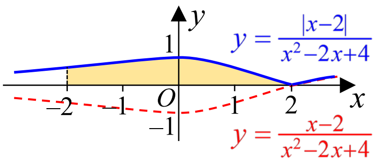
2023 HCI C2 H2 Mathematics Prelim P1 Solutions

1	Solution
	$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 2\mathrm{e}^{-2x}$ $\frac{\mathrm{d}y}{\mathrm{d}x} = -\mathrm{e}^{-2x} + C$ $y = \frac{1}{2}\mathrm{e}^{-2x} + Cx + D$ Hence a possible solution-curve with an oblique asymptote is $y = \frac{1}{2}\mathrm{e}^{-2x} + x + 1$, where $C = 1$, $D = 1$ Oblique asymptote is $y = x + 1$

2	Solution
	By sine rule, $\frac{PQ}{\sin(\frac{\pi}{3} - \theta)} = \frac{PR}{\sin\frac{\pi}{3}}$ $\therefore \frac{PR}{PQ} = \frac{\sin\frac{\pi}{3}}{\sin(\frac{\pi}{3} - \theta)}$ $= \frac{\sin\frac{\pi}{3}}{\sin\frac{\pi}{3}\cos\theta - \cos\frac{\pi}{3}\sin\theta}$ $\approx \frac{\frac{\sqrt{3}}{2}}{\frac{\sqrt{3}}{2}(1 - \frac{\theta^2}{2}) - \frac{1}{2}\theta}$ $= \frac{\frac{\sqrt{3}}{2}}{\frac{\sqrt{3}}{2}[(1 - \frac{\theta^2}{2}) - \frac{1}{\sqrt{3}}\theta]}$ $= \frac{1}{1 - \frac{1}{\sqrt{3}}\theta - \frac{1}{2}\theta^2}$ $= \left[1 - (\frac{1}{\sqrt{3}}\theta + \frac{1}{2}\theta^2)\right]^{-1}$ $\approx 1 + (\frac{1}{\sqrt{3}}\theta + \frac{1}{2}\theta^2) + (\frac{1}{\sqrt{3}}\theta + \frac{1}{2}\theta^2)^2$ $\approx 1 + \frac{1}{\sqrt{3}}\theta + \frac{1}{2}\theta^2 + \frac{1}{3}\theta^2$ $= 1 + \frac{1}{\sqrt{3}}\theta + \frac{5}{6}\theta^2$ where $\alpha = \frac{1}{\sqrt{3}}$, $\beta = \frac{5}{6}$ (shown)

3	Solution
(a)	$x^2y+y^4=5$ $2xy+x^2\dfrac{dy}{dx}+4y^3\dfrac{dy}{dx}=0$ $\therefore\ \dfrac{dy}{dx}=\dfrac{-2xy}{x^2+4y^3}$

3	Solution
(b)	$\dfrac{dy}{dx}=0$ $\therefore\ \dfrac{-2xy}{x^2+4y^3}=0$ $\therefore\ x=0$ or $y=0$ [reject since points of the form $(x,0)$ does not lie on C] When $x=0$, $y=\pm 5^{\frac{1}{4}}$ $\therefore$ required distance $=2(5^{\frac{1}{4}})$

4	Solution
	$\displaystyle\int_{-2}^{2}\frac{\lvert x-2\rvert}{x^2-2x+4}\,dx$  $\displaystyle=\int_{-2}^{2}\frac{2-x}{x^2-2x+4}\,dx$ $\displaystyle=-\frac{1}{2}\int_{-2}^{2}\frac{2x-2}{x^2-2x+4}\,dx+\int_{-2}^{2}\frac{1}{x^2-2x+4}\,dx$ $\displaystyle=-\frac{1}{2}\left[\ln(x^2-2x+4)\right]_{-2}^{2}+\int_{-2}^{2}\frac{1}{(x-1)^2+3}\,dx$ $\displaystyle=-\frac{1}{2}[\ln 4-\ln 12]+\frac{1}{\sqrt{3}}\left[\tan^{-1}\left(\frac{x-1}{\sqrt{3}}\right)\right]_{-2}^{2}$ $\displaystyle=-\frac{1}{2}\left[\ln\frac{1}{3}\right]+\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)-\frac{1}{\sqrt{3}}\tan^{-1}(-\sqrt{3})$ $\displaystyle=\frac{1}{2}\ln 3+\frac{1}{\sqrt{3}}\left(\frac{\pi}{6}\right)-\frac{1}{\sqrt{3}}\left(-\frac{\pi}{3}\right)$ $\displaystyle=\frac{1}{2}\ln 3+\frac{1}{\sqrt{3}}\left[\frac{\pi}{3}+\frac{\pi}{6}\right]$ $\displaystyle=\frac{1}{2}\ln 3+\frac{\pi}{2\sqrt{3}}$ where $p=\frac{1}{2}$, $q=\frac{1}{2\sqrt{3}}$ (shown)

