1 A function is defined as $\mathrm{f}(x)=\dfrac{1-\mathrm{e}^{x}}{\mathrm{e}^{x-2}}$.

(i) Show that $\mathrm{f}(x)$ can be written in the form $\mathrm{e}^{\mathrm{g}(x)}+A$, where A is a constant and $\mathrm{g}(x)$ is a function in x to be determined. [2]

(ii) Hence describe a sequence of transformations that transforms the graph of $y=\mathrm{e}^{x}$ onto the graph of $y=\mathrm{f}(x)$. [3]

2 A curve C has equation $\dfrac{x^3-2y^2}{x^2+3xy}=1$.

(i) The points P and Q on C each have x-coordinate 1. Find the exact gradients of the tangents to C at the points P and Q. [5]

(ii) Find the acute angle between the tangents to C at the points P and Q. [2]

3 The function f is defined as $\mathrm{f}: x \mapsto \dfrac{a}{2-x},\ \ x \in \mathbb{R},\ \ x \neq 0, 2$.

(i) Find f^2 and f^{-1} in terms of a, stating their domains clearly. [5]

(ii) Find the value of a such that $\mathrm{f}^2(x)=\mathrm{f}^{-1}(x)$ for all $x \in \mathbb{R},\ x \neq 0, 2, \dfrac{a}{2}$. [2]

(iii) Using the value of a found in **(ii)**, find $\mathrm{f}^{2021}(x)$. [2]

4 The curve C has equation $y=\mathrm{f}(x)$, where $\mathrm{f}(x)=\dfrac{x(x+a)}{x-a}$ and a is a positive real constant.

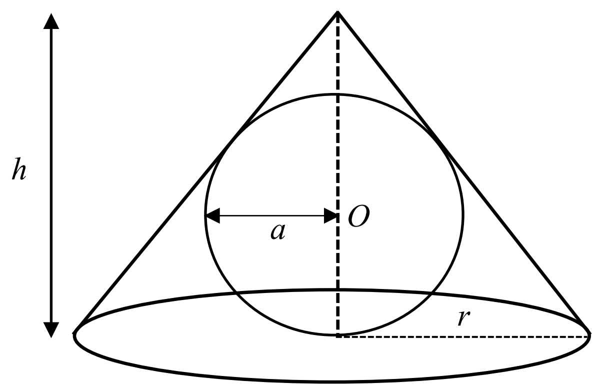
(i) Find algebraically, in terms of a, the set of values that y can take. [4]

(ii) Sketch C, indicating clearly the equations of any asymptotes and the coordinates of the points where the curve crosses the axes. [3]

(iii) By adding a suitable graph to your same diagram in **(ii)** and labelling it clearly, solve the inequality $\dfrac{x(x+a)}{x-a}<|3x|$. [3]

5 [It is given that a cone of radius r and height h has volume $\frac{1}{3}\pi r^2 h$.]

A sphere with fixed radius a and centre O is inscribed in a right cone with base radius r and height h. The sphere is in contact with the base and the curved surface of the cone, as shown in the diagram below.



(i) Show that $r^2 = \dfrac{a^2 h}{h-2a}$. [2]

(ii) Use differentiation to find, in terms of a, the minimum volume of the cone, proving that it is a minimum. [6]

6 It is given that $\mathrm{e}^y = 2 + \sin x$.

(i) Show that $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 2\mathrm{e}^{-y} - 1$. [2]

(ii) By further differentiation of the result in **(i)**, find the Maclaurin series expansion for y in ascending powers of x, up to and including the term in x^3. [3]

(iii) By using the standard series from the List of Formulae (MF26), verify your answer in **(ii)**. [3]

7 Mr Chan is interested in investing in a savings account with an interest rate of 3.2% per year, so that on the last day of each year, the amount in the savings account on that day is increased by 3.2%. He decided to invest \$8 000 on the first day of each year, starting from 2020.

(i) Show that the amount in the account at the end of n years after the interest has been added is given by $\$\,258\,000(1.032^n - 1)$. [3]

After 10 years, Mr Chan stopped investing in this savings account. If he does not withdraw any money from this savings account, the savings account will still continue to generate interest of 1.5% per year on the amount in the account, so that on the last day of each year the amount in the account on that day is increased by 1.5%. Assuming that Mr Chan does not withdraw any money from the savings account,

(ii) by the end of which year will the total interest be first more than \$35 000 from the day that Mr Chan first started his saving account? [3]

At the age of 55, Mr Chan is able to receive a monthly pay-out over a period of 20 years from the savings account if he did not withdraw before. The monthly pay-out in the first year is \$850. The monthly pay-out for each subsequent year is an increment of \$$D$ from the monthly pay-out of the previous year. The total pay-out to Mr Chan at the end of 20 complete years is \$352 200.

