Section A: Pure Mathematics [40 marks]

1 A curve G has equation $y=\dfrac{1}{\cos x}+\cos x$.

The region bounded by G, the line $x=\dfrac{\pi}{3}$, the x-axis and the y-axis is rotated through 4 right angles about the x-axis. Find the exact volume generated, giving your answer in terms of π. [4]

2 **(a)** Use the substitution $u=1-x^3$ to find $\displaystyle\int \frac{x^5}{\sqrt{1-x^3}}\,\mathrm{d}x$. [4]

(b) Hence find $\displaystyle\int \frac{x^8}{\sqrt{1-x^3}}\,\mathrm{d}x$. [4]

3 A complex number z_1 is given by $z_1=a+(a-3)\mathrm{i}$, where a is a positive real constant. It is given that $\arg z_1=\theta$, where $-\dfrac{\pi}{2}<\theta<0$.

(a) Find, leaving your answers in terms of θ,

(i) $\arg(-2z_1)$, [2]

(ii) $\arg(z_1-2a)$. [2]

Another complex number z_2 is given by $z_2=1+3\mathrm{i}$.

(b) Without using a calculator, find the range of values of a such that

$$\frac{|z_1z_2|^2}{\operatorname{Im}(z_1z_2)}\le 10.$$ [5]

4 A curve C has parametric equations

$$x=t^2,\quad y=t^3+1\quad \text{for } t\in\mathbb{R}.$$

(a) Sketch the graph of C, giving the coordinates of any axial intercepts. [2]

(b) Without using a calculator, find the equation of the tangent to C at the point P where $t=1$. [2]

(c) The tangent to C at P meets the curve again at the point Q. Find the exact coordinates of Q. [3]

(d) Hence find the exact area of the region bounded by C and the tangent to C at P. [3]

5 The equations of two planes p_1 and p_2 are

$$\begin{aligned} 2x + sy + z &= -3 \ , \\ x - 2y + 3z &= 1 \ , \end{aligned}$$

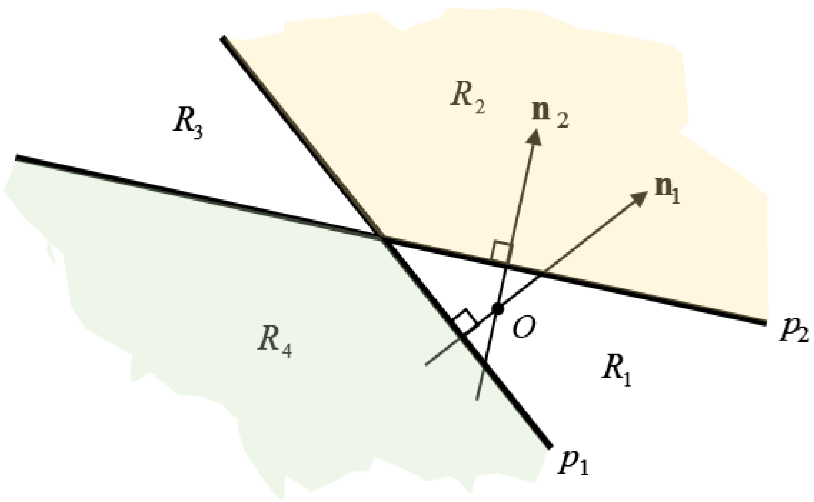
respectively, where s is a negative real constant.

(a) Find a vector parallel to both p_1 and p_2, giving your answer in terms of s. [2]

(b) It is given that p_1 and p_2 intersect in a line l, and l meets the xz-plane at a point A. Find the coordinates of A. [3]

(c) The shortest distance between the origin O and p_1 is $\frac{3}{\sqrt{6}}$. Find the value of s. [2]

It is now given that $s = -2$, with the normal vectors to p_1 and p_2 denoted as $\mathbf{n_1} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ and $\mathbf{n_2} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ respectively. It is given that p_1 and p_2 divide the real space into four regions R_1, R_2, R_3 and R_4 as shown in the following two-dimensional diagram (not drawn to scale).



(d) Relative to the origin O, the point B has position vector $\mathbf{k}$. Determine whether B lies in R_1, R_2, R_3 or R_4, justifying your answer. [2]

Section B: Probability and Statistics [60 marks]

6 Eight students are shortlisted to be prize recipients for a competition. Two students are awarded the first prize and another two students are awarded the second prize.

The eight shortlisted students are to stand in a single row for a photo-taking session with a guest of honour.

Find the number of different possible arrangements if the two first prize awardees must stand together while the two second prize awardees also must stand together during the session. [2]

After the photo-taking session, the eight students proceed for a tea reception and are to be seated at a round table with 10 identical seats. If 4 students are to be seated on each side of the guest of honour at the round table, find the total number of possible seating arrangements. [2]

7 In a school with a large number of students, p%, where $p > 40$, of its students study H2 Biology. In a random sample of 40 students, the probability of no less than nine but no more than twenty students studying H2 Biology is 0.25. Find the value of p. [3]

It is given that $p = 60$. A group of 12 randomly chosen students are to attend an interview session at a meeting room, with only one interviewee being interviewed at a time. Find the probability that the third and tenth interviewee are the first and fourth students respectively among them studying H2 Biology. [3]

8 A biased 4-sided die gives the scores 1, 2, 3, and 4 with the probabilities shown in the table, where a and b are constants. The random variable S denotes the score on the die.

Score (s)	1	2	3	4
$\mathrm{P}(S=s)$	a	a	a	b

It is given that the mean of the score is 2.56.