5	**Solution**
(a)	Method 1: $y = \mathrm{f}(3x-2)+a \xrightarrow[\text{with } y+a]{\text{Replace } y} y=\mathrm{f}(3x-2)$ $\xrightarrow[\text{with } \frac{x}{3}]{\text{Replace } x} y=\mathrm{f}(x-2)$ $\xrightarrow[\text{with } x+2]{\text{Replace } x} y=\mathrm{f}(x)$ 1. Translate in negative y-direction by a units. 2. Scale parallel to x-axis with scale factor 3. 3. Translate in negative x-direction by 2 units. Method 2: $y = \mathrm{f}(3x-2)+a \xrightarrow[\text{with } y+a]{\text{Replace } y} y=\mathrm{f}(3x-2)$ $\xrightarrow[\text{with } x+\frac{2}{3}]{\text{Replace } x} y=\mathrm{f}(3x)$ $\xrightarrow[\text{with } \frac{x}{3}]{\text{Replace } x} y=\mathrm{f}(x)$ 1. Translate in negative y-direction by a units. 2. Translate in negative x-direction by $\frac{2}{3}$ units. 3. Scale parallel to x-axis with scale factor 3.

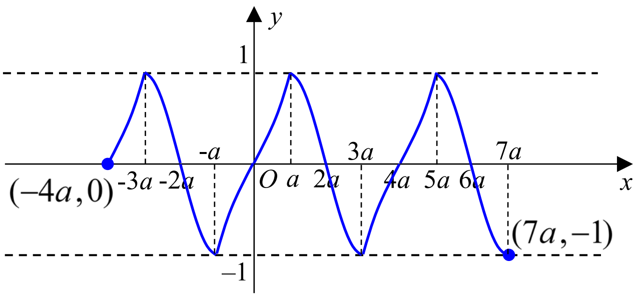
5	**Solution**
(b)	Method 1: $(b,0)$ $\downarrow$ Translate in negative y-direction by a units $(b,-a)$ $\downarrow$ Scale parallel to x-axis with scale factor 3 $(3b,-a)$ $\downarrow$ Translate in negative x-direction by 2 units $(3b-2,-a)$ $\downarrow$ Transform $y=\mathrm{f}(x)$ to $y=\frac{1}{\mathrm{f}(x)}$ $(3b-2,-\frac{1}{a})$ $\therefore\ c=3b-2\ ,\quad d=-\dfrac{1}{a}$ Method 2: $(b,0)$ $\downarrow$ Translate in negative y-direction by a units $(b,-a)$ $\downarrow$ Translate in negative x-direction by $\frac{2}{3}$ units $(b-\frac{2}{3},-a)$ $\downarrow$ Scale parallel to x-axis with scale factor 3 $(3b-2,-a)$ $\downarrow$ Transform $y=\mathrm{f}(x)$ to $y=\frac{1}{\mathrm{f}(x)}$ $(3b-2,-\frac{1}{a})$

