2023 EJC Prelim Paper 2 Solutions

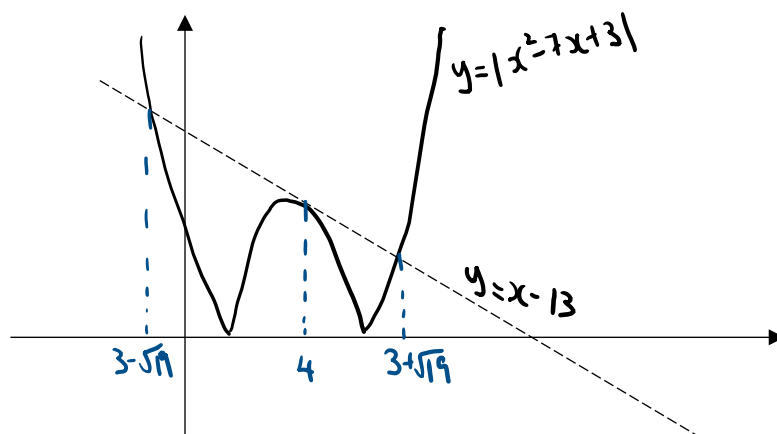| Q1	

	$4z + 5iw = 7$ (1)
	
	$(1-i)z + 8w = 30$ (2)
	
	By (2), $4z + 16(1+i)w = 60(1+i)$ (3)
	
	$(3)-(1):$ $(16+11i)w = 53 + 60i$
	
	               $w = 4 + i$
	
	By (1): $4z + 5i(4+i) = 7$
	
	              $z = 3 - 5i$

Q2

(a) Consider the following cases:

$x^2 - 7x + 3 = 13 - x$ $x^2 - 6x - 10 = 0$ $x = \dfrac{6 \pm \sqrt{6^2 - 4(1)(-10)}}{2(1)}$ $= \dfrac{6 \pm \sqrt{76}}{2} = \dfrac{6 \pm 2\sqrt{19}}{2}$ $= 3 \pm \sqrt{19}$	$-\left(x^2 - 7x + 3\right) = 13 - x$ $x^2 - 8x + 16 = 0$ $x = \dfrac{8 \pm \sqrt{8^2 - 4(1)(16)}}{2(1)}$ $= 4$

The roots are $3 \pm \sqrt{19}$ and 4.

(b)

From the graph, the solution is $3 - \sqrt{19} < x < 4$ or $4 < x < 3 + \sqrt{19}$

Q3	
	(a) $y = z \sec x$ $$\frac{dy}{dx} = z \sec x \tan x + \frac{dz}{dx} \sec x$$ Sub in the DE, $$\pi \left(z \sec x \tan x + \frac{dz}{dx} \sec x \right) + z \sec x (3 - \pi \tan x) = 0$$ $$\pi z \sec x \tan x + \pi \frac{dz}{dx} \sec x + 3z \sec x - \pi z \sec x \tan x = 0$$ $$\pi \frac{dz}{dx} \sec x + 3z \sec x = 0$$ $$\pi \frac{dz}{dx} \sec x = -3z \sec x$$ $$\frac{dz}{dx} = \frac{-3z}{\pi} \quad \text{(shown)}$$ $$\int \frac{1}{z} \, dz = \int \frac{-3}{\pi} \, dx$$ $$\ln|z| = \frac{-3}{\pi} x + C$$ $$z = \pm e^{\frac{-3}{\pi} x + C}$$ $$z = A e^{\frac{-3}{\pi} x} \quad \text{where } A = \pm e^C$$ $$\frac{y}{\sec x} = A e^{\frac{-3}{\pi} x}$$ $$y \cos x = A e^{\frac{-3}{\pi} x}$$ When $x = \frac{\pi}{3}$ and $y = 2$, $$2 \cos \frac{\pi}{3} = A e^{\frac{-3}{\pi} \left(\frac{\pi}{3} \right)}$$ $$A = e$$ $$\therefore y \cos x = e \left(e^{\frac{-3}{\pi} x} \right)$$ $$y \cos x = e^{1 - \frac{3}{\pi} x} \quad \text{where } a = 1 \text{ and } b = -\frac{3}{\pi}$$