(iii) Given that the monthly pay-out is \$1 500 for the mth year of the 20 year period, where m is a positive integer, find m. [3]

8 **(i)** By using the substitution $x = \cos\theta$, find $\int \frac{x^3}{\sqrt{1-x^2}}\,\mathrm{d}x$, expressing your answer in terms of x. [4]

(ii) Verify that the curves with equations $y = \frac{x^3}{\sqrt{1-x^2}}$ and $y = \frac{1}{\sqrt{49-4x^2}}$ intersect at the point with x-coordinate $\frac{1}{2}$. [1]

(iii) Hence find the exact area bounded by the curves $y = \frac{x^3}{\sqrt{1-x^2}}$ and $y = \frac{1}{\sqrt{49-4x^2}}$, and the y-axis. [4]

9 **(a)** **(i)** Show that the cubic polynomial $x^3 + px^2 + p^2x + q$ can be reduced to $y^3 + \left(\frac{2p^2}{3}\right)y + \alpha$ by the substitution $x = y - \frac{p}{3}$, where α is to be determined in terms of p and q. [3]

(ii) Given that -3i is a root of the equation $y^3 + 6y - 9\text{i} = 0$, find the other two roots exactly in the form $a + b\text{i}$. [3]

(iii) Hence find the exact roots of the equation $x^3 + 3x^2 + 9x + 7 - 9\text{i} = 0$. [2]

(b) Given that $z = \text{e}^{\text{i}\theta}$, show that $1 + z + z^2 + z^3 + \cdots + z^{n-1} = z^{\frac{n-1}{2}}\left(\dfrac{\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}\right)$. [3]

10 The **Gompertz differential equation** provides a good model for gauging lung cancer growth. The differential equation is given as

$$\frac{\text{d}V}{\text{d}t} = aV\ln\left(\frac{K}{V}\right),$$

where V mm^3 is the volume of the tumour at time t days after an early discovery, and a and K are positive real constants.

(i) Describe the behaviour of $\frac{\text{d}V}{\text{d}t}$ as $V \to K$. [1]

(ii) By using the substitution $u = \ln\left(\frac{K}{V}\right)$, solve the given differential equation and show that $V = K\text{e}^{-A\text{e}^{-at}}$, where A is an arbitrary constant. [5]

(iii) What happens to V as $t \to \infty$? State the significance of K in the context of this question. [2]

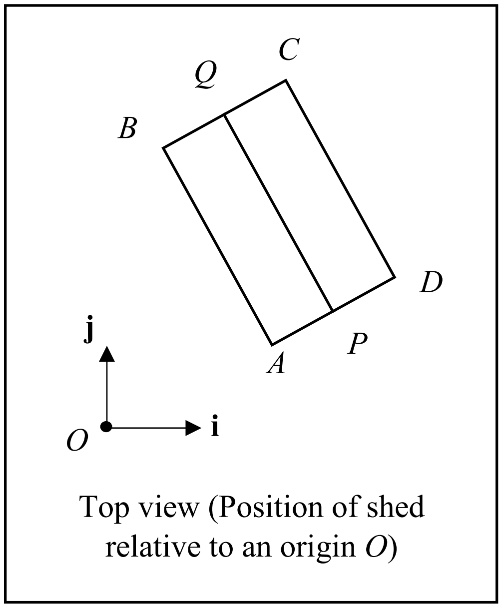
For the rest of this question, you may assume that $a = 0.01$ and $K = 8000$.

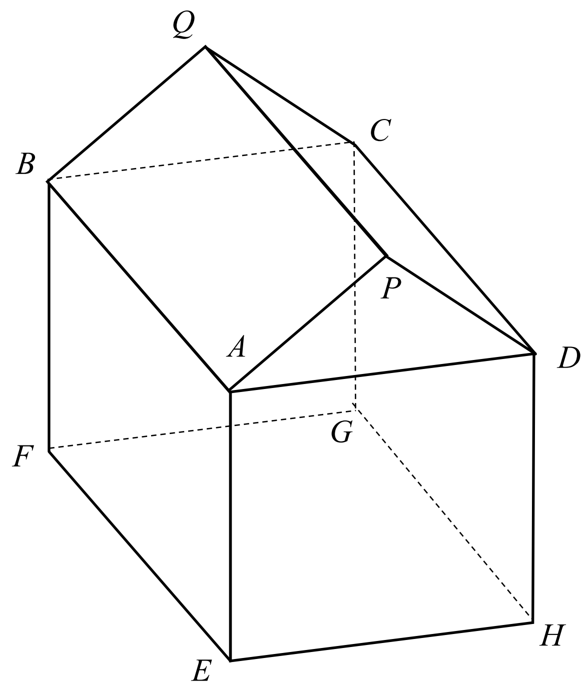
A patient was discovered early to have a lung tumour of volume 100 mm^3.

(iv) Find the size of the tumour after 100 days. Give your answer to the nearest mm^3. [2]

(v) Sketch the graph of V against t. [2]

11 Mr Neo wants to build a shed in his future garden as shown in the diagrams below.





The planes $ABCD$ and $EFGH$ are parallel to the horizontal plane, represented by the xy-plane. The triangles APD and BQC are congruent isosceles triangles and the lengths of the pillars AE, BF, CG and DH are of the same height.

The equation of the plane $ABQP$ is given by $-2x - y + 3z = 2$ and the point P has position vector $4\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$.

(i) Explain why plane $CDPQ$ is perpendicular to $2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$. Hence show that the equation of the plane $CDPQ$ is $2x + y + 3z = 22$. [2]

(ii) Find the angle the roof $ABQP$ makes with the horizontal plane. [2]

(iii) Find the vector equation of the line PQ and hence find the coordinates of the point Q given that PQ has length $3\sqrt{5}$ units. [5]

(iv) When the shed was completed, Mr Neo discovered a hole in the roof $ABQP$. When light shines perpendicularly onto the plane $ABQP$, the light passes through the hole and hits the ground at the point with coordinates (3, 6, 0).

Find the coordinates of the hole in the roof $ABQP$. [3]