5	**Solution**
	$\therefore\ c = 3b - 2\ ,\quad d = -\dfrac{1}{a}$

6	**Solution**
(a)	$\dfrac{1}{(2r+1)(2r+3)} = \dfrac{1}{2(2r+1)} - \dfrac{1}{2(2r+3)}$
	$\displaystyle\sum_{r=1}^{n} \frac{1}{(2r+1)(2r+3)}$
	$\displaystyle = \frac{1}{2}\sum_{r=1}^{n}\left(\frac{1}{2r+1} - \frac{1}{2r+3}\right)$
	$\displaystyle = \frac{1}{2}\left[\begin{array}{l} \frac{1}{3} - \cancel{\frac{1}{5}} \\ + \cancel{\frac{1}{5}} - \cancel{\frac{1}{7}} \\ + \cancel{\frac{1}{7}} - \cancel{\frac{1}{9}} \\ \quad\vdots \\ + \cancel{\frac{1}{2n+1}} - \frac{1}{2n+3} \end{array}\right]$
	$\displaystyle = \frac{1}{2}\left[\frac{1}{3} - \frac{1}{2n+3}\right]$
	$\displaystyle = \frac{1}{6} - \frac{1}{4n+6}$

6	**Solution**
(b)	$\displaystyle\sum_{r=1}^{n}\left[(-\tfrac{1}{3})^r + \frac{1}{(2r+1)(2r+3)}\right]$
	$\displaystyle = \sum_{r=1}^{n}(-\tfrac{1}{3})^r + \sum_{r=1}^{n}\frac{1}{(2r+1)(2r+3)}$
	$\displaystyle\sum_{r=1}^{n}(-\tfrac{1}{3})^r$ is a GP with 1st term $-\frac{1}{3}$ and common ratio $-\frac{1}{3}$.
	$\displaystyle\sum_{r=1}^{n}(-\tfrac{1}{3})^r = \frac{(-\frac{1}{3})[1-(-\frac{1}{3})^n]}{1-(-\frac{1}{3})} = -\frac{1}{4}[1-(-\tfrac{1}{3})^n]$
	As $n \to \infty$, $(-\frac{1}{3})^n \to 0$ and $\therefore\ \displaystyle\sum_{r=1}^{n}(-\tfrac{1}{3})^r \to -\frac{1}{4}$
	Hence it is a convergent series.
	From (a), as $n \to \infty$, $\dfrac{1}{4n+6} \to 0$ and $\therefore\ \displaystyle\sum_{r=1}^{n}\frac{1}{(2r+1)(2r+3)} = \frac{1}{6} - \frac{1}{4n+6} \to \frac{1}{6}$
	Hence it is also a convergent series.

6	**Solution**
	$\therefore \sum_{r=1}^{n}\left[(-\frac{1}{3})^r + \frac{1}{(2r+1)(2r+3)}\right]$ is a sum of 2 convergent series, and hence is also convergent. $\lim_{n\to\infty}\sum_{r=1}^{n}(-\frac{1}{3})^r = \frac{-\frac{1}{3}}{1-(-\frac{1}{3})} = -\frac{1}{4}$ $\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{(2r+1)(2r+3)} = \frac{1}{6}$, $\therefore$ required sum to infinity $= -\frac{1}{4} + \frac{1}{6} = -\frac{1}{12}$

7	**Solution**
(a)	

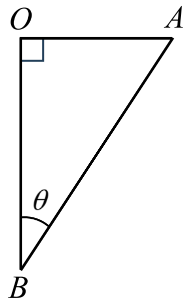
7	**Solution**
(b)	From sketch in (a), $\mathrm{f}(-x) = -\mathrm{f}(x)$

