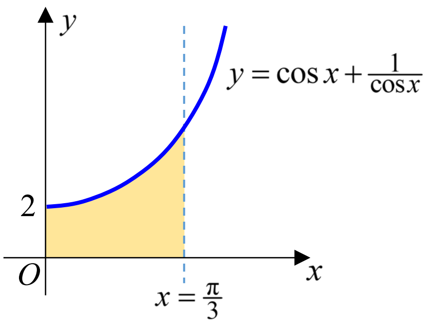
2023 HCI C2 H2 Mathematics Prelim P2 Solutions

1. Solution



$$V_A=\pi\int_0^{\frac{\pi}{3}}\left(\frac{1}{\cos x}+\cos x\right)^2\,\mathrm{d}x$$

$$=\pi\int_0^{\frac{\pi}{3}}\left(\frac{1}{\cos^2 x}+\cos^2 x+2\right)\mathrm{d}x$$

$$=\pi\int_0^{\frac{\pi}{3}}\left(\sec^2 x+\frac{1}{2}(\cos 2x+1)+2\right)\mathrm{d}x$$

$$=\pi\left[\tan x+\frac{1}{4}(\sin 2x)+\frac{5}{2}x\right]_0^{\frac{\pi}{3}}$$

$$=\pi\left(\tan\frac{\pi}{3}+\frac{1}{4}\left(\sin\frac{2\pi}{3}\right)+\frac{5}{2}\left(\frac{\pi}{3}\right)\right)$$

$$=\pi\left(\sqrt{3}+\frac{5\pi}{6}+\frac{\sqrt{3}}{8}\right)$$

$$=\pi\left(\frac{9\sqrt{3}}{8}+\frac{5\pi}{6}\right)\ \text{unit}^3$$

2 a. Solution

$$u=1-x^3 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x}=-3x^2$$

$$\int\frac{x^5}{\sqrt{1-x^3}}\,\mathrm{d}x=-\frac{1}{3}\int\frac{x^3}{\sqrt{1-x^3}}(-3x^2)\,\mathrm{d}x$$

$$=-\frac{1}{3}\int\frac{1-u}{\sqrt{u}}\,\mathrm{d}u$$

$$=-\frac{1}{3}\int\frac{1}{\sqrt{u}}-\sqrt{u}\,\mathrm{d}u$$

$$=-\frac{2}{3}u^{\frac{1}{2}}+\frac{2}{9}u^{\frac{3}{2}}+C$$

$$=-\frac{2}{3}(1-x^3)^{\frac{1}{2}}+\frac{2}{9}(1-x^3)^{\frac{3}{2}}+C$$

2b. Solution

Let $u = x^3$ and $v' = \dfrac{x^5}{\sqrt{1-x^3}}$

$\therefore\ u' = 3x^2$ and $v = -\dfrac{2}{3}(1-x^3)^{\frac{1}{2}} + \dfrac{2}{9}(1-x^3)^{\frac{3}{2}}$

$$\int \frac{x^8}{\sqrt{1-x^3}}\,\mathrm{d}x$$
$$= \int x^3\left(\frac{x^5}{\sqrt{1-x^3}}\right)\mathrm{d}x$$
$$= x^3\left(-\frac{2}{3}(1-x^3)^{\frac{1}{2}} + \frac{2}{9}(1-x^3)^{\frac{3}{2}}\right) - \int 3x^2\left(-\frac{2}{3}(1-x^3)^{\frac{1}{2}} + \frac{2}{9}(1-x^3)^{\frac{3}{2}}\right)\mathrm{d}x$$
$$= -\frac{2}{3}x^3(1-x^3)^{\frac{1}{2}} + \frac{2}{9}x^3(1-x^3)^{\frac{3}{2}} - \frac{2}{3}\int(-3x^2)(1-x^3)^{\frac{1}{2}}\,\mathrm{d}x + \frac{2}{9}\int(-3x^2)(1-x^3)^{\frac{3}{2}}\,\mathrm{d}x$$
$$= -\frac{2}{3}x^3(1-x^3)^{\frac{1}{2}} + \frac{2}{9}x^3(1-x^3)^{\frac{3}{2}} - \frac{4}{9}(1-x^3)^{\frac{3}{2}} + \frac{4}{45}(1-x^3)^{\frac{5}{2}} + C$$

