2021 ACJC H2 Math Prelim P1 Marking Scheme

Qn	Solutions	
1(i)	$\mathrm{f}(x)=\dfrac{1-\mathrm{e}^{x}}{\mathrm{e}^{x-2}}$ $=\dfrac{1}{\mathrm{e}^{x-2}}-\dfrac{\mathrm{e}^{x}}{\mathrm{e}^{x-2}}$ $=\mathrm{e}^{-x+2}-\mathrm{e}^{2}$ $\therefore A=-\mathrm{e}^{2}$ and $\mathrm{g}(x)=-x+2$	
1(ii)	1. A translation of 2 units in the negative x-axis direction. 2. A reflection in the y-axis. 3. A translation of e^{2} units in the negative y-axis direction. **Alternative1:** 1. A reflection in the y-axis. 2. A translation of 2 units in the positive x-axis direction. 3. A translation of e^{2} units in the negative y-axis direction. **Alternative2:** $\mathrm{f}(x)=\mathrm{e}^{-x+2}-\mathrm{e}^{2}=\mathrm{e}^{2}\bullet\mathrm{e}^{-x}-\mathrm{e}^{2}$ 1. A reflection in the y-axis. 2. A Scaling parallel to the y-axis by the scale factor e^{2}. 3. A translation of e^{2} units in the negative y-axis direction. **Alternative3:** $\mathrm{f}(x)=\mathrm{e}^{-x+2}-\mathrm{e}^{2}=\mathrm{e}^{2}\left(\mathrm{e}^{-x}-1\right)$ 1. A reflection in the y-axis. 2. A translation of 1 unit in the negative y-axis direction. 3. A Scaling parallel to the y-axis by the scale factor e^{2}.	
2(i)	$\dfrac{x^{3}-2y^{2}}{x^{2}+3xy}=1$ $x^{3}-2y^{2}=x^{2}+3xy$ Differentiating implicitly wrt x: $3x^{2}-4y\dfrac{\mathrm{d}y}{\mathrm{d}x}=2x+\left(3x\dfrac{\mathrm{d}y}{\mathrm{d}x}+y(3)\right)$ $3x^{2}-2x-3y=\left(3x+4y\right)\dfrac{\mathrm{d}y}{\mathrm{d}x}$ $\Rightarrow\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{3x^{2}-2x-3y}{3x+4y}$ **Alternative (not advised)** Using quotient rule to differentiate:	

$$\frac{\left(x^2+3xy\right)\left(3x^2-4y\frac{\mathrm{d}y}{\mathrm{d}x}\right)-\left(x^3-2y^2\right)\left(2x+3x\frac{\mathrm{d}y}{\mathrm{d}x}+y(3)\right)}{\left(x^2+3xy\right)^2}=0$$

$$\Rightarrow \left(x^2+3xy\right)\left(3x^2-4y\frac{\mathrm{d}y}{\mathrm{d}x}\right)-\left(x^3-2y^2\right)\left(2x+3x\frac{\mathrm{d}y}{\mathrm{d}x}+y(3)\right)=0$$

(because $x^2+3xy\neq 0$)

$$\Rightarrow 3x^2\left(x^2+3xy\right)-\left(x^3-2y^2\right)\left(2x+3y\right)=4y\left(x^2+3xy\right)\frac{\mathrm{d}y}{\mathrm{d}x}+3x\left(x^3-2y^2\right)\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{3x^2\left(x^2+3xy\right)-\left(x^3-2y^2\right)\left(2x+3y\right)}{4y\left(x^2+3xy\right)+3x\left(x^3-2y^2\right)}=\frac{x^4+6x^3y+4xy^2+6y^3}{3x^4+4x^2y+6xy^2}$$

Sub $x=1$,

$$\frac{1^3-2y^2}{1^2+3(1)y}=1$$

$$\Rightarrow 1-2y^2=1+3y$$

$$\Rightarrow 2y^2+3y=0$$

$$\Rightarrow y(2y+3)=0 \quad \therefore y=0 \text{ or } y=-\frac{3}{2}$$

Sub $x=1$ and $y=0$ into $\frac{\mathrm{d}y}{\mathrm{d}x}$:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{3(1)^2-2(1)-3(0)}{3(1)+4(0)}=\frac{1}{3}$$

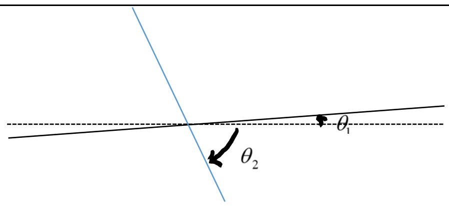
Sub $x=1$ and $y=-\frac{3}{2}$ into $\frac{\mathrm{d}y}{\mathrm{d}x}$:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{3(1)^2-2(1)-3\left(\frac{-3}{2}\right)}{3(1)+4\left(\frac{-3}{2}\right)}=-\frac{11}{6}$$

