2023 H2 Math Prelims Paper 2 Solutions

<table>
<tr><th>No.</th><th>Solutions</th><th></th></tr>
<tr><td>1</td><td>
Substitute $z = k + k\mathrm{i}$ into the equation
$3\left(k+k\mathrm{i}\right)^2 - (5+\mathrm{i})\left(k+k\mathrm{i}\right) + \alpha = 0$
$3k^2\left(1+\mathrm{i}\right)^2 - k(5+\mathrm{i})\left(1+\mathrm{i}\right) + \alpha = 0$
$3k^2\left(1+2\mathrm{i}+\mathrm{i}^2\right) - k(5+5\mathrm{i}+5\mathrm{i}+\mathrm{i}^2) + \alpha = 0$
$3k^2(2\mathrm{i}) - k(4+6\mathrm{i}) + \alpha = 0$ ----------(*)
$\left(\alpha - 4k\right) + \left(6k^2 - 6k\right)\mathrm{i} = 0$

Comparing real and imaginary parts:
$\alpha - 4k = 0$ ----(1)
$6k^2 - 6k = 0$ ----(2)
$6k\left(k-1\right) = 0$
$k = 1$ or $k = 0$ (rejected since k is non-zero)

Substitute $k = 1$ into (1)
$\alpha = 4$
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Let the other root be w
$3z^2 - (5+\mathrm{i})z + 4 = 3\left(z-w\right)\left(z-1-\mathrm{i}\right)$
Comparing constants:
$4 = 3(-w)(-1-i)$
$w = \dfrac{4}{3\left(1+\mathrm{i}\right)}$
$= \dfrac{4\left(1-\mathrm{i}\right)}{3\left(1+\mathrm{i}\right)\left(1-\mathrm{i}\right)}$
$= \dfrac{2}{3} - \dfrac{2}{3}\mathrm{i}$
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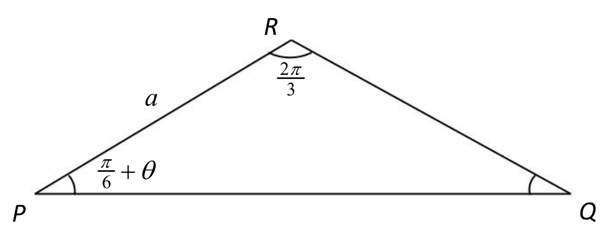
<tr><td>2(i)</td><td>


