EUNOIA JUNIOR COLLEGE

JC2 Preliminary Examination 2023

General Certificate of Education Advanced Level

Higher 2

CANDIDATE NAME	

CIVICS GROUP		INDEX NO.	

MATHEMATICS

9758/01

Paper 1 [100 marks]

12 September 2023

3 hours

Candidates answer on the Question Paper.

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name, civics group and index number on the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.
DO **NOT** WRITE ON ANY BARCODES.

Answer **all** questions.
Write your answers in the spaces provided in the Question Paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **23** printed pages and **1** blank page.

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	Q12	Total

1 The general solution of a differential equation is given by $y = Ae^{x} + Be^{2x} + Ce^{-2x}$, where A, B and C are arbitrary constants. Given that a particular solution satisfies $y = 2$, $\frac{dy}{dx} = -3$ and $\frac{d^2y}{dx^2} = 11$ when $x = 0$, find the values of A, B and C. [4]

2 A sequence of numbers $u_1, u_2, u_3, \ldots$ has sum S_n where $S_n = \sum_{r=1}^{n} u_r$. It is given that $S_n = 18 - \frac{2^{n+1}}{3^{n-2}}$.

(a) By finding an expression for u_n, or otherwise, show that the sequence is a geometric progression. [3]

An arithmetic progression has first term -4. The sum of the first 9 terms of the arithmetic progression is equal to $\sum_{r=1}^{\infty} u_r$.

(b) Find the common difference of the arithmetic progression. [3]

3 With reference to the origin O, the points A, B and C are such that $\overrightarrow{OA} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$ and $\overrightarrow{OC} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$.

Point P lies on AB, between A and B, such that $AP : PB = \lambda : 1 - \lambda$ where $0 < \lambda < 1$. Point Q lies on BC, between B and C, such that $BQ : QC = \lambda : 1 - \lambda$.

(a) Express $\overrightarrow{PQ}$ in terms of λ. [3]

The point X has coordinates $\left(2, -1, -\frac{5}{2}\right)$. Points P, Q and X are collinear.

(b) Find the value of λ. [3]

4 **(a)** Using the method of differences, find $\sum_{r=2}^{n} \frac{2}{r^2 - 1}$. [4]

(b) Hence, find $\sum_{r=11}^{3n-1} \frac{2}{r(r+2)}$. [3]

5 **(a)** Find $\int \sin 3x \cos x \, dx$. [2]

(b) Find $\int e^{2x} \cos 3x \, dx$. [5]

6 A curve R has equation $y = \mathrm{f}(x)$. The point P with coordinates $(7, -9)$ lies on R. The tangent to R at point P has gradient 10.

(a) The curve C is obtained from R by applying the following two transformations:

- A scaling parallel to the y-axis with scale factor 3, and
- A translation in the negative x-direction by 4 units.

Find the coordinates of the point on C corresponding to P, and state the gradient of C at this point. [2]

(b) The curve D has equation $y = \dfrac{1}{\mathrm{f}(x)}$. Find the coordinates of the point on D corresponding to P, and find the gradient of D at this point. [3]

7 The parametric equations of a curve C_1 are $x = 3t^3 + 2t^2$ and $y = -t^3 + 2t^2 + 1$ for $-1 \le t \le 1$.

(a) Sketch C_1, stating the exact coordinates of the endpoints and any axial intercepts. [3]

Another curve C_2 has equation $x^2 + (y-1)^2 = k$ where $k \in \mathbb{R}^+$.

(b) Describe C_2 geometrically. [1]

(c) For $k = 16$, find the coordinates of the point of intersection between C_1 and C_2. [4]

(d) State all the possible number of point(s) of intersection between C_1 and C_2 as k varies. [1]

8 The function f is defined by

$$\mathrm{f} : x \mapsto 4 + \frac{1}{(x-1)^2 + 1}, \quad x \ge 0.$$

(a) Sketch the graph of $y = \mathrm{f}(x)$, indicating clearly the coordinates of any stationary points, the value of the y-intercept, and the equation of any asymptotes. [3]