3ai. Solution

$$\arg(-2z_1) = \arg(-2) + \arg(z_1)$$
$$= \pi + \theta$$

3aii. Solution

$z_1 - 2a$

$= a + (a-3)\mathrm{i} - 2a$

$= -a + (a-3)\mathrm{i}$

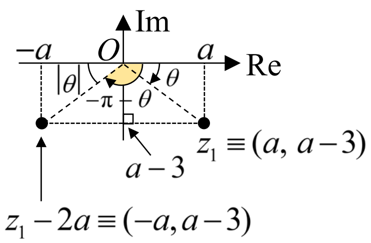
$\arg(z_1 - 2a)$

$= \arg[-a + (a-3)\mathrm{i}]$

$= -\pi + |\theta|$

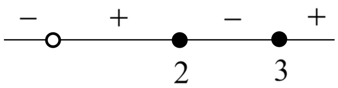
$= -\pi + (-\theta)$

$= -\pi - \theta$



3b. Solution

$$z_1 z_2 = (a+(a-3)\mathrm{i})(1+3\mathrm{i})$$
$$= (a-3a+9)+(3a+a-3)\mathrm{i}$$
$$= (-2a+9)+(4a-3)\mathrm{i}$$
$$\therefore\ \mathrm{Im}(z_1 z_2) = 4a-3$$
$$|z_1 z_2|^2 = |z_1|^2 |z_2|^2 = (a^2+(a-3)^2)(1^2+3^2)$$
$$= 10(2a^2-6a+9)$$
$$\therefore\ \frac{|z_1 z_2|^2}{\mathrm{Im}(z_1 z_2)} \le 10$$
$$\frac{10(2a^2-6a+9)}{4a-3} \le 10$$
$$\frac{2a^2-6a+9}{4a-3} \le 1$$
$$\frac{2a^2-6a+9}{4a-3} - 1 \le 0$$
$$\frac{2a^2-10a+12}{4a-3} \le 0$$
$$\frac{2(a^2-5a+6)}{4a-3} \le 0$$
$$\frac{2(a-2)(a-3)}{4a-3} \le 0$$



$$\therefore\ a < \frac{3}{4} \quad \text{or} \quad 2 \le a \le 3$$

Since $-\frac{\pi}{2} < \arg z_1 < 0$, $\therefore\ \mathrm{Im}(z_1) = a-3 < 0$

Also, it is given that $a > 0$

Hence $0 < a < 3$

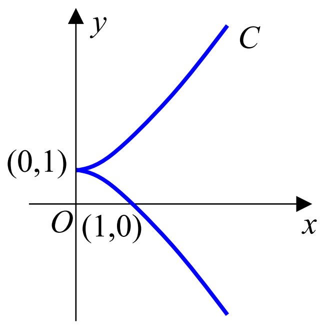
$\therefore$ required range of values of a is

$$0 < a < \frac{3}{4} \quad \text{or} \quad 2 \le a < 3$$

4a. Solution

When $x = 0$, $t = 0$ and $y = 1$

When $y = 0$, $t = -1$ and $x = 1$



4b. Solution

$\frac{dx}{dt} = 2t$, $\frac{dy}{dt} = 3t^2$

$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2}{2t} = \frac{3}{2}t$

When $t=1$, $x=1$, $y=2$ and $\frac{dy}{dx} = \frac{3}{2}$

$\therefore$ equation of tangent at P is

$y-2 = \frac{3}{2}(x-1)$

$y = \frac{3}{2}x + \frac{1}{2}$ ...(*)

4c. Solution

Substitute $x = t^2$ and $y = t^3 + 1$ into (*)::

$$t^3 + 1 = \frac{3}{2}t^2 + \frac{1}{2}$$
$$2t^3 + 2 = 3t^2 + 1$$
$$2t^3 - 3t^2 + 1 = 0$$
$$(t-1)(2t^2 - t - 1) = 0$$
$$(t-1)(2t+1)(t-1) = 0$$

$\therefore t = -\frac{1}{2}$ or $t=1$ (reject since this is P)

When $t = -\frac{1}{2}$, $x = \frac{1}{4}$ and $y = \frac{7}{8}$

Hence coordinates of Q is $\left(\frac{1}{4}, \frac{7}{8}\right)$.