(b) $\quad y = \dfrac{e^{1-\frac{3}{\pi}x}}{\cos x}$

For vertical asymptotes, consider $\cos x = 0$

The asymptotes closest to y-axis are $x = -\dfrac{\pi}{2}$ and $x = \dfrac{\pi}{2}$

Q4	
(a)	$x^3 + y^3 - xy = A$ Differentiating with respect to x, $3x^2 + 3y^2 \dfrac{dy}{dx} - x\dfrac{dy}{dx} - y = 0$ $\left(3y^2 - x\right)\dfrac{dy}{dx} = -3x^2 + y \qquad (*)$ When $\dfrac{dy}{dx} = 0$, $0 = -3x^2 + y$ $y = 3x^2$ (shown)
(b)	Sub $y = 3x^2$ into $x^3 + y^3 - xy = A$, $x^3 + \left(3x^2\right)^3 - x\left(3x^2\right) = A$ $x^3 + 27x^6 - 3x^3 = A$ $27x^6 - 2x^3 - A = 0$ $x^3 = \dfrac{-(-2) \pm \sqrt{(-2)^2 - 4(27)(-A)}}{2(27)}$ $x^3 = \dfrac{2 \pm 2\sqrt{1 + 27A}}{2(27)}$ $x^3 = \dfrac{1 \pm \sqrt{1 + 27A}}{27}$ $x = \sqrt[3]{\dfrac{1 \pm \sqrt{1 + 27A}}{27}} = \dfrac{1}{3}\sqrt[3]{1 \pm \sqrt{1 + 27A}}$

	For more than one stationary point,
	Discriminant $1 + 27A > 0$
	So $A > -\dfrac{1}{27}$
(c)	From (*), $\left(3y^2 - x\right)\dfrac{dy}{dx} = -3x^2 + y$
	Differentiate with respect to x,
	$\left(3y^2 - x\right)\dfrac{d^2 y}{dx^2} + \left(6y\dfrac{dy}{dx} - 1\right)\dfrac{dy}{dx} = -6x + \dfrac{dy}{dx}$
	When $\dfrac{dy}{dx} = 0$, $y = 3x^2$, so we have
	$\left(3\left(3x^2\right)^2 - x\right)\dfrac{d^2 y}{dx^2} = -6x$
	$\left(27x^4 - x\right)\dfrac{d^2 y}{dx^2} = -6x$
	$\dfrac{d^2 y}{dx^2} = -\dfrac{6x}{27x^4 - x} = -\dfrac{6}{27x^3 - 1}$
	When $x^3 = \dfrac{1 + \sqrt{1 + 27A}}{27}$,
	$\dfrac{d^2 y}{dx^2} = -\dfrac{6}{27x^3 - 1} = -\dfrac{6}{\sqrt{1 + 27A}} < 0$ so this is a maximum point.
	When $x^3 = \dfrac{1 - \sqrt{1 + 27A}}{27}$,
	$\dfrac{d^2 y}{dx^2} = -\dfrac{6}{27x^3 - 1} = \dfrac{6}{\sqrt{1 + 27A}} > 0$ so this is a minimum point.

Q5
(a) Since p and q are perpendicular, a direction vector parallel to q is $\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$.

Normal to $q = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \times \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \\ -4 \end{pmatrix} = -2 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

Equation of q: $\mathbf{r} \bullet \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = 7$

$$x + 2y + 2z = 7$$

(b) $2x - z = 10$

$x + 2y + 2z = 7$

Solve using GC:

$x = 5 + \dfrac{1}{2}z$

$y = 1 - \dfrac{5}{4}z$

$z = z$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix} + \frac{1}{4}z \begin{pmatrix} 2 \\ -5 \\ 4 \end{pmatrix}$$

Let $\mu = \dfrac{1}{4}z$, equation of m is $\mathbf{r} = \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -5 \\ 4 \end{pmatrix}$, $\mu \in \mathbb{R}$

(c) Let F be the foot of perpendicular from A to q.