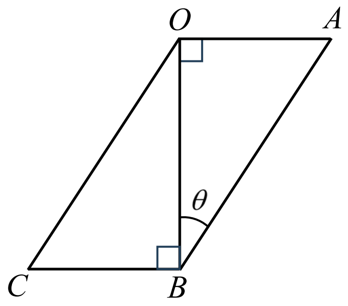
7	Solution
(c)	$\int_{-2a}^{4a} \mathrm{f}(x)\,\mathrm{d}x$ $= \int_{-2a}^{-a} \mathrm{f}(x)\,\mathrm{d}x + \int_{-a}^{a} \mathrm{f}(x)\,\mathrm{d}x + \int_{a}^{3a} \mathrm{f}(x)\,\mathrm{d}x + \int_{3a}^{4a} \mathrm{f}(x)\,\mathrm{d}x$ $= \int_{2a}^{3a} \mathrm{f}(x)\,\mathrm{d}x + \quad 0 \quad + \quad 0 \quad + \int_{-a}^{0} \mathrm{f}(x)\,\mathrm{d}x$ $= \int_{2a}^{3a} \sin\left(\frac{\pi}{2a}x\right)\mathrm{d}x + \int_{-a}^{0} \tan\left(\frac{\pi}{4a}x\right)\mathrm{d}x$ $= \left[\frac{-\cos(\frac{\pi}{2a}x)}{\frac{\pi}{2a}}\right]_{2a}^{3a} + \left[\frac{\ln\left\|\sec(\frac{\pi}{4a}x)\right\|}{\frac{\pi}{4a}}\right]_{-a}^{0}$ $= \frac{2a}{\pi}\left[-\cos(\frac{3\pi}{2}) + \cos(\pi)\right]$ $\quad + \frac{4a}{\pi}\left[\ln\left\|\sec(0)\right\| - \ln\left\|\sec(-\frac{\pi}{4})\right\|\right]$ $= \frac{2a}{\pi}[-1] + \frac{4a}{\pi}[-\ln\sqrt{2}]$ $= \frac{-2a}{\pi}(1 + 2\ln\sqrt{2})$ $= \frac{-2a}{\pi}(1 + \ln 2)$ where $k = -2$, $m = 2$ (shown)

8	Solution
(a)	Method 1: $\overrightarrow{OA} \cdot \overrightarrow{OB} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ \alpha \end{pmatrix} = 0$ $\therefore$ OA and OB are perpendicular. $\therefore \measuredangle AOB = 90^\circ$

8	Solution
(b)	$\sin\theta = \frac{1}{2}$ $\therefore \theta = 30^\circ$ or $\theta = 150^\circ$ Method 1: (using right-angle $\triangle OAB$) Since $OA \perp OB$, $\therefore$ $\triangle OAB$ is a right-angle $\triangle$. Hence $\measuredangle OBA$ and $\measuredangle OAB$ must be acute. $\therefore \theta = 30^\circ$ Method 2: (using dot product $\overrightarrow{AB} \cdot \overrightarrow{OB}$) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 0 \\ 0 \\ \alpha \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ \alpha \end{pmatrix}$ Since $\overrightarrow{AB} \cdot \overrightarrow{OB} = \begin{pmatrix} -2 \\ 1 \\ \alpha \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ \alpha \end{pmatrix} = \alpha^2 > 0$, and $\overrightarrow{AB} \cdot \overrightarrow{OB} = \left\|\overrightarrow{AB}\right\| \left\|\overrightarrow{OB}\right\| \cos\theta$

8	**Solution**
	$\therefore\ \cos\theta > 0 \Rightarrow \theta$ is acute Hence $\theta = 30°$

8	**Solution**
(c)	$\overrightarrow{AB}\,.\,\overrightarrow{OB} = \lvert\overrightarrow{AB}\rvert\lvert\overrightarrow{OB}\rvert\cos 30°$ $\alpha^2 = (\sqrt{5+\alpha^2})\lvert\alpha\rvert(\frac{\sqrt{3}}{2})$ $2\alpha^2 = \lvert\alpha\rvert\sqrt{5+\alpha^2}\sqrt{3}$ $2\alpha^2 = \lvert\alpha\rvert\sqrt{15+3\alpha^2}$ $2\lvert\alpha\rvert^2 = \lvert\alpha\rvert\sqrt{15+3\alpha^2}$ $2\lvert\alpha\rvert = \sqrt{15+3\alpha^2}$ $\Rightarrow\ 4\alpha^2 = 15+3\alpha^2$ $\alpha^2 = 15$ $\alpha = \pm\sqrt{15}$ (reject $+\sqrt{15}$ since $\alpha < 0$) $\therefore\ \alpha = -\sqrt{15}$ 