$\angle PQR = \pi - \dfrac{2\pi}{3} - \left(\dfrac{\pi}{6} + \theta\right)$
$= \dfrac{\pi}{6} - \theta$
Using Sine Rule,
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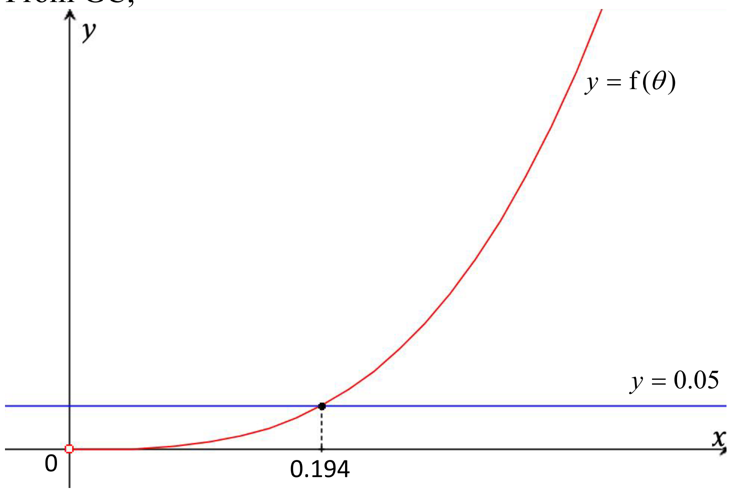
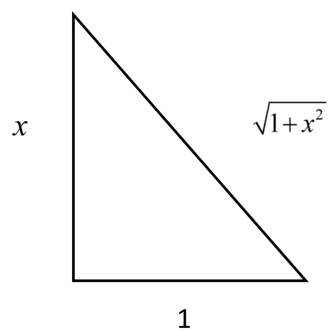
	$\dfrac{PQ}{\sin\frac{2\pi}{3}}=\dfrac{a}{\sin\left(\frac{\pi}{6}-\theta\right)}$ $PQ=\dfrac{a\sin\frac{2\pi}{3}}{\sin\frac{\pi}{6}\cos\theta-\cos\frac{\pi}{6}\sin\theta}$ $=\dfrac{a\left(\frac{\sqrt{3}}{2}\right)}{\frac{1}{2}\cos\theta-\frac{\sqrt{3}}{2}\sin\theta}$ $=\dfrac{\sqrt{3}\,a}{\cos\theta-\sqrt{3}\sin\theta}$	
2(ii)	$PQ=\dfrac{\sqrt{3}\,a}{\cos\theta-\sqrt{3}\sin\theta}$ $\approx\dfrac{\sqrt{3}\,a}{1-\frac{\theta^2}{2}-\sqrt{3}\,\theta}$ $=\sqrt{3}\,a\left(1-\sqrt{3}\,\theta-\frac{\theta^2}{2}\right)^{-1}$ $=\sqrt{3}\,a\left(1+(-1)\left(-\sqrt{3}\,\theta-\frac{\theta^2}{2}\right)+\frac{(-1)(-1)}{2!}\left(-\sqrt{3}\,\theta-\frac{\theta^2}{2}\right)^2+\cdots\right)$ $=\sqrt{3}\,a\left(1+\left(\sqrt{3}\,\theta+\frac{\theta^2}{2}\right)+\left(3\theta^2+\cdots\right)+\cdots\right)$ $\approx\sqrt{3}\,a\left(1+\sqrt{3}\,\theta+\frac{7}{2}\theta^2\right)$	
2(iii)	Let $\mathrm{f}(\theta)=\left\lvert\dfrac{\sqrt{3}\,a\left(1+\sqrt{3}\,\theta+\frac{7}{2}\theta^2\right)-\frac{\sqrt{3}\,a}{\cos\theta-\sqrt{3}\sin\theta}}{\frac{\sqrt{3}\,a}{\cos\theta-\sqrt{3}\sin\theta}}\right\rvert$ $=\left\lvert\dfrac{\left(1+\sqrt{3}\,\theta+\frac{7}{2}\theta^2\right)-\frac{1}{\cos\theta-\sqrt{3}\sin\theta}}{\frac{1}{\cos\theta-\sqrt{3}\sin\theta}}\right\rvert$ $\therefore \mathrm{f}(\theta)<0.05$	

	From GC,  $0 < \theta < 0.194 \quad \text{(3sf)}$	
3(a)	$\int \frac{1}{x^2\sqrt{x^2+1}}\,dx$ $x = \tan t,\ \frac{dx}{dt} = \sec^2 t$ $= \int \frac{1}{\tan^2 t\sqrt{\tan^2 t+1}}\left(\sec^2 t\right)dt$ $= \int \frac{1}{\tan^2 t(\sec t)}\left(\sec^2 t\right)dt$ $= \int \frac{\sec t}{\tan^2 t}\,dt$ $= \int \frac{\cos t}{\sin^2 t}\,dt$ $= \int (\cos t)(\sin t)^{-2}\,dt$  $= -\frac{1}{\sin t} + C$, where C is an arbitrary constant $= -\frac{\sqrt{1+x^2}}{x} + C,$ Alternatively, $\int \frac{\cos t}{\sin^2 t}\,dt = \int \operatorname{cosec} t \cot t\ dt$ $= -\operatorname{cosec} t + C$	
(b)(i)	$\frac{d}{dx}\left(\cos x^3\right) = -3x^2 \sin x^3$	

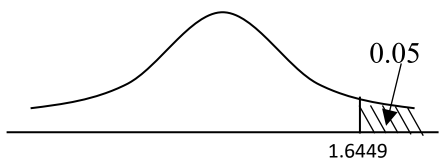
(ii)	$\int x^5 \sin x^3 \, dx$ $\quad u = x^3 \quad v' = x^2 \sin x^3$ $= -\frac{x^3}{3}\cos x^3 + \int x^2 \cos x^3 dx$ $\quad u' = 3x^2 \quad v = -\frac{1}{3}\cos x^3$ $= -\frac{x^3}{3}\cos x^3 + \frac{1}{3}\int 3x^2 \cos x^3 dx$ $= -\frac{x^3}{3}\cos x^3 + \frac{1}{3}\sin x^3 + C$	
4(i)	(i) $2xy + x - 9y = 0$ Differentiate with respect x, $2x\frac{dy}{dx} + 2y + 1 - 9\frac{dy}{dx} = 0$ $\frac{dy}{dx}(2x-9) = -2y-1$ $\frac{dy}{dx} = \frac{2y+1}{9-2x}$	
(ii)	(ii) Let $G = \frac{dy}{dx}$. $\therefore G = \frac{2y+1}{9-2x}$ Diff wrt x, $\frac{dG}{dx} = \frac{2\frac{dy}{dx}(9-2x)-(2y+1)(-2)}{(9-2x)^2}$ When $x=3$, $2xy + x - 9y = 0$ $\Rightarrow 6y + 3 - 9y = 0$ $\Rightarrow y = 1$ $\therefore \frac{dy}{dx} = \frac{2(1)+1}{9-2(3)} = 1$ Hence, when $x = 3$ $\frac{dG}{dt} = \frac{dG}{dx}\cdot\frac{dx}{dt}$ $= \frac{2(1)(9-2(3))-(2(1)+1)(-2)}{(9-2(3))^2} \times 0.02$ $= \frac{2}{75}$ or 0.0267 (3sf) Therefore, required rate is $\frac{2}{75}$ units/s.	