Method 1

$l_{AF} : \mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ -6 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$, $\alpha \in \mathbb{R}$

Since F lies on l_{AF}, $\overrightarrow{OF} = \begin{pmatrix} 2 + \alpha \\ 1 + 2\alpha \\ -6 + 2\alpha \end{pmatrix}$ for some $\alpha \in \mathbb{R}$

Since F also lies on q, $(2 + \alpha) + 2(1 + 2\alpha) + 2(-6 + 2\alpha) = 7$

$$-8 + 9\alpha = 7$$

$$\alpha = \frac{5}{3}$$

$\therefore \overrightarrow{OF} = \begin{pmatrix} 11/3 \\ 13/3 \\ -8/3 \end{pmatrix}$

$k = \left| \overrightarrow{AF} \right| = \left| \begin{pmatrix} 5/3 \\ 10/3 \\ 10/3 \end{pmatrix} \right| = 5$

Method 2:

Since p and q are perpendicular, the foot of perpendicular F lies on m.

$$\overrightarrow{OF} = \begin{pmatrix} 5+2\omega \\ 1-5\omega \\ 4\omega \end{pmatrix} \text{ for some } \omega \in \mathbb{R}$$

$$\overrightarrow{AF} = \overrightarrow{OF} - \overrightarrow{OA} = \begin{pmatrix} 3+2\omega \\ -\omega \\ 6+4\omega \end{pmatrix}$$

Since AF is perpendicular to m, $\begin{pmatrix} 3+2\omega \\ -5\omega \\ 6+4\omega \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -5 \\ 4 \end{pmatrix} = 0$

$$6 + 4\omega + 25\omega + 24 + 16\omega = 0$$

$$45\omega = -30$$

$$\omega = -\frac{2}{3}$$

$$\therefore \overrightarrow{OF} = \begin{pmatrix} 11/3 \\ 13/3 \\ -8/3 \end{pmatrix}$$

$$k = \left| \overrightarrow{AF} \right| = \left| \begin{pmatrix} 5/3 \\ 10/3 \\ 10/3 \end{pmatrix} \right| = 5$$

(d) The other point lying on p with shortest distance 5 units from q is A', the reflection of A about m.

By Ratio Theorem, $\overrightarrow{OF} = \dfrac{\overrightarrow{OA} + \overrightarrow{OA'}}{2}$

$$\overrightarrow{OA'} = 2\overrightarrow{OF} - \overrightarrow{OA} = 2\begin{pmatrix} 11/3 \\ 13/3 \\ -8/3 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ -6 \end{pmatrix} = \begin{pmatrix} 16/3 \\ 23/3 \\ 2/3 \end{pmatrix}$$

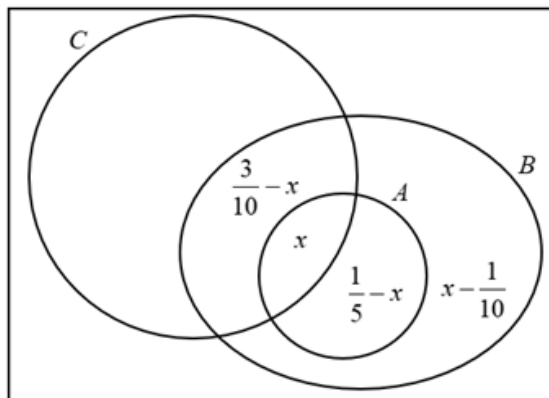Both the lines must be parallel to m.

