ST ANDREW'S JUNIOR COLLEGE

PRELIMINARY EXAMINATION

MATHEMATICS **HIGHER 2** **9758/02**

Wednesday **13 September 2023** **3 hr**

Candidates answer on the Question Paper.
Additional Materials: List of Formulae (MF26)

NAME:____ **(_____) C.G.: __________**

TUTOR'S NAME: ____

SCIENTIFIC / GRAPHIC CALCULATOR MODEL: _______________________

NUMBER OF ADDITIONAL PIECES OF WRITING PAPER :________________

READ THESE INSTRUCTIONS FIRST

Write your name, civics group, index number and calculator models on the cover page.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions. Total marks : **100**

Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing calculator is expected, where appropriate.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.
The number of marks is given in brackets [ ] at the end of each question or part question.

Question	1	2	3	4	5	6	7	8	9	10	11	TOTAL
Marks												
	7	9	9	6	9	8	8	10	10	12	12	100

This document consists of **26** printed pages and **2** blank pages including this page.

Section A: Pure Mathematics [40 marks]

1 **Do not use a calculator in answering this question.**

The complex number z satisfies the equation

$$3z^2 - (5+\mathrm{i})z + \alpha = 0,$$

where α is a real number. It is given that one root is of the form $k + k\mathrm{i}$, where k is real and non-zero. Find α and k, and the other root of the equation. **[7]**

2 In triangle PQR, $\angle PRQ = \dfrac{2\pi}{3}$, $\angle RPQ = \dfrac{\pi}{6} + \theta$, and $PR = a$ units, where a is a positive real constant.

(i) Show that $PQ = \dfrac{\sqrt{3}\,a}{\cos\theta - \sqrt{3}\sin\theta}$. **[4]**

(ii) Given that θ is sufficiently small, show that $PQ \approx \sqrt{3}\,a\left(1 + b\,\theta + c\,\theta^2\right)$, where b and c are real constants. **[3]**

(iii) Given that $\theta > 0$, find the range of values of θ such that the percentage error of the approximation in **(ii)** is less than 5%. **[2]**

3 **(a)** Use the substitution $x = \tan t$, where $0 < t < \dfrac{\pi}{2}$, to find $\displaystyle\int \frac{1}{x^2\sqrt{x^2+1}}\,\mathrm{d}x$. **[4]**

(b) **(i)** Write down $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\cos x^3\right)$. **[1]**

(ii) Hence, find $\displaystyle\int x^5 \sin x^3\, dx$. **[4]**

4 The point P travels along the curve C with equation $2xy + x - 9y = 0$.

(i) Without expressing y in terms of x, find an expression for the gradient of the tangent at P in terms of x and y. **[2]**

The x-coordinate of P is increasing at the rate of 0.02 units per second when $x = 3$.

(ii) Determine the rate at which the gradient is changing at this instant. **[4]**

5 **(i)** Show that $\dfrac{1}{r!}-\dfrac{1}{(r+1)!}=\dfrac{r}{(r+1)!}$. **[1]**

(ii) Using the result in **(i)**, evaluate the sum $S_n=\dfrac{1}{2!}+\dfrac{2}{3!}+\dfrac{3}{4!}+\ldots+\dfrac{n}{(n+1)!}$ in terms of n. **[3]**

(iii) Explain why the series converges as $n\to\infty$ and state its sum to infinity S_∞ . **[2]**

(iv) The Comparison Test for Convergence states that if $\sum_{r=1}^{\infty}a_r$ and $\sum_{r=1}^{\infty}b_r$ are two infinite series with $a_r \geq 0$ and $b_r \geq 0$ and $a_r \leqslant b_r$ for all $r\in\mathbb{Z}^+$, then $\sum_{r=1}^{\infty}b_r$ converges implies that $\sum_{r=1}^{\infty}a_r$ converges.

Using the Comparison Test for Convergence and result from part(**iii**), explain why the series $\sum_{r=1}^{\infty}\dfrac{(r-1)^2}{(r+2)!}$ converges. **[3]**

Section B: Probability and Statistics [60 marks]

6 A bag contains 5 red balls and 3 blue balls. In a game, Anne removes balls at random from the bag, one at a time, without replacement, until she has taken out 2 red balls. The total number of balls Anne removes from the bag is denoted by X.

(i) Find $\mathrm{P}(X=x)$ for all possible values of x. **[3]**

(ii) Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$. **[3]**

Anne pays \$1 every time she removes a ball and she will receive \$$y$ once she gets 2 red balls.

(iii) Calculate the value of y if the game is fair. **[2]**

7 Ben has a collection of baby toys of various shapes and colours. There are four different shapes: Circle, Square, Triangle and Star. Each shape has four different colours: Yellow, Green, Red and Purple. Ben's collection consists of 40 toys and the numbers of each shape and colour are shown in the table.

	Yellow	Green	Red	Purple
Circle	1	2	2	0
Square	3	1	5	2
Triangle	4	4	2	3
Star	2	5	3	1

(i) Ben puts all the toys in a box and chooses one toy at random. Given that this toy is not Yellow, find the probability that this toy is either a Triangle or a Star. **[2]**

(ii) Ben now puts the toy back into the box and chooses two toys at random.

