Section A: Pure Mathematics [40 marks]

1 Three digits are chosen from 0, 1, 2, …, 9 and arranged to form a 3-digit number, with no digit being repeated. The sum of the digits in the 3-digit number is 8. When the digits in the 3-digit number are reversed, the new number is 297 less than the original 3-digit number. Find all possibilities of the 3-digit number. [3]

2 Relative to the origin O, the points A, B, C have position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, such that $OACB$ is a quadrilateral. Let P, Q, R and S be the midpoints of the line segments OA, AC, CB and OB respectively.

(i) Show that $PQRS$ is a parallelogram. [2]

(ii) For any vectors $\mathbf{p}$ and $\mathbf{q}$, state the condition for $|\mathbf{p}+\mathbf{q}| = |\mathbf{p}| + |\mathbf{q}|$. [1]

(iii) Hence, by considering vector products, show that the area of $OACB$ is twice the area of $PQRS$. [3]

3 **(i)** It is given that $\mathrm{f}(r) = \dfrac{1}{(r-1)r!}$ for $r \geq 2, r \in \mathbb{Z}$.

Show that $\mathrm{f}(r) - \mathrm{f}(r+1) = \dfrac{r^2+1}{r(r-1)(r+1)!}$. [1]

The sum $\displaystyle\sum_{r=2}^{n} \frac{r^2+1}{r(r-1)(r+1)!} + 3^{-r}$ is denoted by S_n.

(ii) Find an expression for S_n in terms of n. [4]

(iii) Explain why S_∞ is a convergent series and state its value. [2]

4 A curve C has parametric equations

$$x = \ln(3+\theta), \qquad y = \cos^{-1}\left(\frac{\theta}{3}\right) \qquad \text{for } -3 < \theta \leq 3.$$

(i) Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\sqrt{\dfrac{3+\theta}{3-\theta}}$. What can be said about the tangent to C as $\theta \to 3$? [3]

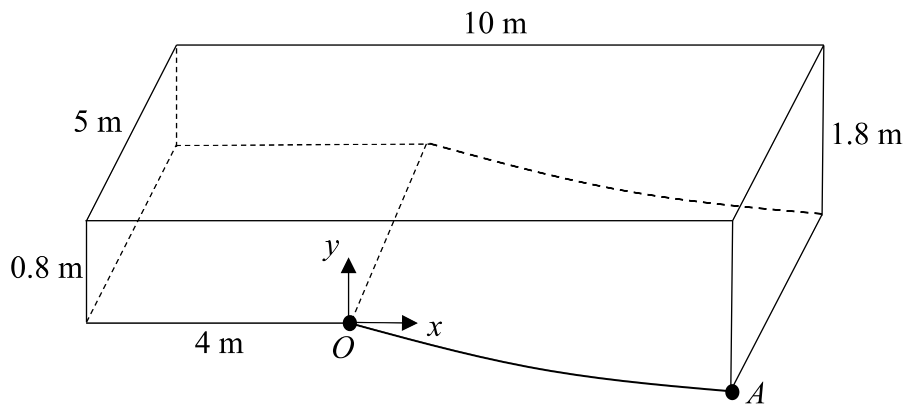
(ii) Sketch the curve C, stating the exact equations of any asymptotes and coordinates of any axial intercepts, showing clearly the feature of the curve at the point where $\theta = 3$. [3]

(iii) Show that the gradient of C at the point with y-coordinate p is $-\cot\dfrac{p}{2}$. [3]

(iv) The normal to C at the point with y-coordinate p passes through the point $(0,1)$.

Find the value of p. [3]

5 Tom wishes to build a pool in his backyard. The following diagram shows the dimensions of the pool he desires.



The surface of the pool is rectangular, while the floor consists of a rectangular piece and a sloped piece with a side from point O to point A as shown (with the depth of the pool gradually deepening from 0.8 m to 1.8 m). Taking O as the origin, the curve OA can be modelled by the equation $y=-\sqrt{hx+k}$, where h and k are real constants.

(i) Find the values of h and k. [2]

(ii) Hence find the volume of water required to fill the pool to the brim. [2]

Tom's wife, Jane, prefers a more artistic pool design, involving the curve C given by the equation $y=\mathrm{e}^{0.04x^2}-3$. Her preferred pool is formed by rotating the region bounded by the curve C and the x-axis π radians about the y-axis.