$$\therefore \text{Equation of lines: } \mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ -6 \end{pmatrix} + \alpha \begin{pmatrix} 2 \\ -5 \\ 4 \end{pmatrix}, \ \alpha \in \mathbb{R}$$

$$\text{and } \mathbf{r} = \begin{pmatrix} 16/3 \\ 23/3 \\ 2/3 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ -5 \\ 4 \end{pmatrix}, \ \beta \in \mathbb{R}$$

Q6	
(a)	Let $P(A \cap C) = x$. $P(A \cap C') = \dfrac{1}{5} - x$

$$P(A' \cap B \cap C) = \frac{3}{10} - x$$

$$P(A' \cap B \cap C') = 2\left(\frac{1}{5}\right) - \frac{1}{5} - \left(\frac{3}{10} - x\right) = x - \frac{1}{10}$$

$$\frac{1}{5} - x \geq 0 \qquad \text{and} \qquad \frac{3}{10} - x \geq 0 \qquad \text{and} \qquad x - \frac{1}{10} \geq 0$$
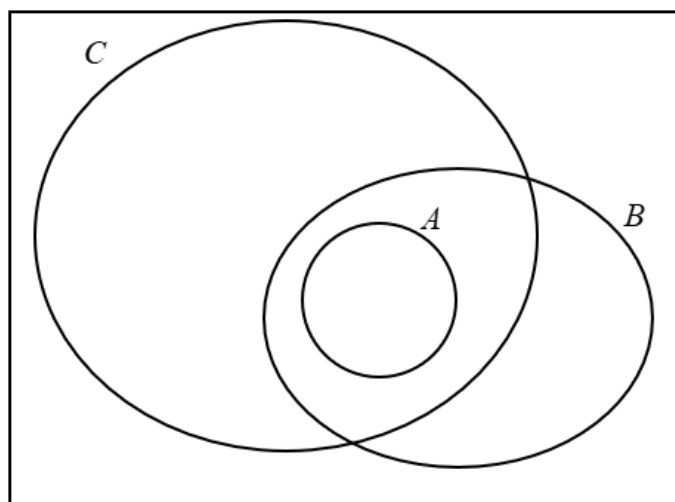
$$x \leq \frac{1}{5} \qquad \text{and} \qquad x \leq \frac{3}{10} \qquad \text{and} \qquad x \geq \frac{1}{10}$$

Hence, $\frac{1}{10} \leq x \leq \frac{1}{5}$.

Greatest value of $P(A \cap C) = \frac{1}{5}$

Least value of $P(A \cap C) = \frac{1}{10}$

(b)	

(c)	$P(A) = \dfrac{1}{5}$

$$P(B' \cap C) = 1 - \frac{1}{12} - 2\left(\frac{1}{5}\right) = \frac{31}{60}$$

$$P(C) = \frac{31}{60} + \frac{3}{10} = \frac{49}{60}$$

For A and C to be independent,

$$P(A \cap C) = P(A) \times P(C) = \frac{1}{5} \times \frac{49}{60} = \frac{49}{300}$$

Q7	
(a)	$E(X+Y) = 1 + \mu$ $\text{Var}(X+Y) = 1 + 2 = 3$ $X + Y \sim N\left(1 + \mu, \sqrt{3}^2\right)$ $P(0 \le X + Y < 3) > 0.44$ Using graph on GC, $-1.1013 < \mu < 2.1013$ $-1.10 < \mu < 2.10$
(b)	$E(X_1 + X_2 + \ldots + X_n - 2Y) = n - 2(10) = n - 20$ $\text{Var}(X_1 + X_2 + \ldots + X_n - 2Y) = n + 2^2(2) = n + 8$ $X_1 + X_2 + \ldots + X_n - 2Y \sim N(n - 20, n + 8)$ $P(X_1 + X_2 + \ldots + X_n - 2Y \ge 10) < 0.03$ From GC, <table><tr><td>n</td><td>$P(X_1 + X_2 + \ldots + X_n - 2Y \ge 10)$</td></tr><tr><td>20</td><td>0.0294</td></tr><tr><td>21</td><td>0.0473</td></tr></table> So largest n is 20
(c)	We need to assume that X and Y are independent.

