ST ANDREW'S JUNIOR COLLEGE

PRELIMINARY EXAMINATION

MATHEMATICS **HIGHER 2** **9758/01**

Tuesday **29 August 2023** **3 hr**

Candidates answer on the Question Paper.
Additional Materials: List of Formulae (MF26)

NAME:____(_____) **C.G.:** __________

TUTOR'S NAME: ____

SCIENTIFIC / GRAPHIC CALCULATOR MODEL: ______________________

NUMBER OF ADDITIONAL PIECES OF WRITING PAPER :_______________

READ THESE INSTRUCTIONS FIRST

Write your name, civics group, index number and calculator models on the cover page.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions. Total marks : **100**

Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing calculator is expected, where appropriate.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.
The number of marks is given in brackets [ ] at the end of each question or part question.

Question	1	2	3	4	5	6	7	8	9	10	11	TOTAL
Marks												
	5	7	7	10	5	12	7	8	12	13	14	100

This document consists of **30** printed pages and **2** blank page including this page.

1 A function f is defined by $\mathrm{f}(x) = ax^3 + bx^2 + cx + d$. The graph of $y = \mathrm{f}(x)$ passes through the points $(-1, -15)$ and (2, 3). The graph has a turning point at $x = 1$, and $\int_0^2 \mathrm{f}(x)\,\mathrm{d}x = 5$.

Find the values of *a, b, c* and *d*. **[5]**

2 **(a)** The sum, S_n, of the first *n* terms of a sequence is given by

$$S_n = n^2 + n$$

Show that the sequence follows an arithmetic progression and state the value of the common difference. **[3]**

(b) An arithmetic series has first term *b* and common difference *d*, where *b* and *d* are non-zero. A geometric series has first term *a* and common ratio *r*, where *a* is non-zero and *r* is positive. It is given that the 3rd, 5th and 7th terms of a geometric series are equal to the 7th, 13th and 25th terms of an arithmetic series respectively.

Find the value of *r* and determine if the geometric series is convergent. **[4]**

3 **Figure 1** shows the net of a square-based pyramid. The net consists of a square of side length *x* cm and four isosceles triangles, each with base *x* cm and fixed sides *k* cm. The net is folded to form a right pyramid which has a square base of side length *x* cm, as shown in **Figure 2**.

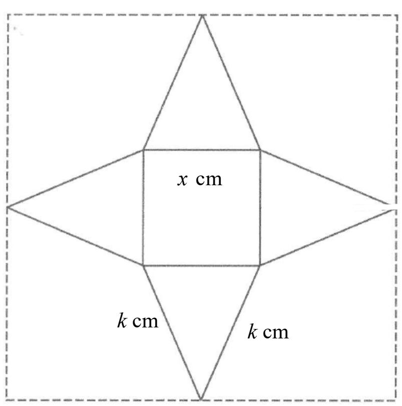


Figure 1

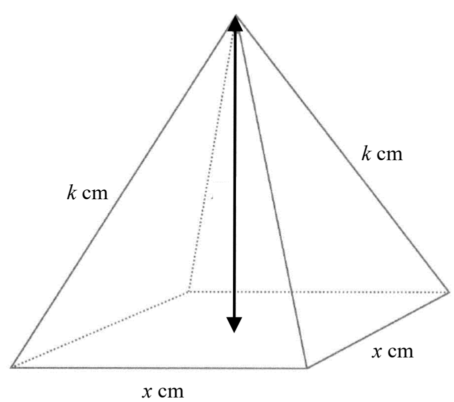


Figure 2

(i) Let the volume of the pyramid be V cm^3. Show that V satisfies the equation $V^2 = \dfrac{x^4}{9}\left(k^2 - \dfrac{x^2}{2}\right)$. **[2]**

(ii) Find, in terms of k, the value of x that will maximise the volume of the pyramid. **[5]**

4 The function f and g are defined by

$$\mathrm{f}(x) = \begin{cases} \dfrac{1}{2}(2^x) - 1 & \text{for } x \in \mathbb{R},\ x < 2, \\ 4 - \dfrac{3}{3x-5} & \text{for } x \in \mathbb{R},\ x \geq 2, \end{cases}$$

$$\mathrm{g}(x) : x \mapsto |\mathrm{f}(x)| \qquad \text{for } x \in \mathbb{R},\ x < a\,.$$