<u>Alternatively:</u>

Solving $y = \frac{3}{2}x + \frac{1}{2}$ and $y = \pm x^{\frac{3}{2}} + 1$ since $t = \pm\sqrt{x}$

Notice that the tangent meets at Q at $y = -x^{\frac{3}{2}} + 1$. $\frac{3}{2}x + \frac{1}{2} = -x^{\frac{3}{2}} + 1$

$3x + 1 = -2x^{\frac{3}{2}} + 2$

$4x^3 - 9x^2 + 6x - 1 = 0$

$(x-1)\left(4x^2 - 5 + 1\right) = 0$

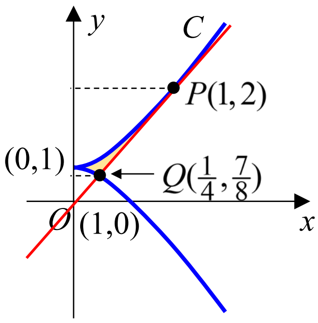
$(x-1)^2(4x-1) = 0$

$x = 1$ (rej) or $x = \frac{1}{4}$

$$y=-\left(\frac{1}{4}\right)^{\frac{3}{2}}+1=\frac{7}{8} \text{ or } y=\frac{3}{2}\left(\frac{1}{4}\right)+\frac{1}{2}=\frac{7}{8}$$

Hence coordinates of Q is $\left(\frac{1}{4},\frac{7}{8}\right)$.

4d. Solution



$$\begin{aligned}\text{Area} &= \frac{1}{2}\left(\frac{1}{4}+1\right)\frac{9}{8}-\int_{\frac{7}{8}}^{2} x \, \mathrm{d}y \\ &= \frac{45}{64}-\int_{-\frac{1}{2}}^{1} t^2\, 3t^2 \, \mathrm{d}t \\ &= \frac{45}{64}-3\int_{-\frac{1}{2}}^{1} t^4 \, \mathrm{d}t \\ &= \frac{45}{64}-3\left[\frac{t^5}{5}\right]_{-\frac{1}{2}}^{1} \\ &= \frac{45}{64}-\frac{3}{5}\left[1-\left(-\frac{1}{32}\right)\right] \\ &= \frac{45}{64}-\frac{99}{160} \\ &= \frac{27}{320} \quad \text{unit}^2\end{aligned}$$

<u>Some alternative methods:</u>

$$\frac{1}{2}\left(\frac{1}{4}+1\right)\left(2-\frac{7}{8}\right)-\int_{\frac{7}{8}}^{2}(y-1)^{\frac{2}{3}} \, \mathrm{d}y$$

<u>OR</u>

$$\int_0^1 x^{\frac{3}{2}}+1 \, \mathrm{d}x-\int_0^{\frac{1}{4}} -x^{\frac{3}{2}}+1 \, \mathrm{d}x-\int_{\frac{1}{4}}^{1}\frac{3}{2}x+\frac{1}{2} \, \mathrm{d}x$$

<u>OR</u>

$$\int_{-\frac{1}{2}}^{1}\left(t^3+1\right)(2t) \, \mathrm{d}t-\int_{\frac{1}{4}}^{1}\frac{3}{2}x+\frac{1}{2} \, \mathrm{d}x$$

(Full credit is awarded for this method only if the graph in part **(a)** is accurate)

5a. Solution
$\begin{pmatrix}2\\s\\1\end{pmatrix}\times\begin{pmatrix}1\\-2\\3\end{pmatrix}=\begin{pmatrix}3s+2\\-5\\-4-s\end{pmatrix}$

5b. Solution
xz-plane $\Rightarrow$ $y=0$ $2x+z=-3$ ...(1) $x+3z=1$ ...(2) $2\times(2)-(1):$ $5z=5 \Rightarrow z=1$ $\therefore\ x=-2$ $\therefore\ \overrightarrow{OA}=\begin{pmatrix}-2\\0\\1\end{pmatrix}$ Hence coordinates of A is $(-2,0,1)$.

