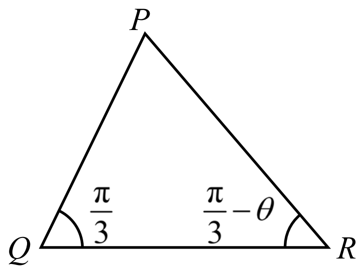
1 The variables x and y are related by the differential equation

$$\frac{d^2y}{dx^2} = 2e^{-2x} .$$

Solve the differential equation. Hence find a possible solution curve that has an oblique asymptote, and state the equation of the oblique asymptote. [4]

2



In the triangle PQR, angle $PQR = \frac{\pi}{3}$ radians and angle $PRQ = \frac{\pi}{3} - \theta$ radians, where θ is small enough for θ^3 and higher powers of θ to be negligible.

Show that $\frac{PR}{PQ} \approx 1 + \alpha\theta + \beta\theta^2$, where α and β are exact real constants to be determined. [5]

3 A curve C has equation $x^2y + y^4 = 5$.

(a) Find $\frac{dy}{dx}$ in terms of x and y. [2]

(b) Find the exact distance between the two tangents to C that are parallel to the x-axis. [3]

4 Without the use of a calculator, show that

$$\int_{-2}^{2} \frac{|x-2|}{x^2 - 2x + 4}\,dx = p \ln 3 + q\pi ,$$

where p and q are exact real constants to be determined. [6]

5 A curve G with equation $y = \mathrm{f}(3x-2)+a$, where a is a positive real constant, is transformed onto the curve with equation $y = \mathrm{f}(x)$ by a sequence of transformations.

(a) Describe fully, in terms of a, the sequence of transformations. [3]

A point on G with coordinates $(b, 0)$ is mapped onto the point with coordinates (c, d) on the curve with equation $y = \dfrac{1}{\mathrm{f}(x)}$.

(b) Find c and d in terms of a and b. [4]

6 **(a)** By writing $\dfrac{1}{(2r+1)(2r+3)}$ in partial fractions, find an expression for $\displaystyle\sum_{r=1}^{n} \frac{1}{(2r+1)(2r+3)}$ in terms of n. [4]

(b) Explain why $\displaystyle\sum_{r=1}^{n} \left[\left(-\frac{1}{3}\right)^{r} + \frac{1}{(2r+1)(2r+3)} \right]$ is a convergent series, and state its sum to infinity. [4]

7 It is given that

$$\mathrm{f}(x) = \begin{cases} \tan\left(\dfrac{\pi}{4a}x\right) & \text{for } -a < x \le a, \\ \sin\left(\dfrac{\pi}{2a}x\right) & \text{for } a < x \le 3a, \end{cases}$$

and that $\mathrm{f}(x) = \mathrm{f}(x-4a)$ for all real values of x, where a is a positive real constant.

(a) Sketch the graph of $y = \mathrm{f}(x)$ for $-4a \le x \le 7a$. [3]

(b) By using the sketch in part **(a)**, write down an equation relating $\mathrm{f}(x)$ and $\mathrm{f}(-x)$. [1]

(c) Show that $\displaystyle\int_{-2a}^{4a} \mathrm{f}(x)\,\mathrm{d}x = \frac{ka}{\pi}(1+\ln m)$, where k and m are constants to be determined. [5]

8 Relative to the origin O, two points A and B have position vectors given by $\mathbf{a} = 2\mathbf{i} - \mathbf{j}$ and $\mathbf{b} = \alpha\mathbf{k}$ respectively, where α is a negative real constant.

(a) Find the angle between OA and OB. [1]

Let $\measuredangle OBA = \theta$, where θ is in degrees. It is given that $\sin\theta = \frac{1}{2}$.

(b) Find the two possible values of θ, and justify which of the two values is the correct value of θ. [2]

(c) By considering a suitable scalar product, find the exact value of α. [3]

A point C has position vector $\mathbf{c}$ such that $\mathbf{c} = \mathbf{b} - \mathbf{a}$.

(d) State the shape of the quadrilateral $OABC$, justifying your answer. Find the exact area of the quadrilateral $OABC$. [3]

9 The function f is defined by

$$\mathrm{f} : x \mapsto \ln(x+1)^2 + 2, \quad x \in \mathbb{R},\ x \geq 0.$$

(a) Show that f has an inverse. [2]

(b) Find $\mathrm{f}^{-1}(x)$ and state the domain of f^{-1}. [3]

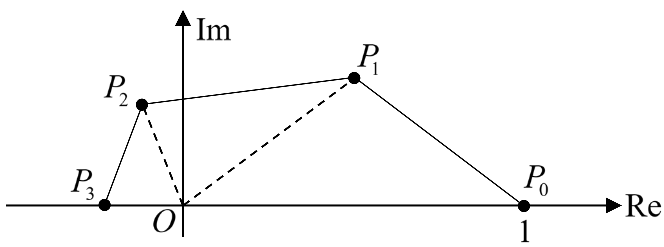
The function g is defined by

$$\mathrm{g} : x \mapsto \frac{1-x}{x+1}, \quad x \in \mathbb{R},\ x \neq -1.$$

(c) Given that g^{-1} exists, show that $\mathrm{g}^{-1}\mathrm{f}^{-1}$ exists, and find the range of $\mathrm{g}^{-1}\mathrm{f}^{-1}$. [3]