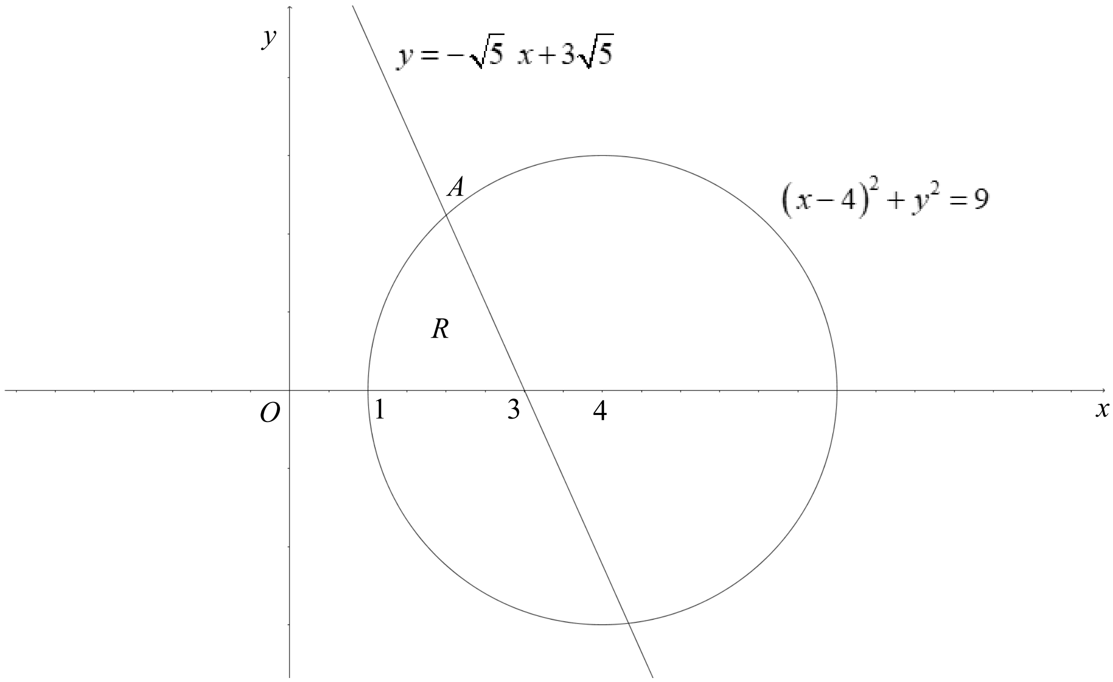
(i) Sketch the graph of $y = \mathrm{f}(x)$, indicating clearly the axial intercepts and equations of asymptotes. **[3]**

(ii) State the maximum value of a such that g has an inverse. **[1]**

(iii) For the value of a found in **(ii)**, find $\mathrm{g}^{-1}(x)$ and state its domain. **[3]**

(iv) Given that the composite function f^2 exists, find f^2. **[3]**
[You do not need to simplify the expressions for f^2.]

5 The diagram below shows the region R bounded by the circle with equation $(x-4)^2+y^2=9$, the line with equation $y=-\sqrt{5}x+3\sqrt{5}$ and the x-axis. The line and the circle meets at the point A with coordinates $(2,\sqrt{5})$.



Show that the volume V, formed when R is rotated through 2π radians about the y-axis is given by

$$\pi\int_0^{\sqrt{5}}\left[\frac{\left(y-3\sqrt{5}\right)^2}{5}-\left(25-y^2-8\sqrt{9-y^2}\right)\right]\mathrm{d}y.$$

Hence evaluate V, leaving your answer correct to 3 decimal places. **[5]**

6 The curve C_1 has equation $y=\dfrac{x(x-a)}{x+a}$ and a is a positive real constant.

(i) Find algebraically, in terms of a, the range of values that y can take. **[3]**

It is now given that $a = 1$ and that curve C_2 has equation $y=-\dfrac{5}{2}+\dfrac{10}{x+3}$.

(ii) Find the exact x-coordinates of the intersection points of C_1 and C_2. **[2]**

(iii) Sketch C_1 and C_2 on the same graph, giving the coordinates of any points where C_1 and C_2 intersect and the equations of any asymptotes. **[4]**

(iv) Find the exact area of the region bounded by the two curves C_1 and C_2, simplifying your answer in the form $\left(a\ln 2+b\ln 5+c\right)$ units^2, where a, b, c are constants to be determined. **[3]**

7 With reference to the origin O, the points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. The point P lies on AB such that $AP : AB = 1 : 4$. The point Q lies on the line passing through the points O and B such that AQ is perpendicular to OP.