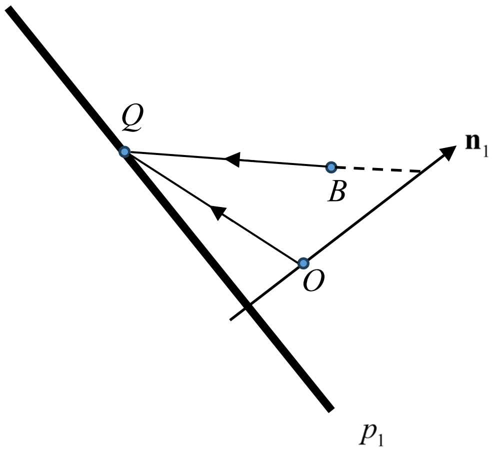
5c. Solution
$p_1:\ \underset{\sim}{r}\cdot\begin{pmatrix}2\\s\\1\end{pmatrix}=-3$ In standard form, $p_1:\ \underset{\sim}{r}\cdot\frac{1}{\sqrt{2^2+s^2+1^2}}\begin{pmatrix}2\\s\\1\end{pmatrix}=\frac{1}{\sqrt{2^2+s^2+1^2}}(-3)=\frac{-3}{\sqrt{s^2+5}}$ $\therefore$ shortest distance between O and p_1 is $\left\|\frac{-3}{\sqrt{s^2+5}}\right\|=\frac{3}{\sqrt{6}}$ $\frac{3}{\sqrt{s^2+5}}=\frac{3}{\sqrt{6}}$ $\sqrt{\frac{6}{s^2+5}}=1$ $\frac{6}{s^2+5}=1$ $6=s^2+5$ $s^2=1$ $s=\pm1$ (reject $s=1$ since $s<0$) $\therefore\ s=-1$

5d. Solution

Given $\overrightarrow{OB} = \underset{\sim}{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

<u>Method 1: Using angles</u>

$p_1 : \underset{\sim}{r} . \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = -3$



$Q(0,0,-3)$ is a point that lies in p_1.

Since $\overrightarrow{OQ} \cdot \mathbf{n}_1 = -3 < 0$,

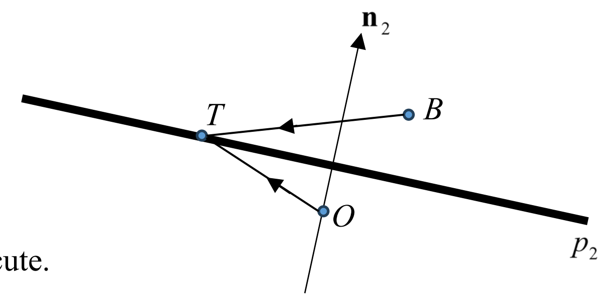
then the angle between OQ and $\mathbf{n}_1$ is obtuse.

$\overrightarrow{BQ} \cdot \mathbf{n}_1 = \begin{pmatrix} 0 \\ 0 \\ -4 \end{pmatrix} . \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = -4 < 0$,

then the angle between BQ and $\mathbf{n}_1$ is obtuse.

$\therefore$ O and B are on same side of p_1. $---(1)$

$p_2 : \underset{\sim}{r} . \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = 1$



$T(1,0,0)$ is a point that lies in p_2.

Since $\overrightarrow{OT} \cdot \mathbf{n}_2 = 1 > 0$,

then the angle between OT and $\mathbf{n}_2$ is acute.

$\overrightarrow{BT} \cdot \mathbf{n}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = -2 < 0$,

then the angle between BT and $\mathbf{n}_2$ is obtuse.

$\therefore$ O and B are on opposite side of p_2. $---(2)$

Combining (1) and (2) with reference to given diagram, $\therefore$ B lies in R_2.

<u>Method 2</u>: (using shortest distance of planes to origin O)

Let p_3 be plane containing B and parallel to p_1.