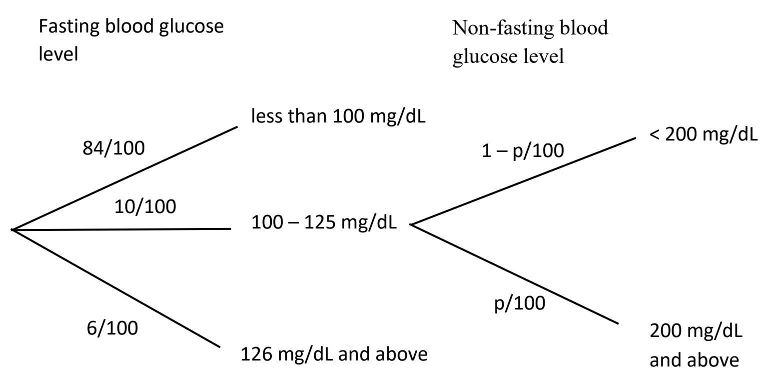
	(or 0.0267 units/s)	
5(i)	L.H.S. $= \dfrac{1}{r!} - \dfrac{1}{(r+1)!}$ $= \dfrac{(r+1)-1}{(r+1)!}$ $= \dfrac{r}{(r+1)!} =$ R.H.S. (Shown)	
(ii)	$S_n = \dfrac{1}{2!} + \dfrac{2}{3!} + \dfrac{3}{4!} + \ldots + \dfrac{n}{(n+1)!}$ $= \displaystyle\sum_{r=1}^{n} \dfrac{r}{(r+1)!} ---(1)$ $= \displaystyle\sum_{r=1}^{n} \left(\dfrac{1}{r!} - \dfrac{1}{(r+1)!} \right) ---(2)$ $= \dfrac{1}{1!} - \cancel{\dfrac{1}{2!}}$ $+ \cancel{\dfrac{1}{2!}} - \dfrac{1}{3!}$ • • • $+ \dfrac{1}{(n-1)!} - \cancel{\dfrac{1}{n!}}$ $+ \cancel{\dfrac{1}{(n)!}} - \dfrac{1}{(n+1)!}$ $= 1 - \dfrac{1}{(n+1)!}$	
(iii)	As $n \to \infty$, $\dfrac{1}{(n+1)!} \to 0$ $\therefore S_n \to 1$ Since the limit 1 is unique and finite, the series converges. $S_\infty = 1$	