2(ii)

$$\theta_1=\tan^{-1}\left(\frac{1}{3}\right)$$

$$\theta_2=\tan^{-1}\left(\frac{11}{6}\right)$$



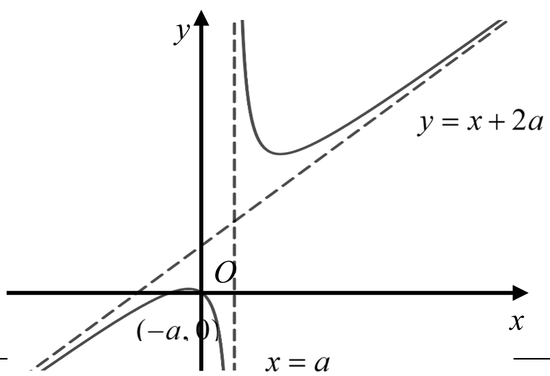
acute angle between tangents :

$$\theta_1+\theta_2=\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{11}{6}\right)=79.8^\circ \text{(to 1d.p.)} \quad \text{(or 1.39 rad)}$$

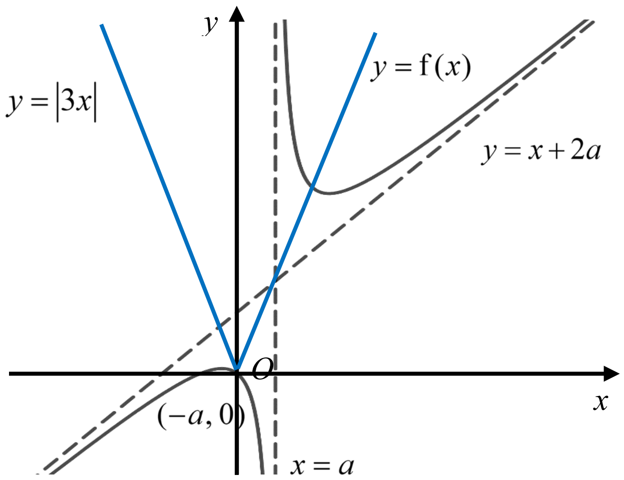
3(i)

$$\mathrm{f}(x)=\frac{a}{2-x}, \quad x\neq 0, 2$$

From graph, $R_\mathrm{f}=\left\{x\in\mathbb{R}: x\neq 0, \frac{a}{2}\right\}$

	$f^2(x) = f(f(x)) = \dfrac{a}{2-\dfrac{a}{2-x}} = \dfrac{a(2-x)}{4-a-2x}$ $f^2 : x \mapsto \dfrac{a(2-x)}{4-a-2x}, x \neq 0, 2$ Let $y = \dfrac{a}{2-x}$, $x \neq 0, 2$ $2 - x = \dfrac{a}{y} \Rightarrow x = 2 - \dfrac{a}{y}$ $\therefore f^{-1} : x \mapsto 2 - \dfrac{a}{x}, x \neq 0, \dfrac{a}{2}$	
3(ii)	For $f^2(x) = f^{-1}(x)$, $\dfrac{a(2-x)}{4-a-2x} = 2 - \dfrac{a}{x} \Rightarrow \dfrac{ax-2}{2x-4+a} = \dfrac{2x-a}{x}$ By observation, $a = 4$.	
3(iii)	$f^2(x) = f^{-1}(x) \Rightarrow f^3(x) = x$ Therefore $f^{2021}(x) = f^2 f^{2019}(x) = f^2(x) = \dfrac{2x-4}{x} = 2 - \dfrac{4}{x}$.	
4(i)	Consider $y = k$, k is a constant $\dfrac{x(x+a)}{x-a} = k$ $x^2 + ax = xk - ak$ $x^2 + (a-k)x + ak = 0$ For the range of y can take, the line $y = k$ and the curve C should have point(s) of intersection. $b^2 - 4ac \geq 0$ $(a-k)^2 - 4ak \geq 0$ $a^2 - 2ak + k^2 - 4ak \geq 0$ $a^2 - 6ak + k^2 \geq 0$ Consider $k^2 - 6ak + a^2 = 0$ $k = \dfrac{6a \pm \sqrt{36a^2 - 4a^2}}{2} = \left(3 \pm 2\sqrt{2}\right)a$ $\therefore k \geq \left(3+2\sqrt{2}\right)a$ or $k \leq \left(3-2\sqrt{2}\right)a$ Hence, $y \geq \left(3+2\sqrt{2}\right)a$ or $y \leq \left(3-2\sqrt{2}\right)a$	
4(ii)		

4(iii)



Consider $\frac{x(x+a)}{x-a} = 3x$

$$x^2 + ax = 3x^2 - 3ax$$
$$2x^2 - 4ax = 0$$
$$x(x-2a) = 0$$
$$x = 0 \quad or \quad x = 2a$$

Hence, $y = |3x|$ and $y = \mathrm{f}(x)$ intersect at $x = 0$ and $x = 2a$.

From the graph, $x < 0$ or $0 < x < a$ or $x > 2a$.