Equation of p_3 is $\underset{\sim}{r} . \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} . \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = 1$

$\therefore$ equation of p_3 in standard form is

$$p_3 : \underset{\sim}{r} . \frac{1}{\sqrt{2^2 + (-2)^2 + 1^2}} \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2^2 + (-2)^2 + 1^2}} = \frac{1}{3} > 0$$

Since equation of p_1 in standard form is $p_1 : \underset{\sim}{r} \cdot \frac{1}{3}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \frac{-3}{3} = -1 < 0$,

$\therefore$ p_1 and p_3 are on opposite sides of origin O. …(1)

Let p_4 be plane containing B and parallel to p_2.

Equation of p_4 is $\underset{\sim}{r} \cdot \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = 3$

$\therefore$ equation of p_4 in standard form is

$$p_4 : r \cdot \frac{1}{\sqrt{1^2 + (-2)^2 + 3^2}}\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = \frac{3}{\sqrt{1^2 + (-2)^2 + 3^2}} = \frac{3}{\sqrt{14}} > 0$$

Since equation of p_2 in standard form is $p_2 : \underset{\sim}{r} \cdot \frac{1}{\sqrt{14}}\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = \frac{1}{\sqrt{14}} > 0$,

$\therefore$ p_2 and p_4 are on same side of origin O. …(2)

Combining (1) and (2) with reference to given diagram, $\therefore$ B lies in R_2.

6. Solution

Group the two first prize awardees as one object, and the two second prize awardees as another object.

$\boxed{1^{st}\text{ Prize}}$ $\boxed{2^{nd}\text{ Prize}}$ $\boxed{\text{GOH}}$ $\boxed{S}$ $\boxed{S}$ $\boxed{S}$ $\boxed{S}$

There are 7 units to be arranged in a row.
Within the unit of 1st Prize and the unit of 2nd Prize, the respective prize winners may arrange themselves.

Number of ways required $= 7!2!2! = 20160$

Solution

For this arrangement, it means the guest of honour's seat opposite is an empty seat.
Thus the 8 students sit at the 8 remaining chairs.

Number of seating arrangement = 8! = 40320

7. Solution

Let X be the number of students who study H2 Biology out of 40 students.

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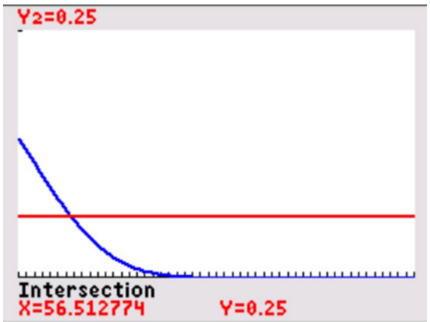
$X \sim B\left(40, \dfrac{p}{100}\right)$

$\mathrm{P}(9 \le X \le 20) = 0.25$

$\mathrm{P}(X \le 20) - \mathrm{P}(X \le 8) = 0.25$

Obtain graphs of these below using GC:

$y = \mathrm{P}(X \le 20) - \mathrm{P}(X \le 8)$ and $y = 0.25$



$p = 56.5$ (3 s.f.)

Solution

Let A be the number of students out of 6 students out of the 4^{th} to 9^{th} interviewees who study H2 Biology.

$A \sim B(6,\ 0.6)$

Required probability

$= (0.4)^2 (0.6) \mathrm{P}(A = 2)(0.6)$

$= 0.00796$ (3 s.f.)

8a. Solution

$3a + b = 1 \qquad \text{-- } (*)$

$\mathrm{E}(S) = 6a + 4b = 2.56 \qquad \text{-- } (*)$

$3a + 2b = 1.28$

Subtracting one eqn from the other,

$b = 1.28 - 1 = 0.28 = \dfrac{7}{25}$

8bi. Solution

From part (a), $a = \dfrac{6}{25}$

$\mathrm{P}(S_1 + S_2 \ge 6 \mid \text{one of the scores is } 3)$

$$=\frac{\mathrm{P}\left((S_1+S_2=3)\ \cap(S_1=3\text{ OR }S_2=3)\right)}{\mathrm{P}\left(S_1=3\text{ OR }S_2=3\right)}$$