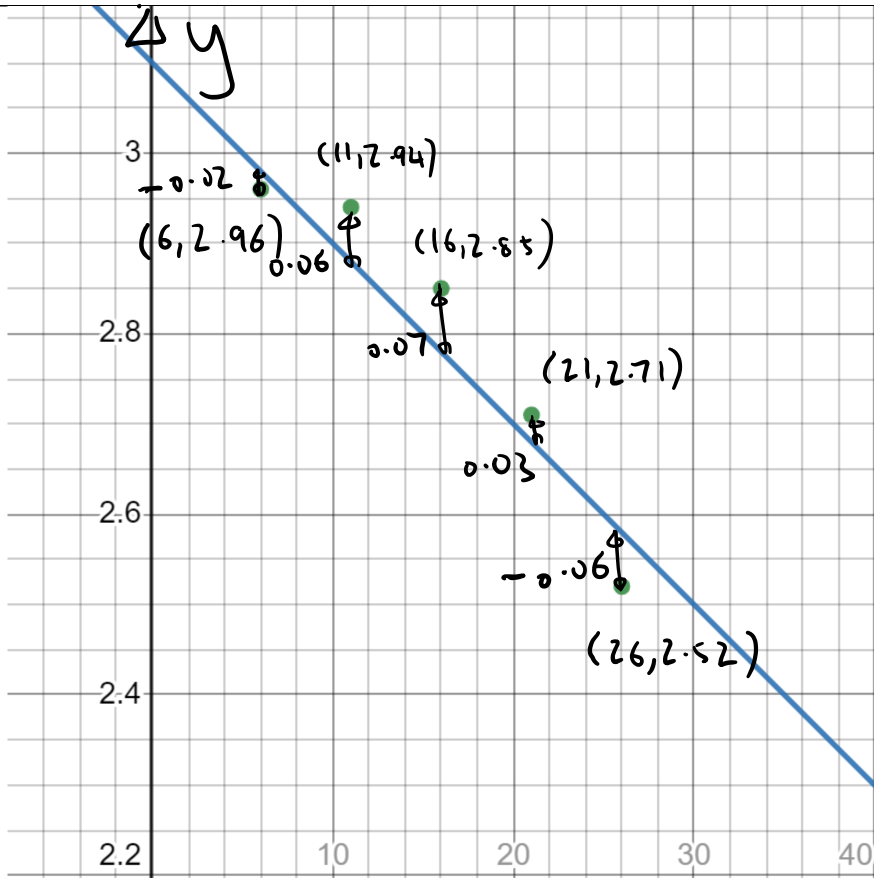
(iv)	Let $a_r = \dfrac{(r-1)^2}{(r+2)!}$ Let $b_r = \dfrac{r}{(r+1)!}$ $a_r \geq 0$ and $b_r \geq 0$ for all $r \in \mathbb{Z}^+$. $---(1)$ $b_r - a_r = \dfrac{r}{(r+1)!} - \dfrac{(r-1)^2}{(r+2)!}$ $= \dfrac{r(r+2)-(r-1)^2}{(r+2)!} = \dfrac{4r-1}{(r+2)!} > 0$ for $r \geq 1$ This implies that $b_r - a_r \geq 0$ for all $r \in \mathbb{Z}^+$. $---(2)$ Hence by using the comparison test, since $\sum_{r=1}^{\infty} \dfrac{r}{(r+1)!}$ converges then $\sum_{r=1}^{\infty} \dfrac{\left(r-1\right)^2}{(r+2)!}$ also converges.	
6(i)	The possible values of X are 2, 3, 4 and 5 $\text{P}(X=2) = \text{P}(RR) = \dfrac{5}{8} \times \dfrac{4}{7} = \dfrac{5}{14}$ $\text{P}(X=3) = \text{P}(RBR \text{ or } BRR) = \dfrac{5}{8} \times \dfrac{3}{7} \times 2! \times \dfrac{4}{6} = \dfrac{5}{14}$ $\text{P}(X=4)$ $= \text{P}(RBBR$, the first 3 balls can be in any order but last one must be $R)$ $= \dfrac{5}{8} \times \dfrac{3}{7} \times \dfrac{2}{6} \times \dfrac{3!}{2!} \times \dfrac{4}{5} = \dfrac{3}{14}$ $\text{P}(X=5)$ $= \text{P}(RBBBR$, the first 4 balls can be in any order but last one must be $R)$ $= \dfrac{5}{8} \times \dfrac{3}{7} \times \dfrac{2}{6} \times \dfrac{1}{5} \times \dfrac{4!}{3!} \times \dfrac{4}{4} = \dfrac{1}{14}$	