Checking:

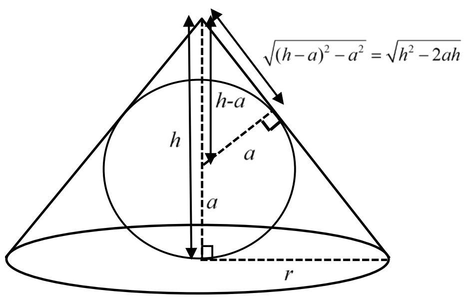
Consider $\frac{x(x+a)}{x-a} = -3x$

$$x^2 + ax = -3x^2 + 3ax$$
$$4x^2 - 2ax = 0$$
$$x(2x-a) = 0$$
$$x = 0 \quad or \quad x = \frac{a}{2} \text{ (N.A. since } a > 0\text{)}$$

5(i)

By similar triangles:

$$\frac{\sqrt{h^2 - 2ah}}{h} = \frac{a}{r}$$
$$\Rightarrow \frac{h^2 - 2ah}{h^2} = \frac{a^2}{r^2}$$
$$\Rightarrow \frac{h - 2a}{h} = \frac{a^2}{r^2}$$
$$\Rightarrow r^2 = \frac{a^2 h}{h - 2a} \text{ (shown)}$$



5(ii)

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{a^2 h}{h-2a}\right)h = \frac{1}{3}\pi a^2\left(\frac{h^2}{h-2a}\right)$$

$$\frac{\mathrm{d}V}{\mathrm{d}h} = \frac{1}{3}\pi a^2 \frac{(h-2a)(2h) - h^2(1)}{(h-2a)^2} = \frac{1}{3}\pi a^2 \frac{h^2 - 4ah}{(h-2a)^2} = \frac{1}{3}\pi a^2 \frac{h(h-4a)}{(h-2a)^2}$$

When V is minimum, $\frac{\mathrm{d}V}{\mathrm{d}h} = 0$

$\Rightarrow h(h-4a)=0$

$\therefore h=0$ (rej $\because h>0$) or $h=4a$

$$\frac{\mathrm{d}V}{\mathrm{d}h}=\frac{1}{3}\pi a^2\frac{h(h-4a)}{(h-2a)^2}=\frac{\pi a^2 h}{3(h-2a)^2}(h-4a)$$

$$\left(\frac{\pi a^2 h}{3(h-2a)^2}>0 \text{ for } h>0\right)$$

h	$(4a)^-$	$4a$	$(4a)^+$
$\frac{\mathrm{d}V}{\mathrm{d}h}$	$\frac{\mathrm{d}V}{\mathrm{d}h}<0$ for $h=(4a)^-$ $\left(\begin{array}{l}\because h<4a \\ \Rightarrow h-4a<0\end{array}\right)$	0	$\frac{\mathrm{d}V}{\mathrm{d}h}>0$ for $h=(4a)^+$ $\left(\begin{array}{l}\because h>4a \\ \Rightarrow h-4a>0\end{array}\right)$
Shape	\	–	/

Or 2$^{\text{nd}}$ derivative test (by quotient rule):

$$\frac{\mathrm{d}^2V}{\mathrm{d}h^2}=\frac{\mathrm{d}}{\mathrm{d}h}\left(\frac{1}{3}\pi a^2\frac{h(h-4a)}{(h-2a)^2}\right)$$

$$=\frac{1}{3}\pi a^2\frac{(h-2a)^2(2h-4a)-h(h-4a)2(h-2a)}{(h-2a)^4}$$

$$=\frac{1}{3}\pi a^2\frac{2(h-2a)\left[(h-2a)(h-2a)-h(h-4a)\right]}{(h-2a)^4}$$

$$=\frac{2}{3}\pi a^2\frac{\left[\cancel{h^2}+4a^2\cancel{-4ah}\cancel{-h^2}+\cancel{4ah}\right]}{(h-2a)^3}$$

$$=\frac{8\pi a^4}{3(h-2a)^3}$$

$$\Rightarrow \left.\frac{\mathrm{d}^2V}{\mathrm{d}h^2}\right|_{h=4a}=\frac{8\pi a^4}{3(4a-2a)^3}=\frac{\pi a}{3}>0 \quad (\because a>0)$$

Or 2$^{\text{nd}}$ derivative test (by implicit differentiation):

$$\frac{\mathrm{d}V}{\mathrm{d}h}=\frac{\pi a^2 h}{3(h-2a)^2}(h-4a)$$

$$3(h-2a)^2\frac{\mathrm{d}V}{\mathrm{d}h}=\pi a^2\left(h^2-4ah\right)$$

$$6(h-2a)\frac{\mathrm{d}V}{\mathrm{d}h}+3(h-2a)^2\frac{\mathrm{d}^2V}{\mathrm{d}h^2}=\pi a^2(2h-4a)$$