$$=\frac{\mathrm{P}\left(S_1=3,S_2=4\right)+\mathrm{P}\left(S_1=4,S_2=3\right)+\mathrm{P}\left(S_1=3,S_2=3\right)}{\mathrm{P}\left(S_1=3,S_2\neq 3\right)+\mathrm{P}\left(S_1\neq 3,S_2=3\right)+\mathrm{P}\left(S_1=3,S_2=3\right)}$$

$$=\frac{2\left(\frac{6}{25}\right)\left(\frac{7}{25}\right)+\left(\frac{6}{25}\right)^2}{2\left(\frac{6}{25}\right)\left(\frac{19}{25}\right)+\left(\frac{6}{25}\right)^2}=\frac{120}{264}=\frac{5}{11}=0.455\text{ (3 s.f.)}$$

8bii. Solution

From part (a), $a=\dfrac{6}{25}$

$\mathrm{Var}\left(2S-\mathrm{E}\left(Y\right)\right)$

$=\mathrm{Var}\left(2S\right)\ \left(\because \mathrm{E}\left(Y\right)\text{ is a constant and }\mathrm{Var}\left(\text{constant}\right)=0\right)$

$=4\mathrm{Var}\left(S\right)$

$=4\left(\mathrm{E}\left(S^2\right)-2.56^2\right)$

$=4\left(a+4a+9a+16b-2.56^2\right)$

$=4\left(14a+16b-6.5536\right)$

$=5.1456$

8biii. Solution

$$\begin{aligned}\mathrm{E}\left(Y-\mathrm{E}\left(Y\right)\right)&=\mathrm{E}\left(Y\right)-\mathrm{E}\left(\mathrm{E}\left(Y\right)\right)\\&=\mathrm{E}\left(Y\right)-\mathrm{E}\left(Y\right)\\&=0\end{aligned}$$

9ai. Solution

P(A wins)

$= \left(\frac{a-1}{8}\right)\left(\frac{a-2}{8}\right) + \left(\frac{a-1}{8}\right)\left(\frac{10-a}{8}\right)\left(\frac{a-3}{8}\right) +$

$\left(\frac{9-a}{8}\right)\left(\frac{a-2}{8}\right)\left(\frac{a-3}{8}\right)$

$= \frac{a^2-3a+2}{8^2} + \frac{-a^3+14a^2-43a+30}{8^3} +$

$\frac{-a^3+14a^2-51a+54}{8^3}$

$= \frac{8a^2-24a+16}{8^3} + \frac{-2a^3+28a^2-94a+84}{8^3}$

$= \frac{-2a^3+36a^2-118a+100}{512}$

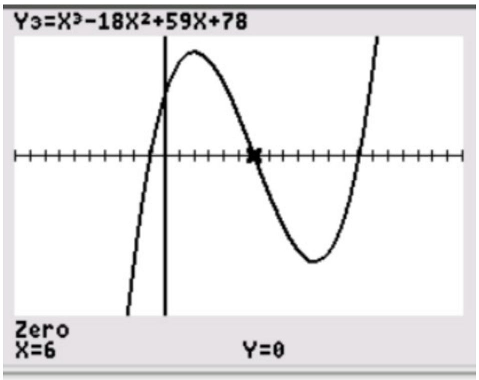
$= \frac{-a^3+18a^2-59a+50}{256}$

9aii. Solution

For the match to be fair,

$\frac{-a^3+18a^2-59a+50}{256} = \frac{1}{2}$

$a^3-18a^2+59a+78=0$



From the GC, $a = -1,\ 6$ or 13

$a = 6$

Since $a > 0$, we reject $a = -1$.

When $a = 13$, for $k = 1,\ 2,\ 3$, we have $10 \le a-k \le 12$

$\frac{10}{8} \le \frac{a-k}{8} \le \frac{12}{8}$

Since probability > 1, we reject $a = 13$.

When $a = 6$, for $k = 1,\ 2,\ 3$, we have $3 \le a-k \le 5$.