8	**Solution**
(d)	$\underset{\sim}{c} = \underset{\sim}{b} - \underset{\sim}{a}$ $\overrightarrow{OC} = \overrightarrow{OB} - \overrightarrow{OA} = \overrightarrow{AB}$ $\therefore\ \overrightarrow{OC} \,\vert\vert\, \overrightarrow{AB}$ $\overrightarrow{CB} = \overrightarrow{OB} - \overrightarrow{OC}$ $= \overrightarrow{OB} - \overrightarrow{AB}$ $= \overrightarrow{OB} - (\overrightarrow{OB} - \overrightarrow{OA})$ $= \overrightarrow{OA}$ $\therefore\ \overrightarrow{CB} \,\vert\vert\, \overrightarrow{OA}$ Since $\overrightarrow{OC} \,\vert\vert\, \overrightarrow{AB}$ and $\overrightarrow{CB} \,\vert\vert\, \overrightarrow{OA}$, $\therefore$ $OABC$ is a parallelogram. 

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$$\begin{aligned}\left|\overrightarrow{OA}\times\overrightarrow{OC}\right| &= \left|\overrightarrow{OA}\times\overrightarrow{AB}\right| \\ &= \left|\begin{pmatrix}2\\-1\\0\end{pmatrix}\times\begin{pmatrix}-2\\1\\-\sqrt{15}\end{pmatrix}\right| \\ &= \left|\begin{pmatrix}\sqrt{15}\\2\sqrt{15}\\0\end{pmatrix}\right| \\ &= \sqrt{15+2^2(15)+0} \\ &= \sqrt{75} \\ &= 5\sqrt{3} \quad \text{unit}^2\end{aligned}$$

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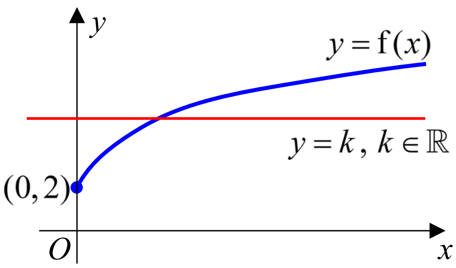
<tr><th>9</th><th>Solution</th></tr>
<tr><td>(a)</td><td>



Let $y=\mathrm{f}(x)$. From graph, since any horizontal line $y=k$, $k\in\mathbb{R}$ cuts $y=\mathrm{f}(x)$ at most once, $\therefore$ f is $1-1$ and hence f^{-1} exists. (shown)

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<tr><th>9</th><th>Solution</th></tr>
<tr><td>(b)</td><td>

$$\begin{aligned}&y=\ln(x+1)^2+2 \\ &(x+1)^2=\mathrm{e}^{y-2} \\ &\therefore\ x=\pm\sqrt{\mathrm{e}^{y-2}}-1 \\ &\qquad\ =\mathrm{e}^{\frac{y-2}{2}}-1 \quad (\text{reject } x=-\sqrt{\mathrm{e}^{y-2}}-1 \text{ since } x\ge 0) \\ &\therefore\ \mathrm{f}^{-1}(x)=\mathrm{e}^{\frac{x-2}{2}}-1 \\ &D_{\mathrm{f}^{-1}}=R_{\mathrm{f}}=[2,\infty)\end{aligned}$$

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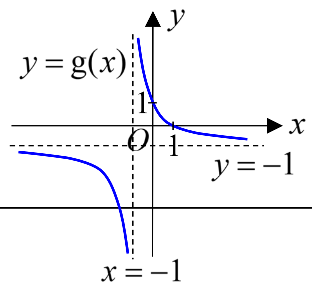
<tr><td>(c)</td><td>

$R_{\mathrm{f}^{-1}}=D_{\mathrm{f}}=[0,\infty)$

$D_{\mathrm{g}^{-1}}=R_{\mathrm{g}}=\mathbb{R}\setminus\{-1\}$

Since $R_{\mathrm{f}^{-1}}\subset D_{\mathrm{g}^{-1}}$,

$\therefore\ \mathrm{g}^{-1}\mathrm{f}^{-1}$ exists.