(d) Find the exact value of x such that $\mathrm{g}^{-1}\mathrm{f}^{-1}(x) = \frac{1}{2}$. [3]

10 A complex number z is given by $z = r\mathrm{e}^{\mathrm{i}\frac{\pi}{n}}$, where $0 < r < 1$ and n is a positive integer with $n \geq 3$. The numbers $1, z, z^2, \ldots, z^n$ can be represented by the points $P_0, P_1, P_2, \ldots, P_n$ respectively in an Argand diagram. The $(n+1)$-sided polygon formed by using $P_0, P_1, P_2, \ldots, P_n$ is called the $(n+1)$-*polygon generated by* z. An example of a $(3+1)$-polygon generated by z is shown in the following Argand diagram (not drawn to scale).



Let $z = \frac{1}{4}(1+\sqrt{3}\,\mathrm{i})$ for parts **(a)** to **(c)**.

(a) Express z in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $0 < \theta \le \pi$. [2]

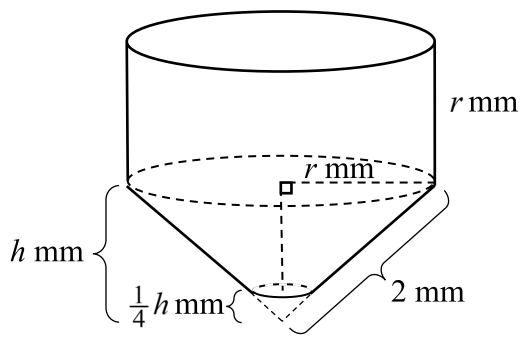
(b) Hence write down z^2 and z^3 in similar form. [2]

(c) Find the exact area of triangle OP_0P_1, where O is the origin. Hence find the exact area of the $(3+1)$-*polygon generated by* z. [4]

Let $z = \frac{1}{2}\mathrm{e}^{\mathrm{i}\frac{\pi}{n}}$ for part **(d)**.

(d) Find the area of an $(n+1)$-*polygon generated by* z in terms of n, leaving your answer in the form $a(1-b^n)\sin\frac{\pi}{n}$, where a and b are real numbers to be determined. [3]

11 [The volume of a right circular cone of radius r and height h is $\frac{1}{3}\pi r^2 h$.]



A student makes a printer nozzle for his self-built 3D printer. The printer nozzle consists of a hollow inverted right circular cone of negligible thickness with radius r mm where $0 < r < 2$, height h mm and slanted edge 2 mm joined to a hollow cylinder of negligible thickness with radius r mm and height r mm. A height of $\frac{1}{4}h$ mm is cut off from the vertex of the cone for the nozzle opening (see diagram).

The volume of the printer nozzle is V mm^3.

(a) Show that $V = \pi r^3 + \frac{21}{64}\pi r^2\sqrt{4-r^2}$. [3]

The student wants V to be a maximum.

(b) It is given that $r = r_1$ gives the maximum value of V. Show that r_1 satisfies the equation $4537r^4 - 18736r^2 + 3136 = 0$. [4]

(c) Show that one of the positive roots of the equation in part **(b)** does not give a stationary value of V. Suggest a reason why this value is a solution to the equation in part **(b)** even though it does not give a stationary value of V. [3]

(d) Given that r_1 is the largest positive root of the equation in part **(b)**, state the value of r_1 and find the corresponding value of h. (You need not show that your answer gives a maximum.) [1]

(e) With reference to the value of r_1 found in part **(d)**, comment on the practicality of having a maximum volume for the printer nozzle. [1]

12 A data processing centre processes data (in suitable units) every day. Let u_n, where n is an integer, $n \geq 1$, represents the amount of data processed each day starting from 1st September. It is given that

$$u_{n+1} = (1+p)u_n \ ,$$

where p is a positive real constant.

(a) Explain why the sequence $\{u_n\}$ is a geometric progression. [1]

On 1st September, 2^m units of data, where m is an integer, $m \geq 6$, were processed.

(b) If the total amount of data processed in the three days from 1st September to 3rd September is $\frac{127}{36}(2^m)$ units, find the value of p. [3]

(c) Explain if there is a limit to the total amount of data the centre can handle in the long run. [1]

Let $p = \frac{1}{4}$ for the rest of the question.

It is desired that data processors at the centre operate at below their maximum capacities to avoid processor downtime due to overheating. In a revised operating procedure, it is proposed that v_r (where r is a positive integer, $r \geq 1$), the amount of data processed each day starting from 1st October, follows a sequence given by

$$v_r = \begin{cases} u_r & \text{for } 1 \leq r \leq 4, \\ v_{r-1} - 25 & \text{for } r \geq 5. \end{cases}$$

(d) Show that

$$v_r = k[5(2^{m-6}) - r + 4] \quad \text{for } r \geq 4,$$

where k is a constant to be determined. [4]

(e) Find the total amount of data processed up to the 15th day, starting from 1st October. Give your answer in the form $s(2^m)+t$, where s and t are constants to be determined. [3]

(f) Assume that the revised operating procedure is adopted. Explain, in context, if it is meaningful to compute the total amount of data processed in the long run. [1]