$\frac{3}{8} \le \frac{a-k}{8} \le \frac{5}{8}$

9b. Solution
Using part (a) with substitution of $a = 7$, P(A wins 2nd set | A wins the match) $= \dfrac{\frac{6}{8}\times\frac{5}{8}+\frac{2}{8}\times\frac{5}{8}\times\frac{4}{8}}{\frac{176}{256}}$ $= \dfrac{35}{44} = 0.795$ (3 s.f.)

10a. Solution
$W \sim \mathrm{N}(250,\ \sigma^2)$ $Z = \dfrac{W-250}{\sigma} \sim \mathrm{N}(0,\ 1)$ $\mathrm{P}\left(W < 245\right) = 0.05$ $\mathrm{P}\left(Z < \dfrac{245-250}{\sigma}\right) = 0.05$ $\dfrac{-5}{\sigma} = -1.644853626$ $\sigma = 3.039784161$ $= 3.0398$ (to 5 s.f.) (shown)

10b. Solution
$W_1 + W_2 + \cdots + W_6 \sim \mathrm{N}\left(1500,\ 55.44172647\right)$ $F \sim \mathrm{N}(300,\ 2.5^2)$ $5F \sim \mathrm{N}(1500,\ 156.25)$ Let $D = 5F - \left(W_1 + W_2 + \cdots + W_6\right)$ $D \sim \mathrm{N}(0,\ 211.6917265)$ $\mathrm{P}\left(0 < D \le 20\right) = 0.4153730399$ $= 0.415$ (to 3 s.f.)

10c. Solution
Let $M = \dfrac{\left(W_1 + W_2 + \cdots + W_n\right) + \left(F_1 + F_2 + \cdots + F_n\right)}{2n}$ $\mathrm{E}\left(M\right) = \dfrac{250n + 300n}{2n} = 275$

$$\text{Var}\left(M\right)=\frac{(3.039784161)^2n+(2.5)^2n}{4n^2}$$
$$=\frac{3.872571936}{n}$$
$$M\sim \text{N}\left(275,\ \frac{3.872571936}{n}\right)$$
$$Z=\frac{M-275}{\sqrt{\frac{3.872571936}{n}}}\sim \text{N}(0,\ 1)$$

<u>Method 1</u>:

$$\text{P}\left(M\geq 278\right)<0.015$$
$$\text{P}\left(Z\geq 1.524479216\sqrt{n}\right)<0.015$$
$$\text{P}\left(Z<1.524479216\sqrt{n}\right)>0.985$$
$$1.524479216\sqrt{n}>2.170090375---(*)$$
$$n>2.026341439$$
$$n\geq 3$$

$\therefore$ smallest value of n = 3

<u>Method 2</u>:

$\text{P}\left(M\geq 278\right)<0.015$

From GC, $n\geq 3$

$\therefore$ smallest value of n = 3

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PRESS ↑ TO EDIT FUNCTION

X	Y1	
1	0.0637	
2	0.0155	
3	0.0041	
4	0.0011	
5	3.3E-4	
6	9.4E-5	

11a. Solution
Let $m = x - 27$, $\sum m = -81$ and $\sum m^2 = 2070.8$. An unbiased estimate of the population mean, $\bar{x} = \bar{m} + 27$ $= \frac{-81}{120} + 27$ $= 26.325$ g An unbiased estimate for the population variance, $s_x{}^2 = \frac{1}{119}\left(\sum m^2 - \frac{\left(\sum m\right)^2}{120}\right)$ $= \frac{1}{119}\left(2070.8 - \frac{(-81)^2}{120}\right)$ $= 16.94222689$ $= 16.9 \text{ g}^2 \quad (\text{to 3 s.f.})$

11b. Solution
Let μ and σ^2 be the population mean and variance of *X*. $H_0 : \mu = 27$ $H_1 : \mu < 27$ Under H_0, since *n* = 120 is large, then by Central Limit Theorem, $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$ approximately. Test statistic: $Z = \frac{\bar{X} - \mu}{\frac{S}{\sqrt{n}}} \sim N(0,1)$ approximately. Level of significance: 5% Reject H_0 if p-value ≤ 0.05 Assuming H_0 is true, using GC, p-value = $0.0362134031 = 0.0362$ (to 3 s.f.). Since p-value = 0.0362 < 0.05, we reject H_0 and conclude that at the 5% level of significance, there is sufficient evidence to say that the population mean mass of polypropylene per 100g that were broken down is less than 27g.