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9	**Solution**

Let $y=\dfrac{1-x}{x+1}$

$xy+y=1-x$

$x(y+1)=1-y$

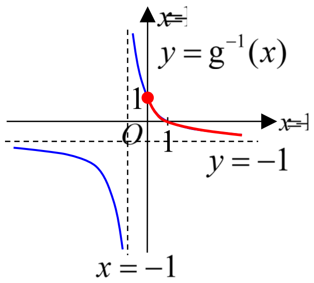
$\therefore\ x=\dfrac{1-y}{y+1}$

$\therefore\ \mathrm{g}^{-1}(x)=\dfrac{1-x}{x+1}$

Using $R_{\mathrm{f}^{-1}}=[0,\infty)$ as restricted domain of g^{-1},

$R_{\mathrm{g}^{-1}}=(-1,1]$

Hence $R_{\mathrm{g}^{-1}\mathrm{f}^{-1}}=(-1,1]$



9	**Solution**
(d)	

$\mathrm{g}^{-1}\mathrm{f}^{-1}(x)=\dfrac{1}{2}$

$\mathrm{g}\mathrm{g}^{-1}\mathrm{f}^{-1}(x)=\mathrm{g}(\frac{1}{2})$

$\therefore\ \mathrm{f}^{-1}(x)=\dfrac{1-\frac{1}{2}}{\frac{1}{2}+1}=\dfrac{1}{3}$

$\mathrm{f}\mathrm{f}^{-1}(x)=\mathrm{f}(\frac{1}{3})$

$\Rightarrow\quad x=\ln(\frac{1}{3}+1)^2+2=\ln\dfrac{16}{9}+2$

10	**Solution**
(a)	

$z=\dfrac{1}{4}(1+\sqrt{3}\,\mathrm{i})$

$\left|1+\sqrt{3}\,\mathrm{i}\right|=\sqrt{1^2+(\sqrt{3})^2}=\sqrt{4}=2$

$\arg(1+\sqrt{3}\,\mathrm{i})=\tan^{-1}\dfrac{\sqrt{3}}{1}=\dfrac{\pi}{3}$

$\therefore\ z=\dfrac{1}{4}(2\mathrm{e}^{\mathrm{i}\frac{\pi}{3}})=\dfrac{1}{2}\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}$

10	**Solution**
(b)	

$z^2=\left(\dfrac{1}{2}\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}\right)^2=\dfrac{1}{4}\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}$

10	**Solution**
	$z^3 = \left(\frac{1}{2}e^{i\frac{\pi}{3}}\right)^3 = \frac{1}{8}e^{i\pi}$

10	**Solution**
(c)	Area of $\Delta OP_0P_1 = \frac{1}{2}(1)\left(\frac{1}{2}\right)\sin\frac{\pi}{3} = \frac{\sqrt{3}}{8}$ unit^2 Area of $\Delta OP_1P_2 = \frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{1}{4}\right)\sin\frac{\pi}{3} = \frac{\sqrt{3}}{32}$ unit^2 Area of $\Delta OP_2P_3 = \frac{1}{2}\left(\frac{1}{4}\right)\left(\frac{1}{8}\right)\sin\frac{\pi}{3} = \frac{\sqrt{3}}{128}$ unit^2 $\therefore$ Area of $(3+1)$-polygon $= \frac{\sqrt{3}}{8} + \frac{\sqrt{3}}{32} + \frac{\sqrt{3}}{128}$ $= \frac{21\sqrt{3}}{128}$ unit^2

10	**Solution**
(d)	Area of $(n+1)$-polygon $\begin{aligned} &= \frac{1}{2}(1)(\frac{1}{2})\sin\frac{\pi}{n} + \frac{1}{2}(\frac{1}{2})(\frac{1}{4})\sin\frac{\pi}{n} + \frac{1}{2}(\frac{1}{4})(\frac{1}{8})\sin\frac{\pi}{n} \\ &\quad + \ldots + \frac{1}{2}(\frac{1}{2^{n-1}})(\frac{1}{2^n})\sin\frac{\pi}{n} \\ &= \frac{1}{2}\left(1\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{4} + \frac{1}{4}\cdot\frac{1}{8} + \ldots + \frac{1}{2^{n-1}}\cdot\frac{1}{2^n}\right)\sin\frac{\pi}{n} \\ &= \frac{1}{2}\underbrace{\left(\frac{1}{2^1} + \frac{1}{2^3} + \frac{1}{2^5} + \ldots + \frac{1}{2^{2n-1}}\right)}_{n \text{ terms}}\sin\frac{\pi}{n} \\ &= \frac{1}{2}\left(\frac{\frac{1}{2}\left(1-\left(\frac{1}{2^2}\right)^n\right)}{1-\frac{1}{2^2}}\right)\sin\frac{\pi}{n} \\ &= \frac{1}{3}\left(1-\left(\frac{1}{4}\right)^n\right)\sin\frac{\pi}{n} \quad \text{where } a=\frac{1}{3},\ b=\frac{1}{4} \end{aligned}$