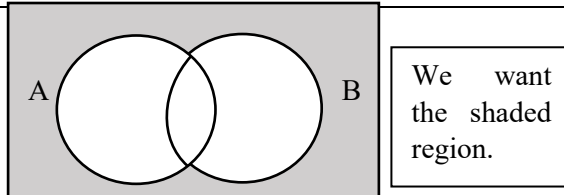| Q8	Solution
(a)	Whether or not a vase is defective is independent of other vases being defective. The probability that a vase is defective is a constant.
(b)	$X \sim B(30, 0.04)$ $\begin{aligned}P(X > 2) &= 1 - P(X \leq 2)\\&= 0.11690 = 0.117 \,(3\,sf)\end{aligned}$
(c)	Let Y be the no. of days where more than 2 defective vases are found, out of 5 days. $Y \sim B(5, 0.11690)$ $P(Y \leq 1) = 0.893$
(d)	Let V be the total no. of defective vases found in 5 days. $V \sim B(150, 0.04)$ Required probability $$= \frac{\begin{array}{c} P(X_1 = 4)P(X_2 = 1)\left[P(X_3 = 0)\right]^3 \times \dfrac{4!}{3!} \times 2! \\[2mm] + P(X_1 = 3)P(X_2 = 2)\left[P(X_3 = 0)\right]^3 \times \dfrac{4!}{3!} \times 2! \end{array}}{P(V = 5)}$$ $$= \frac{0.0056980}{0.16280} = 0.0350$$

Q9
(a) No. of results $= {}^{32}C_5 \times 5! = 24165120$
(b) No. of results $= \begin{pmatrix} \text{no. of results} \\ \text{without restrictions} \end{pmatrix} - \begin{pmatrix} \text{no. of results} \\ \text{with traits from} \\ \text{only 1 domain} \end{pmatrix}$ $= 24165120 - {}^8C_5 \times 5! \times 4 = 24138240$
(c) Case 1: Domain repeated is Cognition No. of results $= \underbrace{{}^8C_2}_{\substack{\text{choose 2}\\\text{Cognition}\\\text{traits}}} \times \underbrace{({}^8C_1)^3}_{\substack{\text{choose 1 trait}\\\text{each from the}\\\text{non-Cognition}\\\text{domain}}} \times \underbrace{{}^3C_2 \times 2!}_{\substack{\text{pick two slots}\\\text{for the Cognition}\\\text{traits among the}\\\text{last 3 and place}\\\text{them there}}} \times \underbrace{3!}_{\substack{\text{no of ways}\\\text{to permute the}\\\text{remaining 3 traits}}} = 516096$

Case 2: Domain repeated is not Cognition

3 possible ways to pick "repeating" domain	Pick 2 traits from "repeating" domain	Pick 1 trait from each of the remaining domains	Place Cognition trait in last three slots	No of ways to permute remaining 4 traits

No of results $= {}^3C_1 \times {}^8C_2 \times ({}^8C_1)^3 \times {}^3C_1 \times 4! = 3096576$

Total no. of results $= 516096 + 3096576 = 3612672$

(d)

We want the shaded region.

A: event that Rapport traits are consecutive

B: event that Planning traits are consecutive

Method 1

$$n\big((A \cup B)'\big) = n(E) - n\big(A \cup B\big) = n(E) - \big(n(A) + n(B \cap A')\big)$$

Case 1: R consecutive

No. of results $= 4! \times 2! = 48$

Case 2: P consecutive, R not consecutive

No. of results $= 2! \times {}^3C_2 \times 2! \times 2! = 24$

Total no. of results

$= \begin{pmatrix} \text{no. of results} \\ \text{without restrictions} \end{pmatrix} - \begin{pmatrix} \text{no. of results with} \\ \text{R consecutive} \end{pmatrix}$

$- \begin{pmatrix} \text{no. of results with} \\ \text{P consecutive, R not} \end{pmatrix}$

$= 5! - 48 - 24$

$= 48$

Alternative 1:

$$n\big((A \cup B)'\big) = n(E) - n\big(A \cup B\big) = n(E) - \big(n(A \cap B') + n(B \cap A') + n\big(A \cap B\big)\big)$$

Complement cases are P consecutive but R not, R consecutive but P not and (R consecutive and P consecutive).