When $h=4a$ $\left(\text{and } \frac{\mathrm{d}V}{\mathrm{d}h}=0\right)$,

	$0+3(4a-2a)^2\dfrac{\mathrm{d}^2V}{\mathrm{d}h^2}=\pi a^2\left(2(4a)-4a\right)$ $\Rightarrow\dfrac{\mathrm{d}^2V}{\mathrm{d}h^2}=\dfrac{4\pi a^3}{12a^2}=\dfrac{\pi a}{3}>0\quad(\because a>0)$ Therefore volume of cone is minimum when $h=4a$. $\therefore$ Minimum volume of the cone $=\dfrac{1}{3}\pi a^2\left(\dfrac{(4a)^2}{4a-2a}\right)=\dfrac{8}{3}\pi a^3$	
6(i)	$\mathrm{e}^y=2+\sin x$ Differentiating w.r.t. x, $\mathrm{e}^y\dfrac{\mathrm{d}y}{\mathrm{d}x}=\cos x$ Differentiating w.r.t. x again, $\mathrm{e}^y\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+\mathrm{e}^y\dfrac{\mathrm{d}y}{\mathrm{d}x}\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)=-\sin x$ $\Rightarrow\mathrm{e}^y\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+\mathrm{e}^y\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2=2-\mathrm{e}^y\quad\left(\because \mathrm{e}^y=2+\sin x\Rightarrow-\sin x=2-\mathrm{e}^y\right)$ $\Rightarrow\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2=2\mathrm{e}^{-y}-1$ (shown) **Alternative Method 1:** $\mathrm{e}^y=2+\sin x\Rightarrow y=\ln(2+\sin x)$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\cos x}{2+\sin x}\Rightarrow(2+\sin x)\dfrac{\mathrm{d}y}{\mathrm{d}x}=\cos x......(1)$ $(2+\sin x)\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+\dfrac{\mathrm{d}y}{\mathrm{d}x}(\cos x)=-\sin x$ ($\because \mathrm{e}^y=2+\sin x$ and fr (1) $\cos x=(2+\sin x)\dfrac{\mathrm{d}y}{\mathrm{d}x}=\mathrm{e}^y\dfrac{\mathrm{d}y}{\mathrm{d}x}$) $\mathrm{e}^y\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+\dfrac{\mathrm{d}y}{\mathrm{d}x}\left(\mathrm{e}^y\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)=2-\mathrm{e}^y$ $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2=2\mathrm{e}^{-y}-1$ (shown) **Alternative Method 2:** $\mathrm{e}^y=2+\sin x\Rightarrow y=\ln(2+\sin x)$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\cos x}{2+\sin x}$ $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=\dfrac{(2+\sin x)(-\sin x)-\cos x(\cos x)}{(2+\sin x)^2}$ $=\dfrac{-2\sin x-\sin^2x-\cos^2x}{(2+\sin x)^2}$ $=\dfrac{-2\sin x-1}{(2+\sin x)^2}$	

	$\text{LHS}=\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2=\dfrac{-2\sin x-1}{(2+\sin x)^2}+\left(\dfrac{\cos x}{2+\sin x}\right)^2$ $=\dfrac{-2\sin x-1+\cos^2 x}{(2+\sin x)^2}$ $=\dfrac{-2\sin x\cancel{-1}\cancel{+1}-\sin^2 x}{(2+\sin x)^2}$ $=\dfrac{-\sin x(2+\sin x)}{(2+\sin x)^2}$ $=\dfrac{-\sin x}{2+\sin x}=-1+\dfrac{2}{2+\sin x}$ $=-1+\dfrac{2}{\mathrm{e}^y}=2\mathrm{e}^{-y}-1=\text{RHS}$ (shown)	
6(ii)	Differentiate $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2=2\mathrm{e}^{-y}-1$ implicitly w.r.t. x: $\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}+2\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=-2\mathrm{e}^{-y}\dfrac{\mathrm{d}y}{\mathrm{d}x}$ When $x=0$, $\mathrm{e}^y=2+\sin 0\Rightarrow y=\ln 2$ $\mathrm{e}^y\dfrac{\mathrm{d}y}{\mathrm{d}x}=\cos x\Rightarrow(2)\dfrac{\mathrm{d}y}{\mathrm{d}x}=\cos 0\Rightarrow\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{1}{2}$ $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2=2\mathrm{e}^{-y}-1\Rightarrow\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(\dfrac{1}{2}\right)^2=2\left(\dfrac{1}{2}\right)-1\Rightarrow\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\dfrac{1}{4}$ $\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}+2\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=-2\mathrm{e}^{-y}\dfrac{\mathrm{d}y}{\mathrm{d}x}$ $\Rightarrow\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}+2\left(\dfrac{1}{2}\right)\left(-\dfrac{1}{4}\right)=-2\left(\dfrac{1}{2}\right)\left(\dfrac{1}{2}\right)\Rightarrow\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}=-\dfrac{1}{4}$ $\therefore y=\ln 2+\left(\dfrac{1}{2}\right)x+\dfrac{\left(-\dfrac{1}{4}\right)}{2!}x^2+\dfrac{\left(-\dfrac{1}{4}\right)}{3!}x^3+\ldots$ $=\ln 2+\dfrac{1}{2}x-\dfrac{1}{8}x^2-\dfrac{1}{24}x^3+\ldots$	
6(iii)	$\mathrm{e}^y=2+\sin x$ $y=\ln(2+\sin x)$ $\approx\ln\left(2+x-\dfrac{x^3}{6}\right)=\ln\left(2\left(1+\dfrac{x}{2}-\dfrac{x^3}{12}\right)\right)$ $=\ln 2+\ln\left(1+\dfrac{x}{2}-\dfrac{x^3}{12}\right)$ $=\ln 2+\left(\dfrac{x}{2}-\dfrac{x^3}{12}\right)-\dfrac{\left(\dfrac{x}{2}-\dfrac{x^3}{12}\right)^2}{2}+\dfrac{\left(\dfrac{x}{2}-\dfrac{x^3}{12}\right)^3}{3}+\ldots$ $\approx\ln 2+\dfrac{x}{2}-\dfrac{x^3}{12}-\dfrac{x^2}{8}+\dfrac{x^3}{24}=\ln 2+\dfrac{x}{2}-\dfrac{x^2}{8}-\dfrac{x^3}{24}$ (verified)	