11	**Solution**
(a)	By Pythagoras' Theorem, $r^2 + h^2 = 2^2$ $\therefore \quad h = \sqrt{4-r^2} \quad$ (reject $h = -\sqrt{4-r^2}$ since $h > 0$) Let r_0 be radius of cut off cone. Using similar $\triangle$s, $\frac{r_0}{r} = \frac{1}{4} \Rightarrow r_0 = \frac{r}{4}$ $\therefore \; V = \pi r^2(r) + \frac{1}{3}\pi r^2 h - \frac{1}{3}\pi r_0^{\,2}(\frac{1}{4}h)$ $= \pi r^3 + \frac{1}{3}\pi r^2\sqrt{4-r^2} - \frac{1}{192}\pi r^2\sqrt{4-r^2}$ $= \pi r^3 + \frac{21}{64}\pi r^2\sqrt{4-r^2} \qquad$ (shown)

11	**Solution**
(b)	$\frac{\mathrm{d}V}{\mathrm{d}r} = 3\pi r^2 + \frac{21}{64}\pi\left[2r\sqrt{4-r^2} + r^2\left(\frac{1}{2\sqrt{4-r^2}}\right)(-2r)\right]$ $= 3\pi r^2 + \frac{21}{64}\pi r\left[2\sqrt{4-r^2} - \frac{r^2}{\sqrt{4-r^2}}\right]$ $= 3\pi r^2 + \frac{21}{64}\pi r\frac{8-3r^2}{\sqrt{4-r^2}}$ At maximum value of V, $\frac{\mathrm{d}V}{\mathrm{d}r} = 0$, since $0 < r < 2$ $\therefore \; 3\pi r^2 + \frac{21}{64}\pi r\frac{8-3r^2}{\sqrt{4-r^2}} = 0$ $3\pi r\left[r + \frac{7}{64}\left(\frac{8-3r^2}{\sqrt{4-r^2}}\right)\right] = 0$ $r + \frac{7}{64}\left(\frac{8-3r^2}{\sqrt{4-r^2}}\right) = 0 \quad$ (since $0 < r < 2$) $r = \frac{7}{64}\left(\frac{3r^2-8}{\sqrt{4-r^2}}\right)$ $64r\sqrt{4-r^2} = 21r^2 - 56$ $4096r^2(4-r^2) = (21r^2-56)^2$ $16384r^2 - 4096r^4 = 441r^4 - 2352r^2 + 3136$ $\therefore \; 4537r^4 - 18736r^2 + 3136 = 0 \quad$ (shown)

11	**Solution**
(c)	Using GC, since $r > 0$, $r = 0.41806129085388 \quad$ or $\quad r = 1.9886743863834$ When $r = 0.41806129085388$, $\frac{\mathrm{d}V}{\mathrm{d}r} = 3.294435715 \neq 0$ Hence $r = 0.41806129085388$ does not give a stationary value of V.

11	**Solution**
	The squaring of the equation to remove the square root created additional roots to the equation. These additional roots may not give rise to stationary values of V .

11	**Solution**
(d)	$r_1 = 1.99$ (3 s.f.) $h = \sqrt{4 - r_1^2} = 0.2125421957 = 0.213$ (3 s.f.)

11	**Solution**
(e)	The value of $r_1 = 1.99$ is almost the same length as the slant height 2 of the inverted cone. This means that the inverted cone is essentially flat and non-existent, leaving the printer nozzle in the shape of a cylinder only. Hence it is not realistic to have a maximum volume for the printer nozzle. Ideal measurements/values/conditions based on theoretical calculations may not be practical or feasible in real life.