The function g is defined by

$$\mathrm{g} : x \mapsto \sin x, \quad x \in [0, 2\pi].$$

(b) State the exact coordinates of two points on the graph of $y = \mathrm{g}(x)$, which demonstrate that g does not have an inverse. [1]

(c) Explain clearly why the composite function gf exists. [2]

(d) Using a suitably-labelled sketch of the graph of $y = \mathrm{g}(x)$, find the range of gf. [3]

9 **(a)** Using standard series, find the Maclaurin series expansion of $\dfrac{1}{\sqrt{1-a^2x^2}}$ up to and including the term in x^4, where a is a positive constant. Find also the range of values of x, in terms of a, for which the expansion is valid. [5]

(b) Hence, by using a suitable value for a, show that $\cos^{-1} x \approx \dfrac{\pi}{2} - x - \dfrac{1}{6}x^3 - \dfrac{3}{40}x^5$. [3]

(c) Using the result in part **(b)**, estimate $\int_0^{\frac{1}{2}} \cos^{-1} x \, dx$ to 5 significant figures. [2]

(d) Comparing your estimate in **(c)** with the actual value of $\int_0^{\frac{1}{2}} \cos^{-1} x \, dx$, comment on the accuracy of your estimate and suggest how it can be further improved. [2]

10 **Do not use a calculator in answering this question.**
The complex numbers z and w have the same modulus r, and have arguments α and β respectively.

(a) Show that $z + w = 2r\cos\dfrac{\alpha-\beta}{2}\left(\cos\dfrac{\alpha+\beta}{2} + \mathrm{i}\sin\dfrac{\alpha+\beta}{2}\right)$ and hence write down expressions for $|z+w|$ and for $\arg(z+w)$. [4]

It is now given that $z = 2\mathrm{e}^{\mathrm{i}\frac{\pi}{12}}$ and $w = 2\mathrm{e}^{\mathrm{i}\frac{5\pi}{12}}$.

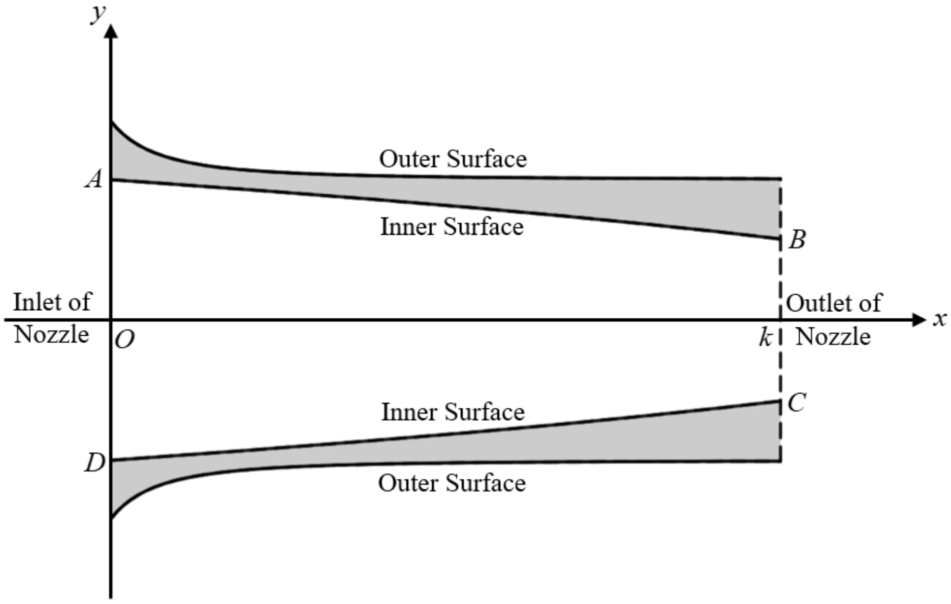
(b) Find the values of $|z+w|$ and $\arg(z+w)$. [2]

Let v be the complex number such that $v = \dfrac{z^2}{w}$.

(c) Find $|v|$ and $\arg(v)$. [3]

(d) Let O be the origin and P and Q be points representing $z+w$ and v respectively on an Argand diagram. Find the area of triangle OPQ. [2]

11 Nozzles are often used at the end of hoses to direct and speed up the flow of water. The diagram below represents the side view of a nozzle.