11c. Solution

It is the probability of wrongly concluding that the population mean mass of polypropylene per 100g that was broken down by Fungus A was less than 27g when the population mean mass is, in fact, 27 g, is 0.05.

11d. Solution

Since the sample size of 120 is large enough, University M can use Central Limit Theorem to approximate the distribution of the sample mean mass of polypropylene per 100g that were broken down to be a normal distribution, hence University M does not need to know the distribution of the mass of polypropylene per 100g that were broken down.

11e. Solution

Given $n = 50$, $\sigma^2 = 4^2$.

Under H_0, since $n = 50$ is large, then by Central Limit Theorem, $\overline{Y} \sim \mathrm{N}\left(27, \dfrac{4^2}{50}\right)$ approximately.

Test statistic: $Z = \dfrac{\overline{Y} - 27}{\sqrt{\dfrac{4^2}{50}}} \sim \mathrm{N}(0,1)$

Level of significance: 5%

Reject H_0 if p-value ≤ 0.05

Given that H_0 is not rejected at 5% level of significance,

p-value $>$ 0.05

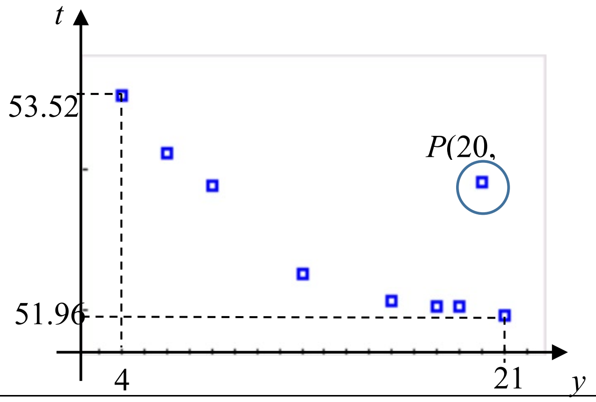
$$\frac{\overline{y} - 27}{\sqrt{\frac{4^2}{50}}} > -1.644853626$$

$$\overline{y} - 27 > -0.9304697224$$

$$\overline{y} > 26.06953028$$

$\therefore$ set of values $= \{\overline{y} \in \mathbb{R} : \ 26.1 \leq \overline{y} \leq 100\}$ (to 3 s.f.)

12a. Solution



12b. Solution
A linear model is inappropriate as it will predict that record time will be 0 seconds or below in the future.

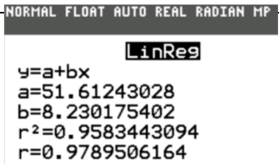
12c. Solution
The scatter diagram shows that as y increases, t decreases and concaves upward. In Model A, as y increases, t decreases and the curve concaves downward. In Model B, as y increases, t decreases and the curve concave upward. Hence Model B is more accurate than Model A.

12d. Solution

Using Model B, from GC,

$r = 0.9789506164$

$\quad = 0.979 \quad$ (to 3 s.f.)



Regression line of t on $\dfrac{1}{y}$:

$$t = 51.61243028 + \frac{8.230175402}{y}$$

$\therefore\ a = 51.612,\ b = 8.2302$ (to 5 s.f.)

12e. Solution

In year 2025, $y = 25$

$$t = 51.61243028 + \frac{8.230175402}{25}$$

$= 51.94163729$

$= 51.9$ s $\quad$ (to 3 s.f.)

Since the year 2025 ($y = 25$) is outside of the data range $4 \le y \le 21$, the estimate is not reliable.

12f. Solution

When $t = 52.45$,

$$52.45 = 51.61243028 + \frac{8.230175402}{y}$$

$y = 9.826257093$

$\quad = 10 \quad$ (to nearest integer)

Since y is the only independent variable, neither the regression line of y on t nor the regression line of $\frac{1}{y}$ on t should be used.