mrs Tan will continue depositing money into the savings plan until the total amount in her savings plan reaches \$200,000. It is given that this savings plan pays no interest.

(i) Find the month in which Mrs Tan's monthly deposit will exceed \$6,550. [2]
(ii) Find the number of months that it will take for Mrs Tan to save \$200,000 and hence find the amount that she would have deposited in the last month. [4]

After Mrs Tan has saved \$200,000, she took a loan of \$500,000 from a finance company. To repay the loan from the finance company, Mrs Tan would have to pay a monthly payment of \$$x$ at the beginning of each month, starting from the first month. An interest of 0.3% per month will be charged on the outstanding loan amount at the end of the month.

(iii) Show that the outstanding amount at the end of n^{th} month, after the interest has been charged, is $A\left(1.003^n\right) - Bx\left(1.003^n - 1\right)$, where A and B are exact constants to be determined. [3]
(iv) Find the amount of \$$x$, to 2 decimal places, if Mrs Tan wants to fully repay her loan in 8 years. [2]
(v) Using the value of x found in part **(iv)**, calculate the total interest that the finance company will earn from Mrs Tan at the end of 8 years. [2]