Case 1: P consecutive, R not consecutive

No. of results $= 2! \times {}^3C_2 \times 2! \times 2! = 24$

Case 2: R consecutive, P not consecutive

No. of results $=$ same as case 1 $= 24$

Case 3: R consecutive and P consecutive

No. of results $= 3! \times 2! \times 2! = 24$
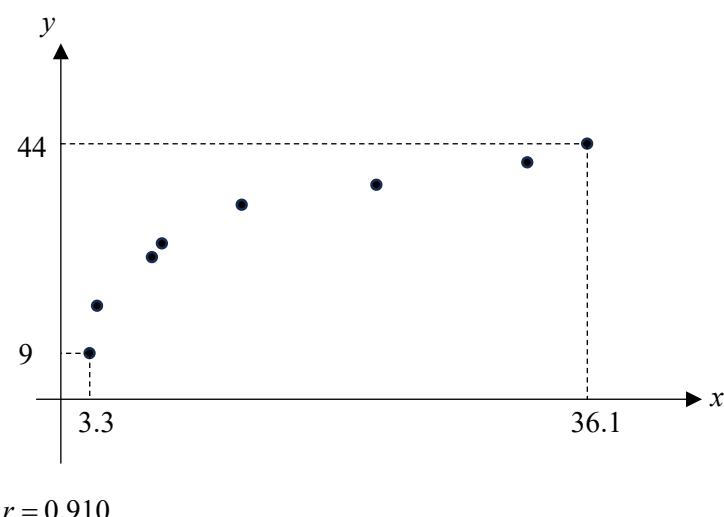
Total no. of results

$= 5! - 24 - 24 - 24$

$= 48$

Alternative 2:

n((A \cup B)') = n(E) - n(A \cup B) = n(E) - \big(n(A) + n(B) - n(A \cap B) \big)
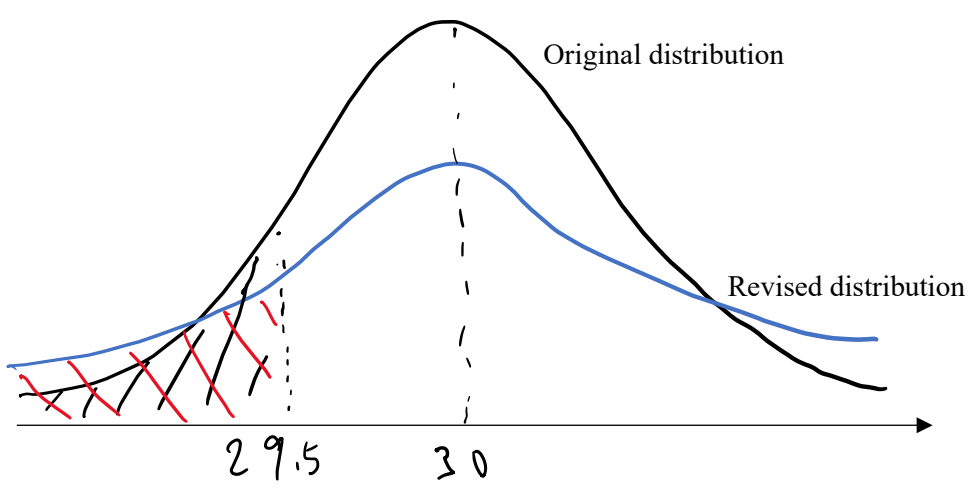
Total no. of results

$$= \begin{pmatrix} \text{no. of results} \\ \text{without} \\ \text{restrictions} \end{pmatrix} - \begin{pmatrix} \text{no. of results} \\ \text{with R} \\ \text{consecutive} \end{pmatrix}$$
$$- \begin{pmatrix} \text{no. of results} \\ \text{with P} \\ \text{consecutive} \end{pmatrix} + \begin{pmatrix} \text{no. of results} \\ \text{with both} \\ \text{consecutive} \end{pmatrix}$$