(i) Explain why Q has position vector in the form $\lambda\mathbf{b}$ where λ is a real constant. **[1]**

(ii) Given that $\mathbf{a}$ and $\mathbf{b}$ are unit vectors and $\angle AOB = \theta$, show that

$$\lambda = \frac{3+\cos\theta}{3\cos\theta+1}.$$ **[4]**

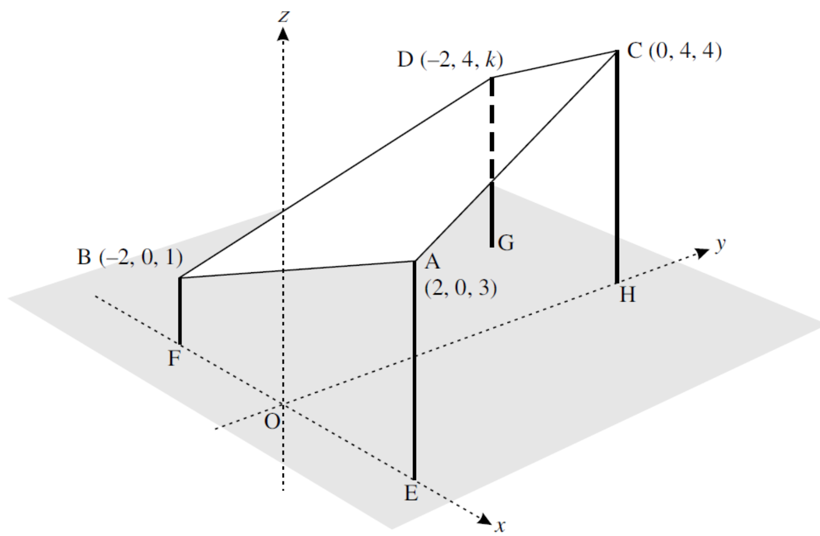
(iii) Given that $0 < \theta < \frac{\pi}{2}$, explain with justification why Q does not lie between O and B. **[2]**

8 The complex number w is given by $w = \frac{z-1}{z+\mathrm{i}}$, where $z = x + \mathrm{i}y$ where $x, y \in \mathbb{R}$ and $z \neq -\mathrm{i}$.

(i) If $z = \mathrm{i}$, find w in the form $r\mathrm{e}^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$. Hence find w^{14}, leaving your answer in cartesian form. **[4]**

(ii) If $z = \cos\theta + \mathrm{i}\sin\theta$, show that w can be expressed as $\dfrac{k}{1+\cot\left(\frac{\theta}{2}\right)}$, where k is a complex number to be found in exponential form. **[4]**

9. A tentage for an outdoor activity is built by attaching a piece of polyester tent fabric to the tops of four vertical poles *AE*, *BF*, *DG* & *CH*, where *E*, *F*, *G* and *H* are positioned at ground level on the *x*-*y* plane. Coordinates of *A*, *B*, *C* and *D* are as shown in the diagram below, with lengths measured in meters. The length of the pole *DG* is *k* meters.



(i) Find the cartesian equation of the plane *ABDC*. **[3]**

(ii) Find the acute angle that plane *ABDC* makes with the horizontal. **[2]**

(iii) Find the value of *k*. **[2]**

(iv) Show that the shape of the polyester tent fabric *ABDC* is a trapezium. Hence, or otherwise, find its exact area in the form of $a^{\frac{3}{2}}$ units^2, where *a* is an integer to be determined. **[5]**

10 Hyperglycaemia, also known as high blood sugar, is a medical condition that can affect a significant portion of the population.

One potential treatment involves regulating blood sugar levels by administering insulin. During a clinical trial, insulin is added to the bloodstream of a patient, and the body's insulin regulation removes excess glucose from the bloodstream at a rate directly proportional to the amount of glucose present in the bloodstream. The rate of increase of glucose level in the bloodstream is inversely proportional to the amount of glucose present in the bloodstream of hyperglycaemic patients.

The amount of glucose present in the bloodstream of the patient at time t (in minutes) is denoted by Q mmol/L (millimoles per litre). It is given that the amount of glucose in the bloodstream remains constant when $Q = 4$.