6(ii)	$\text{E}(X)=\sum_{\text{all } x} x\ \text{P}(X=x)$ $=2\times\frac{5}{14}+3\times\frac{5}{14}+4\times\frac{3}{14}+5\times\frac{1}{14}$ $=3$ $\text{Var}(X)=\text{E}(X^2)-[\text{E}(X)]^2$ $=2^2\times\frac{5}{14}+3^2\times\frac{5}{14}+4^2\times\frac{3}{14}+5^2\times\frac{1}{14}-[3]^2$ $=\frac{69}{7}-9=\frac{6}{7}$	
6(iii)	Expected profit = Expected Loss $\$y=\$2\times\text{P}(X=2)+\$3\times\text{P}(X=3)+\$4\times\text{P}(X=4)+\$5\times\text{P}(X=5)$ $y=2\times\frac{5}{14}+3\times\frac{5}{14}+4\times\frac{3}{14}+5\times\frac{1}{14}$ $y=3$ Or $\$y=\$1\times\text{E}(X)$ $y=3$	
7(i)	P(toy chosen is either a Triangle or a Star given not Yellow) $=\frac{\text{P(Triangle or Star and not Yellow)}}{\text{P(not Yellow)}}$ $=\frac{\frac{(4+2+3)+(5+3+1)}{40}}{\frac{(40-1-3-4-2)}{40}}=\frac{18}{30}=\frac{3}{5}$	
7(ii)a)	P(both toys chosen are purple and different shapes) = P(Purple Square, Purple Triangle) + P(Purple Star, Purple Triangle) +P(Purple Square, Purple Star) $=2\times\frac{2}{40}\times\frac{3}{39}+2\times\frac{1}{40}\times\frac{3}{39}+2\times\frac{2}{40}\times\frac{1}{39}$ $=\frac{11}{780}$	
7b)	Let A and B be Ben's two favourite combinations. Let $n_A\in\mathbb{Z}^+, 1\le n_A\le 5$ and $n_B\in\mathbb{Z}^+, 1\le n_B\le 5$ be the number of A and number of B respectively.	

	$\frac{n_A}{40}\times\frac{n_B}{39}\times 2=\frac{1}{39}$ $n_A n_B = 20$ $\therefore$ the possible cases given that $1\le n_A, n_B \le 5$ are: $n_A = 5, n_B = 4$ $n_A = 4, n_B = 5$ Thus, possible A and B are: Yellow Triangle, Green Star Yellow Triangle, Red Square Green Triangle, Red Square Green Triangle, Green Star	
8(i)	Let X be the random variable denoting the Calculus score of a randomly selected student and μ_1 be the population mean. Test $\mathrm{H}_0 : \mu_1 = 52$ against $\mathrm{H}_1 : \mu_1 \neq 52$ at 5% level of significance Unbiased estimate of the population variance s^2 $=\frac{n}{n-1}\times$sample variance $=\frac{30}{29}\times 15^2$ $=\frac{6750}{29}$ Under H_0, since $n = 30$ is large, by Central Limit Theorem, $\overline{X} \sim \mathrm{N}\left(52, \frac{6750}{(29)(30)}\right)$ approximately. Test statistic $Z=\frac{\overline{X}-52}{\sqrt{\frac{6750}{870}}} \sim \mathrm{N}(0,1)$ approximately. Using a 2-tailed z-test, reject H_0 if p-value ≤ 0.05 Using GC, the test statistic value $\overline{x} = 46$ and $z_{\text{calc}} = -2.1541$ gives p-value = 0.0312 < 0.05 We reject H_0 and conclude that there is sufficient evidence at the 5% level of significance that Professor's claim about the mean score is not valid.	

8(ii)	Let Y be the random variable denoting the Statistics score by a randomly selected student and μ be the population mean. Test $\mathrm{H}_0 : \mu = 48$ against $\mathrm{H}_1 : \mu > 48$ at 5% level of significance Under H_0 , since n is large, by Central Limit Theorem, $\overline{Y} \sim \mathrm{N}\left(48, \frac{13^2}{30}\right)$ approximately. Test statistic $Z = \frac{\overline{Y} - 48}{\sqrt{\frac{13^2}{30}}} \sim \mathrm{N}(0,1)$ approximately. Carry out 1-tailed z-test at the 5% level of significance. Since Professor B has understated the score, we will reject H_0 .  For H_0 to be rejected , $z_{calc} \geq 1.6449$ $\frac{k-48}{\sqrt{\frac{13^2}{30}}} \geq 1.6449$ $k - 48 \leq 3.9041$ $k \geq 51.904$ $\Rightarrow k \geq 52.0$ (Round in) [Accept $k \geq 51.9$]	
8(iii)	Yes. Since it is not known whether the Calculus score and Statistics score by a randomly selected student is normally distributed, it is important that the sample size is at least 30 in order to use Central Limit Theorem so that the mean Calculus score and the mean Statstics score by students are approximately normal.	