12	**Solution**
(a)	$u_{n+1} = (1+p)u_n$ Since $\dfrac{u_{n+1}}{u_n} = 1 + p = \text{constant}$ (since p is a constant) , $\therefore$ $\{u_n\}$ is a geometric progression.

12	**Solution**
(a)	$u_{n+1} = (1+p)u_n$ Since $\dfrac{u_{n+1}}{u_n} = 1 + p = \text{constant}$ (since p is a constant) , $\therefore$ $\{u_n\}$ is a geometric progression.
(b)	

n	u_n
1	2^m
2	$(1+p)u_1 = (1+p)2^m$
3	$(1+p)u_2 = (1+p)[(1+p)2^m]$ $= (1+p)^2 2^m$

12	**Solution**
	$2^m+(1+p)2^m+(1+p)^2 2^m=\frac{127}{36}(2^m)$ $1+(1+p)+(1+2p+p^2)=\frac{127}{36}$ $p^2+3p+3=\frac{127}{36}$ $p^2+3p-\frac{19}{36}=0$ $36p^2+108p-19=0$ Using GC, $p=\frac{1}{6}$ or $p=-\frac{19}{6}$ (rejected since $p>0$) Hence $p=\frac{1}{6}$

12	**Solution**
(c)	Since common ratio $r=\frac{7}{6}>1$, sum to infinity does not exist. As $n\to\infty$, total amount of data $S_n=\frac{a(\frac{7}{6}^n-1)}{\frac{7}{6}-1}$ diverges. Hence there is no limit to total amount of data the data processing centre can handle.

12	**Solution**
(d)	<table><tr><td>n</td><td>v_n</td><td></td></tr><tr><td>1</td><td>2^m</td><td></td></tr><tr><td>2</td><td>$2^m(\frac{5}{4})$</td><td></td></tr><tr><td>3</td><td>$2^m(\frac{5}{4})^2$</td><td></td></tr><tr><td>4</td><td>$2^m(\frac{5}{4})^3$</td><td rowspan="4">AP with 1st term $2^m(\frac{5}{4})^3$, common difference -25</td></tr><tr><td>5</td><td>$2^m(\frac{5}{4})^3-25$</td></tr><tr><td>6</td><td>$2^m(\frac{5}{4})^3-25(2)$</td></tr><tr><td>$\vdots$</td><td></td></tr></table> From v_4 to v_r, no. of terms $=r-4+1=r-3$

12	**Solution**
	$\therefore\ v_r = 2^m\left(\dfrac{5}{4}\right)^3 + [(r-3)-1](-25)$ $= \dfrac{125}{64}(2^m) - 25(r-4)$ $= 125(2^{m-6}) - 25r + 100$ $= 25[5(2^{m-6}) - r + 4]$ where $k = 25$ (shown)

12	**Solution**
(e)	$v_1 + v_2 + v_3 + \ldots + v_{15}$ $= (2^m) + \dfrac{5}{4}(2^m) + \dfrac{25}{16}(2^m) + v_4 + \ldots\ldots + v_{15}$ $= \dfrac{61}{16}(2^m) + \dfrac{15-4+1}{2}(v_4 + v_{15})$ $= \dfrac{61}{16}(2^m) + 6\left[\dfrac{125}{64}(2^m) + \left(25[5(2^{m-6}) - 15 + 4]\right)\right]$ $= \dfrac{61}{16}(2^m) + 6\left[\dfrac{125}{64}(2^m) + \left(\dfrac{125}{64}(2^m) - 275\right)\right]$ $= \dfrac{61}{16}(2^m) + 6\left[\dfrac{125}{32}(2^m) - 275\right]$ $= \dfrac{109}{4}(2^m) - 1650$ where $s = \dfrac{109}{4}$, $t = -1650$

12	**Solution**
(f)	Based on revised operating procedure, amount of data processed after a certain number of days will become negative due to AP with negative common difference. Since it is not possible to process negative amount of data, it is not meaningful to compute such data in the long run.