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CANDIDATE NAME

CLASS

ADMISSION NUMBER

2023 Preliminary Exams
Pre-University 3

MATHEMATICS **9758/01**

Paper 1

14 September 2023

3 hours

Candidates answer on the Question Paper.

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your admission number, name and class on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give your answers in the spaces provided in the Question Paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

Qn No.	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	Q12	*	Total
Score														
Max Score	**4**	**4**	**6**	**6**	**8**	**8**	**8**	**9**	**10**	**12**	**13**	**12**		**100**

1 A function f is defined as $\mathrm{f}(x) = ax^3 + bx + c$. The graph of $y = \mathrm{f}(x)$ passes through the points $(1,\ 10)$ and $(-2,\ 12)$. Given that $\mathrm{f}(x)$ is divisible by $x-2$, find the values of a, b and c. [4]

2 Solve the differential equation

$$x^4 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = x + 3,$$

given that the point $\left(1,\ \frac{23}{6}\right)$ is a stationary point of the solution curve. [4]

3 One of the roots of the equation $x^3 + ax^2 - 7x + b = 0$, where a and b are real numbers, is $x = -2 + \mathrm{i}$.

(i) Find the values of a and b. [3]

(ii) Hence, without using a calculator, find the other roots of the equation. [3]

4 **(i)** Without using a calculator, solve the inequality $\dfrac{x^2 - 14}{x - 5} \geq 2 - x$. [4]

(ii) Hence solve the inequality $\dfrac{x^2 - 14}{|x| - 5} \geq 2 - |x|$. [2]

5 **(i)** Sketch, on the same diagram, the graphs of

$$y = \frac{2\ln x}{\cos x} \quad \text{and} \quad y = 3\cos x \ln x \quad \text{for } \frac{\pi}{2} < x < \frac{3\pi}{2},$$

showing clearly the equations of any asymptotes and the coordinates of any axial intercepts. [3]

(ii) The region R is bounded by the two curves in part **(i)**. Find the area of R, giving your answer to 2 decimal places. [3]

(iii) The region R in part **(ii)** is rotated 2π radians about the x-axis to form a solid S. Find the volume of S. [2]

6 The function f is defined by

$$\mathrm{f}: x \mapsto -\frac{x}{2x+1}, \qquad x \in \mathbb{R},\ x \neq -\frac{1}{2}.$$

(i) Find $\mathrm{f}^{-1}(x)$ and state its domain. [3]

(ii) Hence, or otherwise, find $\mathrm{f}^{2}(x)$. [1]

(iii) Deduce the exact value of $\mathrm{f}^{2023}(5)$. [2]

The function g is defined by

$$\mathrm{g}: x \mapsto \frac{1}{ax}, \qquad x \in \mathbb{R},\ x > a,$$

where a is a positive constant.

(iv) Determine whether the composite function fg exists, justifying your answer. [2]

7 **Do not use a calculator in answering this question.**

The roots of the equation $z^2 = 18\mathrm{i}$ are z_1 and z_2.

(i) Find z_1 and z_2 in cartesian form $x + y\mathrm{i}$, showing your working clearly. [4]

The complex number w is given by $-1-\mathrm{i}\sqrt{3}$.

(ii) Find the complex conjugate of $\dfrac{w}{z^2}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$.

[4]

8 Relative to the origin O, the position vectors of the fixed points A and B are $\mathbf{a}$ and $\mathbf{b}$ respectively.

(a) It is given that $|\mathbf{a}| = 1, |\mathbf{b}| = 2$ and $|3\mathbf{b} - 2\mathbf{a}| = \sqrt{34}$.

(i) By considering the scalar product $(3\mathbf{b} - 2\mathbf{a})\cdot(3\mathbf{b} - 2\mathbf{a})$, find the value of $\mathbf{a}\cdot\mathbf{b}$. [3]

(ii) Hence find the exact value of $|\mathbf{a}\times\mathbf{b}|$ and give a geometrical meaning of this value. [3]

(b) A point P divides AB in the ratio $2:1$ and a point M is the mid-point of BC, where the point C has position vector $\mathbf{c}$. It is given that the point N is such that $OBMN$ forms a rhombus and P is also the mid-point of MN. Show that $2\mathbf{a} + 4\mathbf{b} - 3\mathbf{c} = \mathbf{0}$. [3]

9 **(a)** A curve $y = \mathrm{h}(x)$ intersects the x–axis at point A with coordinates $(a,\ 0)$, where $a \neq 0$, and has a horizontal asymptote with equation $y = b$, where b is a non-zero constant.