(a) Find the probability that the two toys chosen are purple in colour and of different shapes. **[3]**

(b) Ben has two favorites among the 16 possible colour-shape combinations. The probability of choosing these two at random from the 40 toys is $\frac{1}{39}$. Write down all his possible favourite colour-shape combinations. **[3]**

8 Professor A claims that the mean score of the students in his Calculus class is 52. His student Barbie decides to test this claim. She randomly selects a sample of 30 students from the Calculus class and finds that their mean score is 46 with a standard deviation of 15.

(i) Test, at the 5% level of significance, whether Professor A's claim is valid. You should state your hypotheses clearly and define any symbols that you use. **[5]**

Barbie also attends the Statistics class conducted by Professor B, who claims that the mean score of the students in his Statistics class is 48. It is known that the standard deviation of the Statistics scores is 13. Barbie also decides to test whether Professor B has understated the mean Statistics score. She randomly selects a sample of 30 students from the Statistics class and finds that their mean score is k. Testing at the 5% level of significance, she concludes that the professor has indeed understated the mean score.

(ii) Find the range of values of k. **[4]**

(iii) Explain if it is necessary for Barbie to select a sample of at least 30 students in both **(i)** and **(ii)**. **[1]**

9 On average, 1 in 15 patients who visit polyclinics have diabetes mellitus. On a certain day, 50 patients are randomly selected from a polyclinic. The number of these patients having diabetes mellitus is the random variable X.

(i) State, in the context of this question, two assumptions needed to model X by a binomial distribution. **[2]**

Assume now that X has a binomial distribution.

(ii) Find the probability that the 50^{th} patient is the 4^{th} patient with diabetes mellitus. **[1]**

The test for diabetes mellitus consists of two stages. In the first stage, a fasting blood glucose test is conducted. The percentage of patients with the respective fasting blood glucose levels is given in the table below.

Fasting blood glucose level	Percentage
less than 100 mg/dL	84 %
100 – 125 mg/dL	10 %
Above 125 mg/dL	6 %

A patient with a fasting blood glucose level of less than 100 mg/dL is considered non-diabetic and a patient with fasting blood glucose level above 125 mg/dL is diagnosed with diabetes mellitus.

If a patient has a fasting blood glucose level of 100 – 125 mg/dL, the first test is inconclusive and a follow-up non-fasting blood glucose test is conducted on the next day. In the follow-up test, patients with a blood glucose level of 200 mg/dL and above is diagnosed with diabetes mellitus, otherwise they are considered non-diabetic. It is known that p % of the patients with a fasting blood glucose level of 100 – 125 mg/dL has a non-fasting blood glucose level of 200 mg/dL and above.

A patient diagnosed with diabetes mellitus is considered diabetic.

On another day, the polyclinic conducted the fasting blood glucose test on a group of n patients.

(iii) With the aid of a probability tree, find the probability that a patient chosen at random from this group will be diagnosed with diabetes mellitus. **[2]**

(iv) Find the probability that a patient has a fasting blood glucose level of 100 – 125 mg/dL given that the patient is **not** diagnosed with diabetes mellitus. **[2]**

(v) Given that $p = 20$, find the least value of n such that the probability of having at least 6 diabetic patients is greater than 0.3. **[3]**

10 Tom records the maximum speeds of an athlete running in a temperature controlled environment. The table below shows the maximum speed, y metres per second, recorded for various specific temperatures, x degree Celsius.

x	6	11	16	21	26
y	2.96	2.94	2.85	2.71	2.52

(i) Tom decides to model the data using the line $y = -0.02x + 3.1$.

(a) On the grid opposite

- draw a scatter diagram of the data, and
- draw the line $y = -0.02x + 3.1$. **[3]**

(b) For a line of best fit $y = \mathrm{f}(x)$, the residual for a point (a, b) plotted on the scatter diagram is the vertical distance between $(a, \mathrm{f}(a))$ and (a, b). Mark the residual for each point on your diagram. **[1]**

(c) Calculate the sum of the squares of the residuals for Tom's line. **[1]**

(d) Explain why, in general, the sum of the squares of the residuals rather than the sum of the residuals is used. **[1]**

(ii) State the value of the product moment correlation coefficient. **[1]**

(iii) Use your calculator to find the equation of a suitable regression line to estimate the temperature for a given maximum speed. Hence estimate the temperature when the maximum speed is 2.48 metres/second. Explain whether you would expect this value to be reliable. **[4]**

(iv) Explain whether the product moment correlation coefficient would differ if Tom converted the speed to kilometres per hour. **[1]**

11 **In this question you should state the parameters of any normal distributions you use**.

On Kaypoh Island, the heights of men and women, in cm, are modelled as having independent normal distributions with means and standard deviations as shown in the table below.

	Mean (cm)	Standard deviation (cm)
Men	173	10
Women	165	σ

(i) Find the probability that the height of a randomly chosen man is within 4 cm of the mean. **[2]**

(ii) Two women are chosen at random. Suppose the probability that one of them has a height less than 165 cm and the other has a height more than 160 cm is 0.7, calculate the value of σ. **[3]**

It is given instead that $\sigma = 9$ for the rest of the question.

(iii) Find the probability that a randomly chosen woman is taller than a randomly chosen man. **[2]**

(iv) Do you think that the probability calculated in **(iii)** is equivalent to the probability that a married woman is taller than her husband? Explain your answer clearly. **[1]**

(v) Find the probability that the total height of 3 randomly chosen women exceeds twice the height of a randomly chosen man by at most 1 metre. **[4]**