$= 5! - 48 - 48 + 24$

$= 48$

Q10	
(ai)	165
(ii)	$\dfrac{1}{7}\left[\left(7510 + t^2\right) - \dfrac{(216+t)^2}{8} \right]$ i.e., $A = 7$ $B = 9^2 + 25^2 + 27^2 + 33^2 + 37^2 + 41^2 + 44^2 = 7510$ $C = 9 + 25 + 27 + 33 + 37 + 41 + 44 = 216$ $D = 8$

(b)	
	$r = 0.910$
(c)	No, from the sketch there is a non-linear relationship between the variables.
(d)	$p = 12.9$, $q = -2.27$, $r = 0.984$
(e)	When average root hair length increases by e times, the amount of potassium found in the roots increases by 12.9 mg. *OR* For every unit increase in the natural log of the average root hair root length, the amount of potassium found in the roots increases by 12.9 mg.
(f)	41.8 mg Yes, it is reliable because $|r|$ is close to 1 showing a strong linear relationship between y and $\ln x$, and the value of 30.0 mm lies within the range of values of x ($3.3 < x < 36.1$) and is an interpolation.
(g)	No, because there is no clear independent & dependent variable, so a regression line of $\ln x$ on y would be more appropriate.

Q11	Solution
a	If shopping times follow a normal distribution, the median, $Q_2 = 18.5$, should be close to the mean of 30. But the mean is actually closer to Q_3 than Q_2. **Alternatively** Since a normal distribution is symmetrical, if shopping times follow a normal distribution, we should expect $Q_3 - Q_2$ and $Q_2 - Q_1$ to be roughly equal. But the latter is much larger than the former.
b	Jack should carry out a one-tail test. Null hypothesis or $H_0 : \mu = 30$ Alternative hypothesis or $H_1 : \mu < 30$ μ is the mean duration customers <u>spend in his clothing shop</u> (in min).

c	Sample mean $= 1475 \div 50 = 29.5$ Unbiased estimate of popn variance $= \dfrac{1}{49}\left(45029 - \dfrac{1475^2}{50}\right) = 30.949$ Test $H_0 : \mu = 30$ against $H_1 : \mu < 30$ at 10% level of significance. Under H_0, since $n = 50$ is large, $\bar{X} \sim N\left(30, \dfrac{30.949}{50}\right)$ approximately by Central Limit Theorem From G.C., p-value $= 0.263$ Since p-value > 0.10, we do not reject H_0 and conclude that there is insufficient evidence at 10% level of significance that the mean duration customers spend in his shop is less than 30 min.
d	It means that there is a 10% chance of concluding that the mean shopping time in Jack's store is less than 30 min when it is in fact 30 min.
e	Regardless of the distribution of shopping times in his shop, since n = 50 is large, by Central Limit Theorem, the <u>mean</u> shopping time is still approximately normally distributed. (So the hypothesis test can still be carried out.)
f	Since $\sum x^2$ is larger, the revised value of s^2, i.e. the revised unbiased estimate of the population variance, will be larger. So the revised p-value – the proportion of the distribution that is less than 29.5 – will be larger than before. But the original p-value was already larger than 0.1, so the new p-value will continue to be larger than 0.1. --- Original p-value $= 0.263$ --- New p-value > 0.263 (Compare the unshaded parts) OR

Since the revised z-value, $\dfrac{29.5 - 30}{s}$, has a larger denominator than before, the new z-value will be less negative, i.e. larger, than before. But the original z-value was already larger than the critical value, So the new z-value will continue to be larger than the critical value.

Hence, Jack's conclusion in part (c) is unchanged.