The nozzle itself consists of solid material between an outer surface and an inner surface (shaded in the diagram). Water flows through the hollow region $ABCD$ enclosed by the inner surface. The outer surface of the nozzle is obtained by rotating the curve with equation

$$y^2 = k + \frac{2k}{(x+1)(x+2)}, \qquad \text{for } 0 \le x \le k \text{, where } k \in \mathbb{R}, k \ge 1,$$

completely about the x-axis.

Similarly, the inner surface is obtained by rotating the curve with equation

$$y^2 = k - \frac{x}{3}, \qquad \text{for } 0 \le x \le k,$$

completely about the x-axis. The units of x and y are centimetres.

(a) Show that the volume of material used to make the nozzle is $\pi\left[2k\ln\left(\frac{2k+2}{k+2}\right)+\frac{k^2}{6}\right]$ cm^3. [6]

(b) Find the area of $ABCD$ in terms of k. [3]

Using the Principle of Conservation of Mass, the following relationship can be found.

$$A_{\text{in}}v_{\text{in}} = A_{\text{out}}v_{\text{out}}$$

A_{in} and A_{out} are the cross-sectional areas of the circular openings at the inlet and outlet of the nozzle respectively, and

v_{in} and v_{out} are the speeds of the water at the inlet and outlet of the nozzle respectively.

(c) Calculate the ratio of relative speeds $\frac{v_{\text{out}}}{v_{\text{in}}}$ for the given nozzle. [3]

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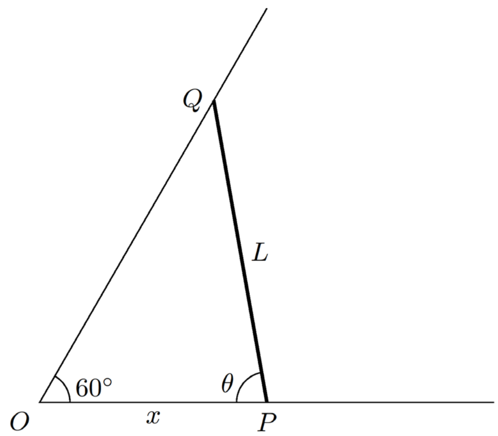


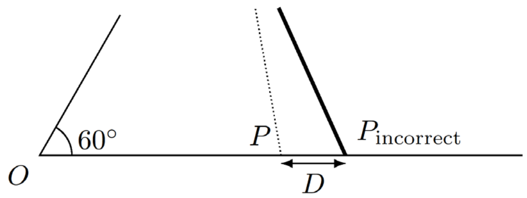
Trinity de Rale is a budding installation artist. Her latest mobile artwork comprises of two fixed rails placed at an angle of 60° to each other at *O* (refer to diagram). A rod *PQ* of fixed length *L* cm is placed such that its ends *P* and *Q* glide freely along each rail.

It is given that the length *OP* is x cm, and that the angle *OPQ* is θ.

(a) Show that $x = L\left(\cos\theta + \frac{1}{\sqrt{3}}\sin\theta\right)$. [2]

A motor is placed at *P* to move that end of the rod along the rail, allowing θ to vary between 30° and 90°. The motor is controlled such that the point *P* moves at a constant rate for 60 seconds, starting from the position where $\theta = 90°$ and ending at the position where $\theta = 30°$.

(b) Find the exact rate of change of θ at the moment when $\theta = 60°$. State clearly the units for your answer. [5]



However, when installing the work, the exhibit curator Mick Schupp programmed the motor incorrectly, causing θ to decrease at a constant rate from $90°$ to $30°$ in 60 seconds. As a result, at time *t* seconds after the motor is started, point *P* is further along its rail by a distance of *D* cm compared to its correct position (refer to diagram).

(c) Find an expression for *D* in terms of *L* and *t*. [3]

(d) Find, to 4 decimal places, the maximum value of $\frac{D}{L}$ during the motion. (You need not justify that the value is maximum.) [2]