(i) Show that the rate of change of the amount of glucose present in the bloodstream at time t satisfies the differential equation

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = k\left(\frac{16 - Q^2}{Q}\right),$$

where k is a constant. **[3]**

(ii) Initially, the patient's blood glucose level is 7 mmol/L, which reduces to 6.8 mmol/L 15 minutes into the trial. Solve the differential equation from part **(i)** leaving the answers in the form $Q^2 = 16 + Ae^{Bt}$, where A and B are constants to be determined. **[6]**

(iii) Determine the amount of glucose present in the bloodstream 1 hour into the clinical trial. **[1]**

(iv) It is given that the normal glucose level in the bloodstream is between 4 to 5.9mmol/L. The clinical trial claims that it is able to help regulate the glucose level in patients' bloodstream within 1 hour. Using the result from **(iii),** comment if the treatment that the patient has undergone is as effective as it claims to be. **[1]**

(v) Deduce the amount of glucose in the patient's bloodstream in the long run as described by the model.
Comment on the limitation of the model in the long run. **[2]**

11 Planet P orbits around the Sun. Using the Sun as the origin O, the orbit follows the parametric equations

$$x = a(\cos\theta - e), \qquad y = b\sin\theta, \qquad \text{where } 0 < a < b$$

where $0 < e < 1$ is known as the eccentricity, and θ is the angle (in radians) between the line joining the Sun to the planet's current position P (see **Figure 1** below) with respect to the positive x-axis.

[The eccentricity is a constant that measures how much a conic section varies from being circular.]

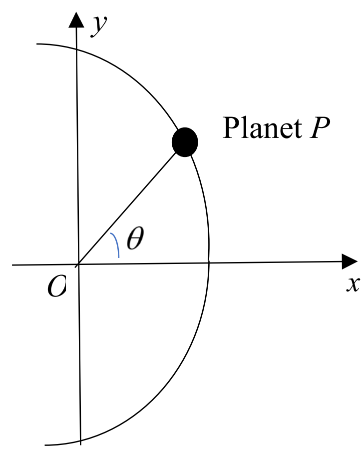


Figure 1

(i) Show that cartesian equation of the planet's orbit in relation to the Sun is given by:

$$\frac{(x+ea)^2}{a^2} + \frac{y^2}{b^2} = 1.$$ **[2]**

Take $e = \frac{1}{2}$ for the remaining parts of the question.

(ii) The planet P is initially at the position $P_0\left(\frac{a}{2}, 0\right)$ where $\theta = 0$. After some time, the planet is at $P_1\left(a\left(\cos\theta_1 - \frac{1}{2}\right), b\sin\theta_1\right)$ when $\theta = \theta_1$. The area, A, swept out during that period as the planet travels from P_0 to P_1 is defined by the area enclosed as shown in **Figure (2)** below.

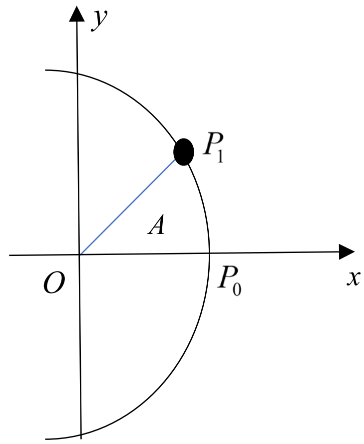


Figure 2

Show that A is given by $\frac{ab}{4}\left(\sin 2\theta_1 - \sin\theta_1\right) + ab\int_0^{\theta_1} \sin^2\theta \, \mathrm{d}\theta$. **[4]**

Hence find in terms of a, b and θ_1, the area A swept out by the planet. **[3]**

(iii) Using your result in part **(ii)**, find the area swept out by the planet as it travels from P_0 (where $\theta = 0$) to P_2 (where $\theta = \frac{\pi}{4}$). **[1]**

(iv) **Kepler's Second Law** states that the line joining the planet and the Sun sweeps out equal areas during equal intervals of time.

Assume that the planet is now at a position P_3(where $\theta = \theta_3$), where $\frac{\pi}{2} < \theta_3 < \pi$ and it is moving towards the position P_4(where $\theta = \pi$). Using Kepler's Second Law, find the value of θ_3 such that the planet will take an equal amount of time for the planet to travel from P_3 to P_4 as it takes to travel from P_0 to P_2. **[4]**