For each of the following curves, state, if it is possible to do so, the equation of the horizontal asymptote and the coordinates of the point where the curve intersects the x–axis.

(i) $y = 2\mathrm{h}(x+1)$

(ii) $y = \mathrm{h}(-x)$

(iii) $y = \dfrac{1}{\mathrm{h}(x)}$ [5]

(b) It is given that

$$\mathrm{f}(x) = \begin{cases} \sqrt{x} & \text{for } 0 \le x \le 1, \\ -\dfrac{1}{2}x + \dfrac{3}{2} & \text{for } 1 < x \le 3, \end{cases}$$

and that $\mathrm{f}(x) = \mathrm{f}(x+3)$ for all real values of x.

(i) Sketch the graph of $y = \mathrm{f}(x)$ for $-3 \le x \le 5$. [3]

(ii) Hence find the value of $\int_{-3}^{5} \mathrm{f}(x)\,\mathrm{d}x$. [2]

10 **(a)** **(i)** Differentiate $\sin^{-1}\left(e^{2x}\right)$ with respect to x. [2]

(ii) Hence find $\displaystyle\int \frac{2e^{2x}+e^{4x}}{\sqrt{1-e^{4x}}}\,dx$. [3]

(b) **(i)** Show that $\sin^{n-2}\theta\cos^2\theta=\sin^{n-2}\theta-\sin^n\theta$, where $n\in\mathbb{Z}^+$, $n\geq 2$. [1]

(ii) By considering $\sin^n\theta=\sin^{n-1}\theta\sin\theta$, show that

$$\int_0^{\frac{\pi}{2}}\sin^n\theta\ d\theta=\frac{n-1}{n}\int_0^{\frac{\pi}{2}}\sin^{n-2}\theta\ d\theta,$$

where $n\in\mathbb{Z}^+$, $n\geq 2$. [4]

(iii) Hence find the exact value of $\displaystyle\int_0^{\frac{\pi}{2}}\sin^3\theta\ d\theta$. [2]

11 Linda decides to open a certain savings account.

In one of her plans, she deposits \$500 on 1 January 2022 into the savings account which pays compound interest at a rate of 0.1% per month on the last day of each month. She deposits a further \$500 into the account on the first day of each subsequent month.

(i) Find the total amount of money in the account at the end of December 2022. [3]

(ii) Find the minimum number of complete months needed for the account to exceed a total of \$20 000. [4]

In another plan, Linda decides to deposit a lump sum of \$50 000 on 1 January 2022. She wishes to withdraw \$*k* at the end of each month after the interest is paid, starting from January 2022. It is also given that the interest rate per month is still 0.1%.

(iii) Show, in terms of *m* and *k*, that the total amount of money in the account by the end of the *m*th month is

$$\$\left[1.001^m\left(50000-1000k\right)+1000k\right].$$ [3]

(iv) Find the maximum amount of money, to the nearest dollar, that can be withdrawn monthly such that the last possible withdrawal occurs at the end of 2024. [3]

12 A curve C has parametric equations

$$x = t - \frac{10}{t}, \qquad y = 6t - t^2, \qquad \text{for } \sqrt{10} \le t \le 6.$$

(i) Sketch the curve C, labelling the coordinates of any points of intersection with the axes. [2]

It is given that a plot of flat land is represented by the region bounded by C and both axes, where units are measured in kilometres.

(ii) Find the area of the plot of land. [3]

The landowner considers using a portion of this land to rear cattle. This portion is a rectangular piece of land, $OPRQ$, where O is the origin, R is a point on the curve, P and Q are the points on the x-axis and y-axis respectively.

(iii) By using differentiation, find the value of t that maximises the area of rectangle $OPRQ$. (You need not show that your answer gives a maximum.) [3]

It is given instead that the landowner decides to divide the plot of flat land equally into two by building a straight fence through the land, parallel to the y-axis. It is assumed that the thickness of the fence is negligible.

(iv) Using the result from part **(ii)**, find the equation of the fence that divides the plot of land equally into two parts. [4]

End of Paper