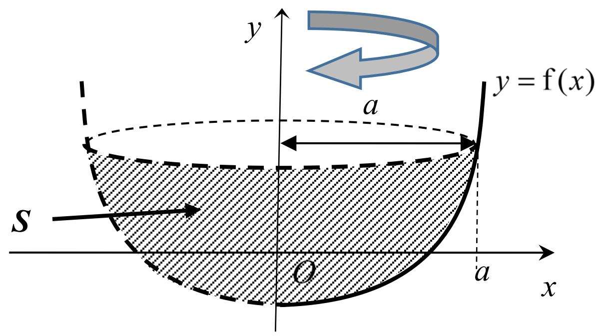
(iii) What is the exact volume of water required to fill Jane's preferred pool? Give your answer in the form $(a\ln b-c)\pi$ m^3, where a, b and c are real constants to be determined. [5]

5 [Continued]

For a curve $y = \mathrm{f}(x)$, where $0 \le x \le \alpha$, to be rotated 2π radians about the y-axis, the formula for the **surface area of the revolution** is given by

$$S = 2\pi\int_0^{\alpha} x\sqrt{1+\left(\mathrm{f}'(x)\right)^2}\,\mathrm{d}x\,,$$

where S is the surface area of the revolution and $\mathrm{f}'(x)$ is the derivative of $\mathrm{f}(x)$ with respect to x (see diagram below for illustration).



(iv) To ensure that there is no seepage in her preferred pool, Jane wants to laminate the entire curved side/floor of the pool. By using the formula given, find the area of laminating material required. Give your answer to the nearest m^2. [3]

Section B: Probability and Statistics [60 marks]

6 (a) The events A and B are such that $\mathrm{P}(A \cap B) = 0.03$ and $\mathrm{P}(A \cup B) = 0.37$.

(i) Find $\mathrm{P}\left[(A \cap B)' \mid (A \cup B)\right]$. [2]

It is given that events A and B are independent.

(ii) If $\mathrm{P}(A) > \mathrm{P}(B)$, find the possible values of $\mathrm{P}(A)$ and $\mathrm{P}(B)$. [3]

(b) The events X, Y and Z are such that events X and Z are mutually exclusive, $\mathrm{P}(X) = 0.2$, $\mathrm{P}(Y \cap Z) = 0.05$ and $\mathrm{P}(X' \cap Y' \cap Z') = 0.28$. Find the maximum and minimum possible values of $\mathrm{P}(Z)$. [3]

7 A group of 12 students consists of 3 students from class A, 4 students from class B and 5 students from class C.

(i) Find the number of ways in which a committee of 8 students can be chosen from the 12 students if it includes at least 1 student from each class. [2]

(ii) The 12 students from the 3 classes sit at random at a round table. Albert is a student from class A and Bob is a student from class B. Find the probability that Albert and Bob are seated together and no two students from class A are next to each other. [2]

(iii) Each of the 12 students attends one of 3 leadership programmes X, Y and Z. The table below shows the number of students from each class attending the various leadership programmes.

	Leadership Programmes		
	X	***Y***	***Z***
Class *A*	3	0	0
Class *B*	0	3	1
Class *C*	0	0	5

4 students are selected from the 12 students to participate in a group interview about the leadership programmes. They are arranged to sit in a row of 8 labelled seats such that there is exactly one empty seat between every 2 students as part of safe management measures.

Find the number of possible arrangements if each arrangement must include students from all 3 classes with representation from all 3 leadership programmes. [4]

8 3 discs are taken, at random and without replacement, from a bag containing 5 red discs and n white discs, where $n \geq 3$. The random variable R is the number of red discs taken and the random variable W is the number of white discs taken.

(i) Determine the probability distribution of W. [3]

(ii) Show that $\mathrm{E}(W) = \dfrac{3n}{n+5}$ and $\mathrm{Var}(W) = \dfrac{\mathrm{g}(n)}{(n+5)^2(n+4)}$, where $\mathrm{g}(n)$ is a quadratic polynomial to be determined. [5]

(iii) Hence write down an expression for $\mathrm{Var}(R)$. [1]

9 The Particle Filtration Efficiency (PFE) of a mask is a measure of how well a mask filters airborne particles such as pollen or dust. A mask with higher PFE is deemed to be of better efficiency as it filters more particles. A mask with a PFE of 95% would have met the requirement for surgical masks.