7(i)	*n*th year / Beginning / End table below		

nth year	Beginning	End
1	8000	8000(1.032)
2	8000(1.032)+8000	$8000(1.032)^2$+8000(1.032)
3	$8000(1.032)^2$+8000(1.032) +8000	$8000(1.032)^3+8000(1.032)^2$ +8000(1.032)
…		…
n		$8000(1.032)^n+8000(1.032)^{n-1}$ +….$8000(1.032)^2$+8000(1.032)

$$\text{Total amount} = 8000(1.032)+8000(1.032)^2+....+8000(1.032)^n$$

$$= \frac{8000(1.032)\left(1.032^n-1\right)}{1.032-1}$$

$$= 258000\left(1.032^n-1\right)$$

7(ii)

After 10 years, total amount in the account $= 258000\left(1.032^{10}-1\right)$

$= 95522.18995$

Let k be the additional number of years required.

$$95522.18995\left(1.015\right)^k \geq 8000\times10+35000$$

$$\left(1.015\right)^k \geq 1.2039$$

$$k \geq \frac{\ln 1.2039}{\ln 1.015} \quad \Rightarrow \quad k \geq 12.5$$

$$2020+10+13-1=2042$$

Hence, he will have a total interest be first more than $35 000 at the end of 2042.

7(iii)

Year	Pay-out per month	Pay-out per year
1	850	12(850)
2	850+D	12(850+D)
3	850+2D	12(850+2D)
⋮	⋮	

$$\text{Total pay-out} = 12\times\frac{20}{2}\left(2\times850+(20-1)D\right)$$

$$120\left(1700+(20-1)D\right)=352200$$

$$19D=1235$$

$$D=65$$

To find the year m with a pay-out of $1500:

$$850+(m-1)(65)=1500$$

$$(m-1)(65)=650$$

$$m=11$$

$\therefore$ The pay-out is $1500 in the 11^{th} year.

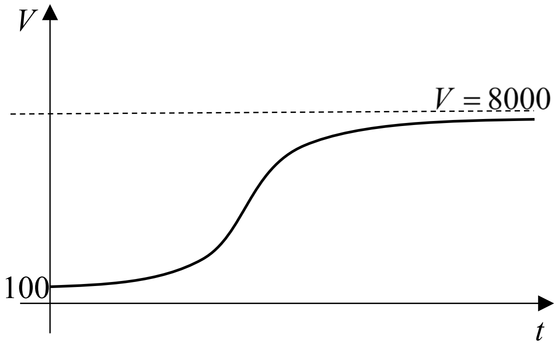
8(i)

$$x=\cos\theta \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}\theta}=-\sin\theta$$