9(i)	The probability that a patient has diabetes mellitus is a constant at $\frac{1}{15}$ for each patient. A patient having diabetes mellitus is independent of any other patient having diabetes mellitus.	
(ii)	Let Y be the random variable "number of patients having diabetes mellitus out of 49. $Y \sim \mathrm{B}\left(49, \frac{1}{15}\right)$ Required Probability $= \mathrm{P}(Y=3)\,\mathrm{P}(\text{the 50th patient is the fourth patient who has diabetes})$ $= 0.2284556 \times \frac{1}{15} = 0.0152$	
(iii)	 Required Probability $= \frac{6}{100} + \frac{10}{100} \times \frac{p}{100}$ $= 0.06 + 0.001p$	
(iv)	P(100 – 125 mg/dL | not diagnosed with diabetes mellitus) $= \frac{\text{P(100 – 125 mg/dL} \cap \text{not diagnosed with diabetes mellitus)}}{\text{P(not diagnosed with diabetes mellitus)}}$ $= \frac{\frac{10}{100} \times \left(1 - \frac{p}{100}\right)}{1 - (0.06 + 0.001p)}$ $= \frac{0.1 - 0.001p}{0.94 - 0.001p} = \frac{100 - p}{940 - p}$	
(v)	Let W be the random variable "number of patients having diabetes mellitus out of n. When $p = 20$, $0.06 + 0.001p = 0.08$ $W \sim \mathrm{B}(n, 0.08)$	

	$P(W \geq 6) > 0.3$ $1 - P(W \leq 5) > 0.3$ $P(W \leq 5) < 0.7$ From GC, n / $P(W \leq 5)$: 56 / $0.7102 > 0.7$ 57 / $0.696 < 0.7$ 58 / $0.6816 < 0.7$ Least value of n = 57	
10(i) a) & b		
(c)	Sum of squares of residuals $= (-0.02)^2 + 0.06^2 + 0.07^2 + 0.03^2 + (-0.06)^2 = 0.0134$	
(d)	Residuals may be positive or negative. If we find the sum of residuals, it may not indicate how close the data lies to the line of best fit. Squaring the residuals prevent this occurrence by ensuring all values are positive, and the closer to 0 the sum of squares is, the closer the data lies to the line of best fit.	
(ii)	Product moment correlation coefficient = –0.958 (3s.f.)	

(iii)	By GC, $y = -0.0222x + 3.1512 \approx -0.0222x + 3.15$ $2.48 = -0.0222x + 3.1512$ $x = 30.2$ (3s.f) Estimated tempertature is 30.2 degree Celsius The value may not be reliable due to extrapolation as x = 30 is not within the range of data given.	
(iv)	Changes in scale or units of measurement will not affect the value of the product moment correlation coefficient.	
11(i)	Let M be the random variable denoting the height of a man. $M \sim \text{N}(173, 10^2)$ $\text{P}(173-4 < M < 173+4)$ $= \text{P}(169 < M < 177)$ $= 0.31084 \approx 0.311$ (to 3 s.f.)	
(ii)	Let W be the random variable denoting the height of a woman. $W \sim \text{N}(165, \sigma^2)$ $\text{P}(W < 165) \times \text{P}(W > 160) \times 2! = 0.7$ -----(*) Since P(W < 165) = 0.5, $\text{P}(W > 160) = 0.7$ $\text{P}\left(Z > \frac{160-165}{\sigma}\right) = 0.7$ where $Z \sim \text{N}(0,1)$ $\frac{-5}{\sigma} = -0.5244005$ $\sigma = 9.5346 \approx 9.53$ (to 3 s.f.)	
(iii)	$\text{P}(W > M) = \text{P}(W - M > 0)$ $\text{E}(W - M) = \text{E}(W) - \text{E}(M) = 165 - 173 = -8$ $\text{Var}(W - M) = \text{Var}(W) + \text{Var}(M)$ $= 9^2 + 10^2 = 181$ $\therefore W - M \sim N(-8, 181)$ Hence $\text{P}(W - M > 0)$=0.27604 ≈ 0.276 (to 3 s.f.)	
(iv)	No, people do not choose their spouses at random. The heights of a husband and wife may not be independent	

(v)	$\mathrm{P}\left(0<W_1+W_2+W_3-2M\le 100\right)$ ------(#) Let $A=W_1+W_2+W_3-2M$ $\mathrm{E}(A)=3\mathrm{E}(W)-2\mathrm{E}(M)=3(165)-2(173)=149$ $\mathrm{Var}(A)=3\mathrm{Var}(W)+2^2\mathrm{Var}(M)$ $=3\left(9^2\right)+4\left(10^2\right)=643$ $\therefore A\sim \mathrm{N}(149,643)$ $\mathrm{P}\left(0<W_1+W_2+W_3-2M\le 100\right)=0.026657\approx 0.0267$ (3 s.f)	