A factory manufactures Brand BEY surgical masks that is known to have expected PFE of 95.8%. During a routine check of the manufacturing process, the quality control manager suspects that the efficiency of the Brand BEY surgical masks produced is compromised such that the mean PFE is reduced. The PFE, x%, of a random sample of 50 masks is taken and the summarised results are as follows.

$$\sum(x-90) = 289 \qquad \sum(x-90)^2 = 1670.56$$

(i) State what it means for a sample to be random in this context. [1]

The manager carries out a hypothesis test at 1% level of significance.

(ii) Explain whether there is a need for the manager to make any assumption about the population distribution of the PFE of the masks. [1]

(iii) State the appropriate hypotheses, defining any symbols that you use. Test whether the manager's suspicion is justified at 1% level of significance. [5]

The manager discovered an error in the data collection process of the sample and discarded 10 out of the 50 readings. From the remaining random sample of 40 masks, the manager found that the mean PFE is 95.5% and the standard deviation is k %. Given that the manager concludes that there is insufficient evidence to justify his suspicion at 1% level of significance, find the range of possible values of k used in calculating the test statistic. [3]

10 On average, 91% of the people who have taken a vaccine will develop immunity to a particular virus. A random sample of n people who have taken the vaccine is chosen. The number of people in the sample who develop immunity after taking the vaccine is denoted by A.

(i) State, in context, what must be assumed for A to be well modelled by a binomial distribution. [2]

Assume now that A has a binomial distribution.

(ii) Given that the most likely number of people who develop immunity after taking the vaccine is 19, find the value of n without using a graphing calculator. [4]

(iii) Find the probability that in a randomly chosen sample of 25 people, more than 3 people did not develop immunity after taking the vaccine. [1]

A sample of 25 people with at most 21 people who developed immunity after taking the vaccine is deemed to be undesirable.

(iv) Find the probability that out of 30 randomly chosen samples of 25 people, there are at most 2 undesirable samples and the eighth chosen sample is the first undesirable sample. [3]

When a randomly chosen patient develops immunity to the virus after vaccination, there is a 30% chance that he exhibits an allergic reaction. When a randomly chosen patient does not develop immunity to the virus after vaccination, there is a 10% chance that he exhibits an allergic reaction.

(v) Find the probability that a randomly chosen patient who exhibited an allergic reaction after vaccination has developed immunity to the virus. [2]

11 In this question you should state the parameters of any distribution that you use. You should also assume that X, Y and S follow independent normal distributions.

Billy leaves his workplace X minutes past 6 pm daily from Monday to Friday to pick his son up from the childcare centre before its closing time at 7 pm. X follows the distribution $\mathrm{N}(12,3^2)$. The time taken for the journey from his workplace to the childcare centre, Y minutes, follows the distribution $\mathrm{N}(m,10^2)$, where m is a positive constant. The childcare centre imposes a penalty fine on parents who arrive at the childcare centre after 7 pm.

Given that Billy pays a penalty fine 22.176% of the time, show that $m = 40.0$. [3]

For the rest of this question, assume that $m = 40.0$.

(i) The time in minutes after 6 pm at which Billy arrives at the childcare centre each day is denoted by W.

Sketch the distribution of W for the period from 6.20 pm to 7.20 pm. [2]

(ii) Find the latest time Billy has to leave his workplace in order for him to have at least a 98% chance that he will not have to pay a penalty fine. [2]

(iii) In the month of July, the childcare centre is in operation on 22 workings days. It is given that on n randomly chosen working days in July, the probability that Billy's mean journey time from his workplace to the childcare centre exceeds 42 minutes is at most 0.2. Find the possible values of n. [3]

Billy's workplace is relocated to a new address and the time taken for the journey from his new workplace to the childcare centre, S minutes, now follows an independent distribution $\mathrm{N}(65,6^2)$. The childcare centre imposes a penalty fine of \$1.50 per late minute on parents who arrive at the childcare centre after 7 pm. Each day, Billy leaves his workplace X minutes past 6 pm to pick his son up from the childcare centre.

(iv) Find the distribution of T, where T is the time in minutes after 7 pm at which Billy arrives at the childcare centre on a randomly chosen day. Hence find the probability that the total penalty fine paid by Billy for 10 randomly chosen days is more than \$180. [3]