	$\int \frac{x^3}{\sqrt{1-x^2}}\,\mathrm{d}x = \int \frac{\cos^3\theta}{\sqrt{1-\cos^2\theta}}(-\sin\theta)\,\mathrm{d}\theta$ $= -\int \frac{\cos^3\theta}{\sin\theta}(\sin\theta)\,\mathrm{d}\theta$ $= -\int \cos^3\theta\,\mathrm{d}\theta = -\int \cos^2\theta\cos\theta\,\mathrm{d}\theta$ $= -\int \left(1-\sin^2\theta\right)\cos\theta\,\mathrm{d}\theta = -\int \cos\theta - \cos\theta\sin^2\theta\,\mathrm{d}\theta$ $= -\left(\sin\theta - \frac{\sin^3\theta}{3}\right) + c = \frac{\sin^3\theta}{3} - \sin\theta + c$ $\cos\theta = x \;\Rightarrow\; \cos^2\theta = x^2 \;\Rightarrow\; 1-\sin^2\theta = x^2$ $\therefore \sin\theta = \sqrt{1-x^2}$ $\int \frac{x^3}{\sqrt{1-x^2}}\,\mathrm{d}x = \frac{\sin^3\theta}{3} - \sin\theta + c$ $= \frac{1}{3}\left(1-x^2\right)^{\frac{3}{2}} - \left(1-x^2\right)^{\frac{1}{2}} + c$	
8(ii)	$y = \frac{\left(\frac{1}{2}\right)^3}{\sqrt{1-\left(\frac{1}{2}\right)^2}} = \frac{\left(\frac{1}{2}\right)^3}{\frac{\sqrt{3}}{2}} = \frac{1}{4\sqrt{3}}$ $\quad y = \frac{1}{\sqrt{49-4\left(\frac{1}{2}\right)^2}} = \frac{1}{\sqrt{49-1}} = \frac{1}{4\sqrt{3}}$ Hence, the 2 curves intersect at the point $\left(\frac{1}{2}, \frac{1}{4\sqrt{3}}\right)$.	
8(iii)	[Using GC to identify which graph is 'on top'] $\int_0^{\frac{1}{2}} \frac{1}{\sqrt{49-4x^2}} - \frac{x^3}{\sqrt{1-x^2}}\,\mathrm{d}x$ $= \int_0^{\frac{1}{2}} \frac{1}{2\sqrt{\left(\frac{7}{2}\right)^2 - x^2}} - \frac{x^3}{\sqrt{1-x^2}}\,\mathrm{d}x$ $= \left[\frac{1}{2}\sin^{-1}\frac{2x}{7} - \left(-\frac{1}{3}\left(\sqrt{1-x^2}\right)\left(x^2+2\right)\right)\right]_0^{\frac{1}{2}}$ $= \left[\frac{1}{2}\sin^{-1}\frac{2x}{7} + \frac{1}{3}\left(\sqrt{1-x^2}\right)\left(x^2+2\right)\right]_0^{\frac{1}{2}}$ $= \frac{1}{2}\sin^{-1}\frac{1}{7} + \frac{1}{3}\sqrt{\frac{3}{4}}\left(\frac{9}{4}\right) - 0 - \frac{2}{3}$ $= \left(\frac{1}{2}\sin^{-1}\frac{1}{7} + \frac{9\sqrt{3}-16}{24}\right)\text{units}^2 \quad \left(\text{or } \frac{1}{2}\sin^{-1}\frac{1}{7} + \frac{9}{8\sqrt{3}} - \frac{2}{3}\right)$	
9(a)(i)	Sub $x = y - \frac{p}{3}$ into $x^3 + px^2 + p^2x + q$	

	$\left(y-\frac{p}{3}\right)^3+p\left(y-\frac{p}{3}\right)^2+p^2\left(y-\frac{p}{3}\right)+q$ $=\left(y^3-3\left(\frac{p}{3}\right)y^2+3\left(\frac{p}{3}\right)^2y-\left(\frac{p}{3}\right)^3\right)$ $\quad+p\left(y^2-2\left(\frac{p}{3}\right)y+\left(\frac{p}{3}\right)^2\right)$ $\quad\quad+p^2\left(y-\frac{p}{3}\right)+q$ $=y^3+y^2\left(-p+p\right)+y\left(\frac{p^2}{3}-\frac{2p^2}{3}+p^2\right)+\left(-\frac{p^3}{27}+\frac{p^3}{9}-\frac{p^3}{3}+q\right)$ $=y^3+\frac{2p^2}{3}y+\left(q-\frac{7p^3}{27}\right)$ $\therefore\alpha=q-\frac{7p^3}{27}$	
9(a)(ii)	$-3\mathrm{i}$ is a root of the equation $y^3+6y-9\mathrm{i}=0$ $y^3+6y-9\mathrm{i}=(y-(-3\mathrm{i}))\left(y^2+ay+b\right)=(y+3\mathrm{i})\left(y^2+ay+b\right)$ Compare coefficients $y^0:-9i=3\mathrm{i}b\Rightarrow b=-3$ $y^2:0=a+3\mathrm{i}\Rightarrow a=-3\mathrm{i}$ $y^3+6y-9\mathrm{i}=(y+3\mathrm{i})\left(y^2-3\mathrm{i}y-3\right)$ For roots of $y^2-3\mathrm{i}y-3=0$ $y^2-3\mathrm{i}y-3$ $y=\frac{-(-3\mathrm{i})\pm\sqrt{(-3\mathrm{i})^2-4(1)(-3)}}{2}=\frac{3\mathrm{i}\pm\sqrt{3}}{2}$ Therefore, the other two roots are $\frac{3\mathrm{i}+\sqrt{3}}{2}$ and $\frac{3\mathrm{i}-\sqrt{3}}{2}$.	
9(a)(iii)	For $x^3+3x^2+9x+7-9\mathrm{i}=0$, let $p=3, q=7-9\mathrm{i}$, then $\frac{2p^2}{3}=6,\quad \alpha=\left(7-9\mathrm{i}\right)-\frac{7(3)^3}{27}=-9\mathrm{i}$ gives the equation in (ii). Hence roots of $x^3+3x^2+9x+7-9\mathrm{i}=0$ are $x=y-1=-3\mathrm{i}-1,\frac{3\mathrm{i}+\sqrt{3}-2}{2},\frac{3\mathrm{i}-\sqrt{3}-2}{2}$	