(a) Show that $b = \dfrac{7}{25}$. [2]

(b) S_1 and S_2 are two independent observations of S.

(i) Find the probability that the sum of the two independent observations of S is at least 6 given that one of the observed scores is 3, [3]

Let $Y = |S_1 - S_2|$. Without finding the probability distribution table of Y,

(ii) find $\mathrm{Var}(2S - \mathrm{E}(Y))$. [3]

(iii) state the value of $\mathrm{E}(Y - \mathrm{E}(Y))$. [1]

9 Players A and B play against each other in a racquet match consisting of at most 3 sets. Each set is won by either Player A or B, and the match is won by the first person to win two sets. Player A has a probability of $\frac{a-k}{8}$, where a is a real positive constant, of winning at the k^{th} set where $k = 1, 2, 3$.

(a) **(i)** Show that the probability that player A will win the match is $\frac{-a^3+18a^2-59a+50}{256}$. [3]

(ii) Find all the values of a for the two players to have an equal chance of winning. Hence explain why only one value is suitable in this context. [3]

(b) Given that $a = 7$ and player A wins the match, find the probability that player A wins the second set. [2]

10 **In this question you should, where applicable, state clearly all the distributions that you use, together with the values of the appropriate parameters.**

A supermarket produces its own packs of white sesame seeds and fennel seeds for sale under its house brand. The masses (in grams) of a randomly chosen pack of white sesame seeds and that of a randomly chosen pack of fennel seeds, denoted by W and F respectively, have independent normal distributions. The means and standard deviations of these distributions are shown in the following table.

Mass of a pack of	Mean (g)	Standard deviation (g)
white sesame seeds (W)	250	σ
fennel seeds (F)	300	2.5

It is given that 5% of the packs of white sesame seeds have a mass of less than 245.

(a) Show that the value of σ is 3.0398, correct to 5 significant figures. [2]

(b) Find the probability that the total mass of six randomly chosen packs of white sesame seeds is less than five times the mass of a randomly chosen pack of fennel seeds by not more than 20 grams. [3]

(c) n packs of white sesame seeds and n packs of fennel seeds are chosen at random. Find the smallest value of n such that the probability that the mean mass of these $2n$ packs being at least 278 grams is less than 0.015. [4]

11 Researchers at University S claimed that *Fungus A*, a strain of fungi that is typically found in soil and plants, was able to break down polypropylene, a plastic that is hard to recycle, that had been pre-treated with either UV light or heat by at least 27 grams per 100 grams of polypropylene over 90 days.

Researchers at University M believe that the researchers at University S overstated their claim and decided to perform a hypothesis test on a random sample of 120 batches of polypropylene, each of mass 100 grams. The researchers recorded the amount x grams of polypropylene for each batch that *Fungus A* was able to break down over a 90-day period. The results are summarised as follows:

$$\sum(x-27)=-81,\ \sum(x-27)^2=2070.8$$

(a) Calculate the unbiased estimates of the population mean and variance. [3]

(b) Test, at 5% level of significance, the claim that the population mean mass of polypropylene per 100 grams that were broken down is at least 27 grams. You should state your hypotheses and define the symbols you use. [4]

(c) State, in the context of the question, what you understand by the expression 'at 5% level of significance' found in part **(b)**. [1]

(d) Explain if it is necessary for University M to assume a normal distribution for the mass of polypropylene per 100 grams that *Fungus A* was able to break down for this test to be valid. [1]

University S responded by experimenting on 50 batches of polypropylene, each of mass 100 grams, and recorded the amount y grams of polypropylene for each batch that *Fungus A was* able to break down over a 90-day period. The null and alternative hypotheses remain the same as the ones stated by University M in part **(b)**.

Based on the research findings from University S, the population standard deviation of the mass of polypropylene per 100 grams that *Fungus A* was able to break down is known to be 4 grams.

(e) Find the set of values of sample mean mass, $\bar{y}$ grams, of polypropylene per 100 grams from University S such that the null hypothesis is not rejected in favour of the alternative hypothesis at 5% level of significance. [3]

12 The table gives the record time, t seconds, for a particular 100-metre freestyle swimming long course event for women for the number of years, y, after Year 2000.

y	4	6	8	12	16	18	19	20	21
t	53.52	53.11	52.88	52.25	52.06	52.03	52.03	52.9	51.96

(a) Draw the scatter diagram for these values, labelling the axes clearly. Circle the point on the scatter diagram that seems to have the value of t recorded wrongly and label it as P. [3]

Omit point *P* identified in part (a) for the rest of this question.

(b) It is desired to predict the record time in future years. Explain why, in this context, a linear model is likely to be inappropriate. [1]

The following models are proposed to describe the relationship between t and y:
For $a > 0$ and $b > 0$,

$$\text{Model A: } t = a - by^2; \qquad \text{Model B: } t = a + \frac{b}{y}.$$

(c) Explain, without further calculation, why Model B provides a more accurate model of the relationship between t and y than Model A. [1]

(d) For Model B, find the product moment correlation coefficient between t and $\dfrac{1}{y}$.

State the values of a and of b correct to 5 significant figures. [3]

(e) Use your equation to estimate the record time in 2025, correct to 3 significant figures. Explain whether your estimate is reliable. [2]

(f) It is required to estimate the value of y for which t = 52.45. Use a suitable equation to find the estimated value of y, correct to the nearest integer. Explain why neither the regression line of y^2 on t nor the regression line of $\dfrac{1}{y}$ on t should be used. [2]