9(b)	$1+z+z^2+z^3+\cdots+z^{n-1}$ $=\dfrac{z^n-1}{z-1}=\dfrac{\mathrm{e}^{\mathrm{i}n\theta}-1}{\mathrm{e}^{\mathrm{i}\theta}-1}$ $=\dfrac{\mathrm{e}^{\mathrm{i}\frac{n\theta}{2}}\left(\mathrm{e}^{\mathrm{i}\frac{n\theta}{2}}-\mathrm{e}^{-\mathrm{i}\frac{n\theta}{2}}\right)}{\mathrm{e}^{\mathrm{i}\frac{\theta}{2}}\left(\mathrm{e}^{\mathrm{i}\frac{\theta}{2}}-\mathrm{e}^{-\mathrm{i}\frac{\theta}{2}}\right)}$ $=\dfrac{\mathrm{e}^{\mathrm{i}\frac{(n-1)\theta}{2}}\left(2\mathrm{i}\sin\dfrac{n\theta}{2}\right)}{2\mathrm{i}\sin\dfrac{\theta}{2}}\quad\left(\begin{array}{c}\because \mathrm{e}^{\mathrm{i}\alpha}-\mathrm{e}^{-\mathrm{i}\alpha}=\cos\alpha+\mathrm{i}\sin\alpha\\ -(\cos\alpha-\mathrm{i}\sin\alpha)\\ =2\mathrm{i}\sin\alpha\end{array}\right)$ $=z^{\frac{n-1}{2}}\left(\dfrac{\sin\dfrac{n\theta}{2}}{\sin\dfrac{\theta}{2}}\right)$	
10(i)	As $V\to K$, $\ln\left(\dfrac{K}{V}\right)\to 0$. $\therefore \dfrac{\mathrm{d}V}{\mathrm{d}t}\to 0$	
10(ii)	$u=\ln\left(\dfrac{K}{V}\right)\Rightarrow \dfrac{\mathrm{d}u}{\mathrm{d}t}=\left(\dfrac{V}{K}\right)\left(-\dfrac{K}{V^2}\right)\dfrac{\mathrm{d}V}{\mathrm{d}t}$ $\Rightarrow \dfrac{\mathrm{d}V}{\mathrm{d}t}=-V\dfrac{\mathrm{d}u}{\mathrm{d}t}$ $\dfrac{\mathrm{d}V}{\mathrm{d}t}=aV\ln\left(\dfrac{K}{V}\right)$ $-V\dfrac{\mathrm{d}u}{\mathrm{d}t}=aVu$ $\dfrac{\mathrm{d}u}{\mathrm{d}t}=-au$ $\int\dfrac{1}{u}\,\mathrm{d}u=\int -a\,\mathrm{d}t$ $\int\dfrac{1}{u}\,\mathrm{d}u=\int -a\,\mathrm{d}t$ $\ln\lvert u\rvert=-at+c\Rightarrow \lvert u\rvert=\mathrm{e}^{-at+c}$ $u=A\mathrm{e}^{-at}$, where $A=\pm\mathrm{e}^{c}$ $\ln\left(\dfrac{K}{V}\right)=A\mathrm{e}^{-at}$ $\dfrac{K}{V}=\mathrm{e}^{A\mathrm{e}^{-at}}\Rightarrow V=K\mathrm{e}^{-A\mathrm{e}^{-at}}$	
10(iii)	As $t\to\infty$, $\mathrm{e}^{-at}\to 0$ and hence $\mathrm{e}^{-A\mathrm{e}^{-at}}\to \mathrm{e}^{0}=1$ $\therefore V\to K$ The size of the lung tumour approaches K $\left(\mathrm{mm}^3\right)$. K is the maximum possible size of the lung tumour that can be achieved.	
10(iv)	$V=8000\mathrm{e}^{-A\mathrm{e}^{-0.01t}}$	

	When $t=0,\ V=100$. $100=8000e^{-Ae^{0}} \Rightarrow \dfrac{1}{80}=e^{-A} \Rightarrow A=\ln 80$. When $t=100$, $V=8000e^{-\ln 80 e^{-1}}=1595.81\approx 1596\text{ mm}^3$	
10(v)	 $V=8000e^{-\ln 80 e^{-0.01t}}$	
11(i)	*ABQP* and *CDPQ* are symmetric about the vertical plane passing through *PQ*. Hence direction of the normal is $2\mathbf{i}+\mathbf{j}+3\mathbf{k}$. $\mathbf{r}.\begin{pmatrix}2\\1\\3\end{pmatrix}=\begin{pmatrix}4\\2\\4\end{pmatrix}.\begin{pmatrix}2\\1\\3\end{pmatrix}=22$. $\quad 2x+y+3z=22$.	
11(ii)	Normal of horizontal plane is **k**. Therefore $\cos\theta=\dfrac{\begin{pmatrix}-2\\-1\\3\end{pmatrix}.\begin{pmatrix}0\\0\\1\end{pmatrix}}{\sqrt{2^2+1^2+3^2}\sqrt{1}}=\dfrac{3}{\sqrt{14}}\Rightarrow\theta=36.7°$	
11(iii)	*PQ* is the line of intersection of the planes *ABQP* and *CDPQ*. $-2x-y+3z=2$ $2x+y+3z=22$ Solving using GC gives $\mathbf{r}=\begin{pmatrix}5\\0\\4\end{pmatrix}+\lambda'\begin{pmatrix}-\frac{1}{2}\\1\\0\end{pmatrix}=\begin{pmatrix}5\\0\\4\end{pmatrix}+\lambda\begin{pmatrix}-1\\2\\0\end{pmatrix}$ Given $\left\|\overrightarrow{PQ}\right\|=3\sqrt{5}$,	

	$\overrightarrow{OQ}=\begin{pmatrix}5\\0\\4\end{pmatrix}+\lambda\begin{pmatrix}-1\\2\\0\end{pmatrix}\Rightarrow\overrightarrow{PQ}=\begin{pmatrix}5\\0\\4\end{pmatrix}+\lambda\begin{pmatrix}-1\\2\\0\end{pmatrix}-\begin{pmatrix}4\\2\\4\end{pmatrix}=\begin{pmatrix}1\\-2\\0\end{pmatrix}+\lambda\begin{pmatrix}-1\\2\\0\end{pmatrix}$ $\therefore\left\|\begin{pmatrix}1\\-2\\0\end{pmatrix}+\lambda\begin{pmatrix}-1\\2\\0\end{pmatrix}\right\|=3\sqrt{5}\Rightarrow\left\|(\lambda-1)\begin{pmatrix}-1\\2\\0\end{pmatrix}\right\|=3\sqrt{5}\Rightarrow\|\lambda-1\|\left\|\begin{pmatrix}-1\\2\\0\end{pmatrix}\right\|=3\sqrt{5}$ $\Rightarrow\left\|(\lambda-1)\right\|=3\Rightarrow\lambda=4\text{ or }-2$ $\Rightarrow\overrightarrow{OQ}=\begin{pmatrix}1\\8\\4\end{pmatrix}\text{ or }\begin{pmatrix}7\\-4\\4\end{pmatrix}$ (reject, from top view y-coordinate > 2) Alternatively, $\overrightarrow{OQ}=\overrightarrow{OP}\pm3\sqrt{5}\left(\widehat{\overrightarrow{PQ}}\right)$ $=\begin{pmatrix}4\\2\\4\end{pmatrix}\pm3\sqrt{5}\left(\frac{1}{\sqrt{1^2+2^2}}\begin{pmatrix}-1\\2\\0\end{pmatrix}\right)$ $=\begin{pmatrix}4\\2\\4\end{pmatrix}\pm3\begin{pmatrix}-1\\2\\0\end{pmatrix}=\begin{pmatrix}1\\8\\4\end{pmatrix}\text{ or }\begin{pmatrix}7\\-4\\4\end{pmatrix}$ (reject, from top view y-coordinate > 2)	
11(iv)	Let l be the line passing through $(3, 6, 0)$ and perpendicular to the plane *ABQP*. $l:\mathbf{r}=\begin{pmatrix}3\\6\\0\end{pmatrix}+\mu\begin{pmatrix}-2\\-1\\3\end{pmatrix}$ For foot of perpendicular on the *ABQP*, $\left(\begin{pmatrix}3\\6\\0\end{pmatrix}+\mu\begin{pmatrix}-2\\-1\\3\end{pmatrix}\right).\begin{pmatrix}-2\\-1\\3\end{pmatrix}=2$ $\Rightarrow(-6-6)+\mu(4+1+9)=2$ $\Rightarrow\mu=1$ $\Rightarrow\begin{pmatrix}3\\6\\0\end{pmatrix}+\begin{pmatrix}-2\\-1\\3\end{pmatrix}=\begin{pmatrix}1\\5\\3\end{pmatrix}$ Hence hole on roof has coordinates $(1, 5, 3)$. Alternatively, Let the point $(3, 6, 0)$ be R, then $\overrightarrow{RP}=\begin{pmatrix}4\\2\\4\end{pmatrix}-\begin{pmatrix}3\\6\\0\end{pmatrix}=\begin{pmatrix}1\\-4\\4\end{pmatrix}$	

	$\overrightarrow{RN} = \overrightarrow{RP}.\dfrac{\begin{pmatrix}-2\\-1\\3\end{pmatrix}}{\sqrt{2^2+1^2+3^2}}\dfrac{\begin{pmatrix}-2\\-1\\3\end{pmatrix}}{\sqrt{2^2+1^2+3^2}} = \dfrac{\begin{pmatrix}1\\-4\\4\end{pmatrix}.\begin{pmatrix}-2\\-1\\3\end{pmatrix}}{14}\begin{pmatrix}-2\\-1\\3\end{pmatrix} = \begin{pmatrix}-2\\-1\\3\end{pmatrix}$ Therefore, $\overrightarrow{ON} = \overrightarrow{OR} + \overrightarrow{RN} = \begin{pmatrix}3\\6\\0\end{pmatrix} + \begin{pmatrix}-2\\-1\\3\end{pmatrix} = \begin{pmatrix}1\\5\\3\end{pmatrix}$ Hence hole on roof has coordinates $(1, 5, 3)$.	